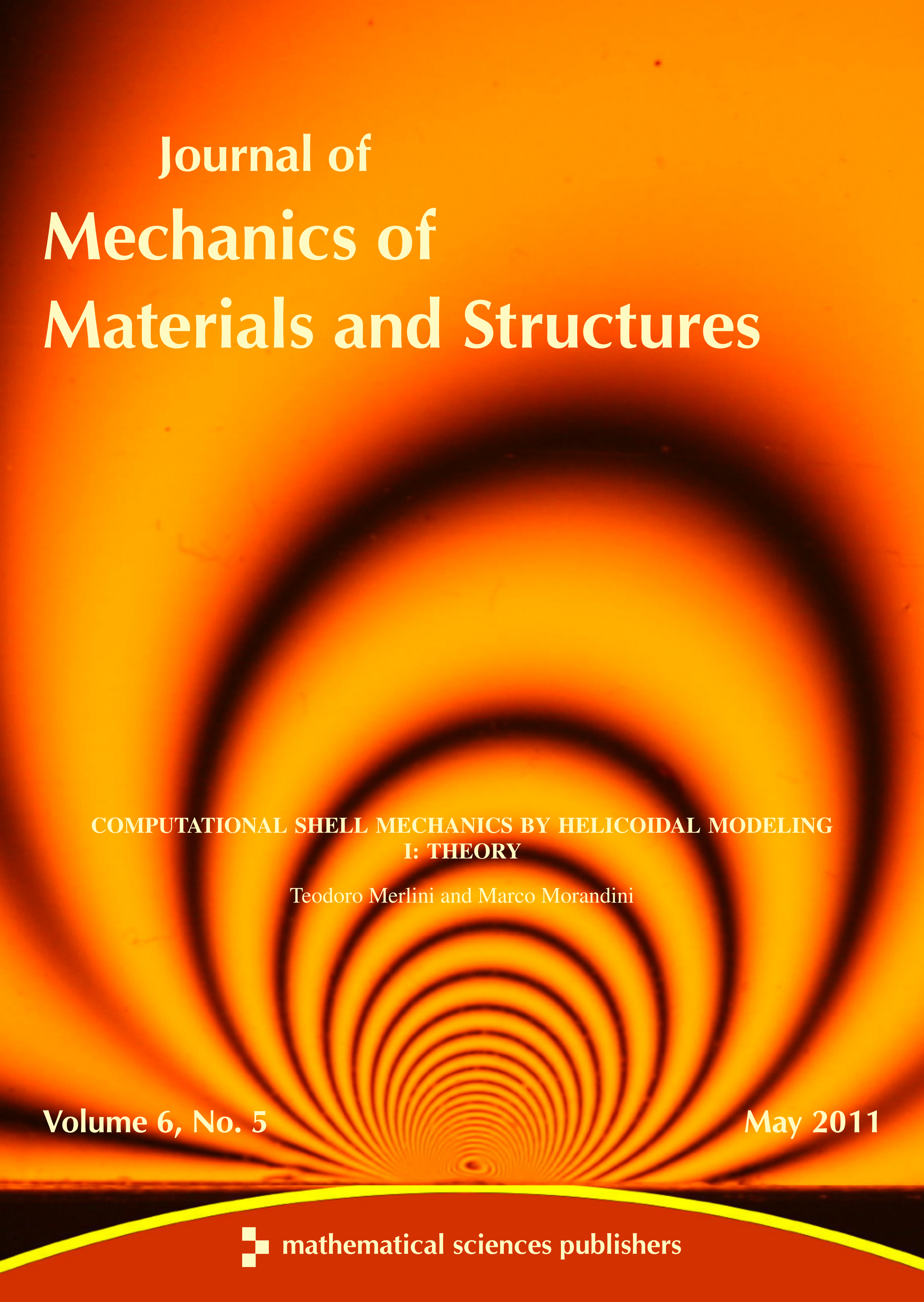

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COMPUTATIONAL SHELL MECHANICS BY HELICOIDAL MODELING I: THEORY

TEODORO MERLINI AND MARCO MORANDINI

Starting from recently formulated helicoidal modeling in three-dimensional continua, a low-order kinematical model of a solid shell is established. It relies on both the six degrees of freedom (DOFs) on the reference surface, including the drilling DOF, and a dual director — six additional DOFs — that controls the relative rototranslation of the material particles within the thickness. Since the formulation pertains to the framework of the micropolar mechanics, the solid shell mechanical model includes a workless stress variable — the axial vector of the Biot stress tensor, referred to as the Biot-axial — that allows us to handle nonpolar materials. The local Biot-axial is approximated with a linear field across the thickness and relies on two vector parameters. On the reference surface, the dual director is condensed locally together with one Biot-axial parameter, leaving the surface strains and the other Biot-axial parameter as the basic variables governing the two-dimensional internal work functional.

The continuum-based shell mechanics are cast in weak incremental form from the beginning. They yield the two-dimensional nonlinear constitutive law of the shell in incremental form, built dynamically along the solution process. Poisson thickness locking, related to the low-order kinematical model, is prevented by a dynamical adaptation of the local constitutive law. No hypotheses are introduced that restrict the amplitudes of displacements, rotations, and strains, so the formulation is suitable for computations with strong geometrical and material nonlinearities, as shown in Part II.

1. Introduction

This study belongs to the research field, very prolific in the eighties and nineties, aimed at writing nonlinear shell mechanics with a full and consistent account of the rotation of the material surface particles. In the present decade, however, the scientific community has seemed to desist from seeking a consistent settlement of the drilling rotation and to focus again on more classical approaches to shell mechanics. A recent work by the authors in three-dimensional finite elasticity has motivated a renewed interest in formulations based on an explicit full three-parametric rotation field, which can be addressed now with greater chance of success. Our present contribution to this research field is characterized by, and can be said to be original in, two respects: on the one hand, a total adhesion to a consistent mechanical formulation based on the micropolar description; on the other, the broad usage of an integral kinematic field.

According to the micropolar description, the particle rotations are retained as primary unknowns, even in the case of nonpolar materials. In three-dimensional solid mechanics, variational formulations assuming independent rotation fields were pioneered in [Reissner 1965] and developed in [Fraeijs de

Keywords: nonlinear shell theory, micropolar shell variational mechanics, helicoidal modeling, geometric invariance, shell constitutive equations, finite rotations and rototranslations, dual tensor algebra.

Veubeke 1972; Atluri 1984; Reissner 1984; Bufler 1985; 1995; Atluri and Cazzani 1995]. Then, finite elements were formulated starting, mostly, from the regularized principles proposed in [Hughes and Brezzi 1989] and extended to finite elasticity in [Simo et al. 1992]; see also [Ibrahimbegović and Frey 1995; Sansour et al. 1996]. In such formulations, the rotation is introduced using the polar decomposition theorem of the deformation gradient mostly like an appended constraint equation. Application of these concepts to shell mechanics was pursued by several authors, mainly with the explicit motivation of accommodating the drilling rotation; those principles were applied to either the two-dimensional domain of the shell surface [Gruttmann et al. 1992; Wriggers and Gruttmann 1993; Ibrahimbegović 1994; Zhu and Zacharia 1996], or the three-dimensional domain across the shell thickness [Li and Zhan 2000]. A number of remarkable works were delivered, such as [Chróścielewski et al. 1992; Sansour and Bufler 1992; Sansour and Bednarczyk 1995; Wisniewski 1998]; these works also include important references to the early approaches to finite rotations in shell mechanics. In [Wisniewski and Turska 2000; 2001; 2002] a formulation was proposed with a linear drilling rotation across the thickness to account for in-plane twist. It should be noted that almost all the aforementioned works rely on strain measures of the Biot type instead of the Green type and exploit a full three-parametric rotation tensor, different than the two-parametric one used in the constrained rotation approach [Simo and Fox 1989]. Significant papers on the parameterization of the rotation tensor are [Betsch et al. 1998; Ibrahimbegović et al. 2001; Wang and Thierauf 2001]. Interestingly, in an early paper Badur and Pietraszkiewicz [1986] developed a nonlinear Kirchhoff–Love shell theory from micropolar continuum variational mechanics, which is in spirit similar to our approach, as described next.

Meanwhile, Merlini [1997] proposed a variational formulation for three-dimensional solid mechanics based on a different approach. Starting from micropolar mechanics and invoking a constitutive postulate, three equations governing the mechanics of nonpolar materials, that is, linear balance, angular balance, and internal kinematical constraint, are consistently deduced. Correspondingly, two more variables — the particle rotation and the workless axial vector of the Biot stress tensor, hereafter called the *Biot-axial* — become clearly identified and join the displacement as primary unknown fields. Displacement, rotation, and the Biot-axial are the mixed — and balanced — unknowns of an irreducible variational principle referred to as the internally constrained principle of virtual work of the nonpolar medium. In the continuum-based shell theory proposed in this paper, we adhere to this approach both for the reduction from three-dimensional mechanics and for the subsequent two-dimensional mechanics of the material surface. The local Biot-axial within the shell thickness is retained as a primary unknown, to be solved as a function of an unknown field on the shell surface; then, this Biot-axial parameter surface field is retained as a primary unknown of the material surface mechanics. The drilling DOF, as well as a drilling twist, are implicitly accounted for by this extension to the shell surface of a mechanical formulation consistently based on the micropolar description.

The second important feature that characterizes this work is related to the helicoidal modeling of the continuum. Customarily, the micropolar description relies on two independent and uncoupled fields: the displacements and the rotations. Alternatively, displacements and rotations may be coupled together into a unique comprehensive kinematic field, referred to as the integral field of the rototranslations. This alternative representation of motion is already used nowadays in computational multibody dynamics [Borri et al. 2000], and descends seemingly from the modeling of sections along space-curved beams [Borri and Bottasso 1994]; in either case, however, the rototranslation field is defined over a one-coordinate domain, the time or a curvilinear abscissa as the case may be. The difficulties inherent in the representation of the rototranslation field over a multicoordinate domain, as in the case of three-dimensional solids, were overcome in [Merlini and Morandini 2004a], who proposed the helicoidal modeling of the continuum as an alternative to the classical Euclidean modeling and provided the relevant variational mechanics. Their work was completed with an original objective interpolation scheme [Merlini and Morandini 2004b] and the formulation of a successful finite element [Merlini and Morandini 2005].

The present shell theory and its numerical implementation resort extensively to helicoidal modeling. First, the three-dimensional solid across the thickness is modeled helicoidally. Though we rely on the simplest scheme, with a constant generalized curvature through the thickness, the resulting low-order solid shell model proves to be quite flexible: the proposed integral kinematic field gives the material particles twelve coupled DOFs, six more than the underlying particle belonging to the shell surface, and this freedom proves advantageous in modeling saddles and buckles. Secondly, the two-dimensional material surface is modeled helicoidally as well — an approach that proves a natural asset with curved shells. Thirdly, the surface elements are interpolated helicoidally, so helping curved and curving low-order elements.

An outline of the proposed formulation follows. In this outline, we sometimes use the words "vector" and "tensor" loosely, to capture the essence of certain objects whose precise nature will become clear when they are introduced rigorously in the body of the paper.

- *Material surface kinematics.* Each material particle of the shell reference surface is identified by an orientation-position tensor $\boldsymbol{A}$ (a total of 6 independent parameters). Deformation of the material surface is described by a kinematical strain field $\boldsymbol{\omega}$ (12 parameters).
- *Solid shell kinematics.* Each material particle of the shell body is identified via a constant-curvature rototranslation along the transverse coordinate from the orientation-position of the parent particle on the material surface. The curvature is related to a 6-parameter vector $\boldsymbol{\theta}$ (referred to as the shell dual director) in such a way that the shell kinematics does not depend on the gradient of $\boldsymbol{\theta}$ on the reference surface; so, the director can be assumed to belong to a piecewise constant field of the material surface. Deformation of the solid shell yields a local kinematical strain depending on the material surface strain field $\boldsymbol{\omega}$ and on the field of the current dual director $\boldsymbol{\theta}'$ (a total of 18 parameters).
- *Solid shell mechanics.* From the internally constrained principle of virtual work [Merlini and Morandini 2005], the term of the internal work is considered; this term, for a hyperelastic nonpolar material, is a function of the local strain and of the local Biot-axial stress. The latter is assumed to be linear across the shell thickness, with the vector $\hat{\boldsymbol{\tau}}$ as the constant part and the vector $\hat{\boldsymbol{\mu}}$ controlling the linear part. So, the internal work virtual functional is finally a function of $\boldsymbol{\omega}$, $\boldsymbol{\theta}'$, $\hat{\boldsymbol{\tau}}$, and $\hat{\boldsymbol{\mu}}$ (a total of 24 parameters on the reference surface domain), and of the relevant virtual variation variables. Linearization of the virtual functional yields the virtual tangent functional, a function of the incremental variation variables, too.
- *Reduction to shell surface mechanics.* The integral, over the shell thickness, of the volume density of the internal work functionals gives the relevant surface densities. In such expressions, some quantities work-conjugate to $\boldsymbol{\omega}$ are found, as well as some quantities work-conjugate to $\boldsymbol{\theta}'$, $\hat{\boldsymbol{\tau}}$, and $\hat{\boldsymbol{\mu}}$; also, the tangent operators mapping the incremental variation variables on the incremental work-conjugate fields are found. All such integral quantities are nonlinear functions of the unknown surface fields,

and are meant to be computed dynamically by numerical quadrature during the whole shell problem solution.

- *Shell constitutive equations.* The shell director $\boldsymbol{\theta}'$ and the Biot-axial parameter $\hat{\boldsymbol{\mu}}$ are recognized to be local surface variables. From the incremental form of the internal work virtual functional, the equations to solve the local incremental variations for the remaining incremental variations can be written. The main surface problem is statically condensed to a form where only the variation variables relevant to $\boldsymbol{\omega}$ and $\hat{\boldsymbol{\tau}}$ appear (a total of 15 parameters). In this form, we may recognize the true stress resultants work-conjugate to the component vectors of the material surface strain $\boldsymbol{\omega}$ and the integrated form of the polar decomposition theorem work-conjugate to the Biot-axial parameter $\hat{\boldsymbol{\tau}}$, as well as the relevant tangent operators. It will be shown that the former integrals represent the shell constitutive equations and the kinematical constraint of the material surface, and the latter integrals the associated tangent map.
- *Shell finite element.* The internally constrained principle of virtual work is finally stated for the material surface in its incremental form, using as internal work contributions the functionals discussed above. Then, this principle is approximated by the finite element method and a four-node quadrilateral shell element is formulated in a similar way to the eight-node hexahedral solid element in [Merlini and Morandini 2005]. The kinematic field is modeled by the helicoidal interpolation between the six-DOF nodes and the Biot-axial parameter is assumed uniform over the element domain.

The whole formulation is computation oriented. A sound variational formulation is applied to the solid shell body and the weak form of the nonlinear governing equations is written in the incremental form from the beginning. The approximation of the solid mechanics is of the finite element type, but is performed in two separate steps: first, a helicoidal model is used through the thickness and a numerical integration reduces the problem to a two-dimensional one; then, a helicoidal nonlinear interpolation is used on the element surface to bring the problem to a discrete form. The numerical integration across the thickness is performed at the quadrature points of the surface element. The surface problem is statically condensed and the local variables $\boldsymbol{\theta}'$ and $\hat{\boldsymbol{\mu}}$ are allocated and updated at the quadrature points themselves. The governing equations of the material surface mechanics are not explicitly written in strong form nor are they necessary to the solution of the shell problem; the stress resultants as well as their derivatives with respect to the material surface strains are computed dynamically during the nonlinear solution process. A vectorial parameterization of the rototranslation and its consistent differentiation allows us to track rotations unrestricted in size.

In the form presented above, however, the solid shell kinematics suffers a serious drawback doomed to impair most shell analyses. The proposed constant-curvature rototranslation model allows for a transverse normal strain that is uniform across the thickness, and is thus inconsistent with the alternate strain induced by the Poisson effect in shell bending; in such circumstances, the so-called Poisson thickness locking manifests, unless the local constitutive law is adjusted for plane-stress; see, for example, [Bischoff and Ramm 2000] for an accurate discussion. In order to allow for unmodified three-dimensional constitutive laws, the method of the enhanced assumed strains has been widely used to build effective solid shell elements based on kinematics of the Reissner–Mindlin kind (see, for example, [Büchter et al. 1994; Sansour and Kollmann 2000; Klinkel et al. 2006], among others). Instead, a different approach is followed here, which also allows for unmodified three-dimensional nonlinear constitutive laws: the transverse normal strain, say ε_{33}, is regarded as a local, independent variable, disjoint from its kinematical counterpart; from the incremental form of the constitutive equation the increment of ε_{33} is solved for the other strain increments and the constitutive equation is condensed locally; ε_{33} is allocated at each quadrature point across the thickness and updated at each iteration. A similar technique was suggested in [de Borst 1991], whereas in [Klinkel and Govindjee 2002] it was proposed to start a separate iterative process to force plane stress at each quadrature point; see also the implementation in [Campello et al. 2003].

The paper is organized as follows. The present article (Part I) begins with an introduction to helicoidal modeling in three-dimensional solids (Section 2). Then, the material surface kinematics are formulated in Section 3, and used in Section 4 to build the solid shell kinematical model. The solid shell mechanics in the incremental form follow in Section 5, which includes the adaptation of the local constitutive law. In Section 6, the integration across the thickness leads to the shell surface mechanics, and the shell constitutive equations are formulated in incremental form. The companion article (Part II) contains the shell element formulation and a broad selection of nonlinear test cases. Index-free notation is favored throughout, but when necessary, Latin indexes are used for components spanning the range 1 to 3, and Greek indexes for the surface components in the range 1 to 2. The Einsteinian rule of implicit summation over repeated indexes is understood.

2. Overview of three-dimensional modeling

A quick introduction to helicoidal modeling is given in this section; for a more comprehensive discussion refer to [Merlini and Morandini 2004a]. According to the polar description in continuum mechanics, a solid body is regarded as a continuum set of infinitesimal yet three-dimensional material particles. We limit ourselves to a special case of polar description, referred to as micropolar description, where the directors embedded within each particle are not allowed to stretch and rotate relative to each other (see, for example, [Kafadar and Eringen 1971; Eringen and Kafadar 1976] and references therein). In that case, the material particles behave as infinitesimal rigid bodies.

2.1. ***Motion of a particle.*** The configuration of a material particle at point P in space is identified by its position and by its orientation. Let's embed a triad of vectors $\boldsymbol{\alpha}_j$ $(j = 1, 2, 3)$ within the particle. With reference to an absolute frame made of an orthonormal triad of dimensionless unit vectors $\boldsymbol{i}_j \equiv \boldsymbol{i}^j$ and to the associated Cartesian axes x^j departing from the origin O, the particle configuration can be measured by the pair $(\boldsymbol{x}, \boldsymbol{\alpha})$ of the position vector $\boldsymbol{x} = x^j \boldsymbol{i}_j$ and the orientation tensor of the embedded triad, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_j \otimes \boldsymbol{i}^j$. Throughout this work, vectors $\boldsymbol{\alpha}_j$ are assumed to be mutually orthogonal unit vectors, so $\boldsymbol{\alpha}$ is an orthogonal tensor itself, namely a rotation from the identity $\boldsymbol{I} = \boldsymbol{i}_j \otimes \boldsymbol{i}^j$. This assumption is not actually necessary, but it greatly simplifies things without losing generality.

Besides this commonly used way of describing the particle configuration, an alternative exists which relies on the coupled pair $(\boldsymbol{\alpha}, \boldsymbol{x} \times \boldsymbol{\alpha})$, where $\boldsymbol{x} \times \boldsymbol{\alpha}$ is the *moment* of the triad $\boldsymbol{\alpha}$ with respect to the origin O, chosen as the *pole*. That the pole is brought to coincide with the absolute origin is just a matter of convenience. The dual algebra helps managing this kind of geometric pairs of vectors and tensors (see, for example, [Angeles 1998] and references therein, or the recent survey [Pennestrì and Stefanelli 2007]). In fact, the pair $(\boldsymbol{\alpha}, \boldsymbol{x} \times \boldsymbol{\alpha})$ is conveniently represented by the dual tensor $\boldsymbol{A} = \boldsymbol{\alpha} + \varepsilon \boldsymbol{x} \times \boldsymbol{\alpha}$, where ε is the dual unity, a number such that $\varepsilon \neq 0$ and $\varepsilon^2 = \varepsilon^3 = \cdots = 0$. A dual tensor, like a dual

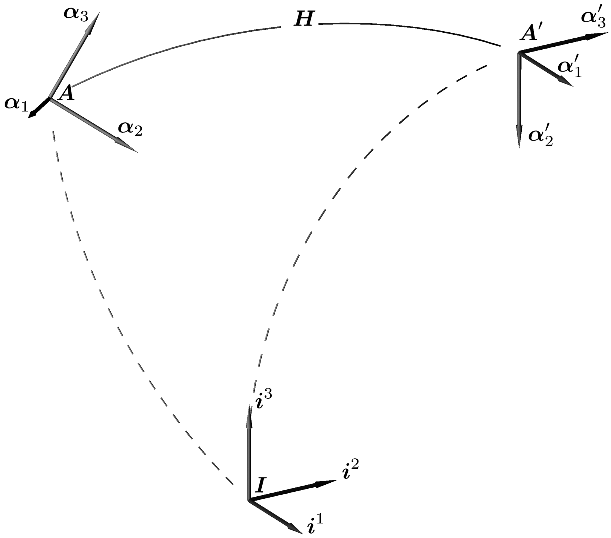

Figure 1. Rototranslation of a particle between the orientopositions in two different configurations.

number, is composed of a primal part plus a dual part multiplied by ε. Note that the dual part is always dimensionally greater than the primal part by a length: within a kinematic context, such parts will be often referred to as the *angular* part and the *linear* part, respectively. The opposite will occur in the cokinematic vector space, where, for example, an applied force-and-couple $(\boldsymbol{f}, \boldsymbol{c})$ shall be represented by a dual vector $\boldsymbol{f} + \varepsilon(\boldsymbol{c} + \boldsymbol{x} \times \boldsymbol{f})$, whose primal part is the linear part (a force) and whose dual part is the angular part (a moment).

The dual tensor

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\alpha} = (\boldsymbol{I} + \varepsilon\boldsymbol{x}\times)\boldsymbol{\alpha} \tag{1}$$

will be called hereafter the *orientoposition* tensor of the material particle. (Here and throughout the paper, the notation $\boldsymbol{v}\times$ denotes a skew-symmetric tensor having $\boldsymbol{v}$ as axial vector. It transforms a vector $\boldsymbol{w}$ into the vector $\boldsymbol{v} \times \boldsymbol{w}$.) In the factorization (1) of $\boldsymbol{A}$, the tensor $\boldsymbol{X}$ is called the dual position tensor and is easily seen to be orthogonal: $\boldsymbol{X}^{-1} = \boldsymbol{I} - \varepsilon\boldsymbol{x}\times = \boldsymbol{X}^{\mathrm{T}}$. Therefore, $\boldsymbol{A}$ is orthogonal too: $\boldsymbol{A} = \boldsymbol{A}^{-\mathrm{T}}$. (Of course, the alternative form $\boldsymbol{A} = \boldsymbol{\alpha}\boldsymbol{X}^{\circ} = \boldsymbol{\alpha}(\boldsymbol{I} + \varepsilon\boldsymbol{x}^{\circ}\times)$ also holds, with $\boldsymbol{x}^{\circ} = \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x}$.)

When the body moves to a new configuration, the particle's orientoposition changes to, say, $\boldsymbol{A}' = \boldsymbol{X}'\boldsymbol{\alpha}'$; see Figure 1. The particle rotates by $\boldsymbol{\Phi} = \boldsymbol{\alpha}'\boldsymbol{\alpha}^{\mathrm{T}}$ and undergoes a *rototranslation*

$$\boldsymbol{H} = \boldsymbol{A}'\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{X}'\boldsymbol{\Phi}\boldsymbol{X}^{\mathrm{T}}. \tag{2}$$

$\boldsymbol{H}$ is an orthogonal dual tensor too and can be factorized into two orthogonal tensors in sequence, specifically the *rotation* $\boldsymbol{\Phi}$ followed by a *translation* dual tensor $\boldsymbol{T}$,

$$\boldsymbol{H} = \boldsymbol{T}\boldsymbol{\Phi} = (\boldsymbol{I} + \varepsilon\boldsymbol{t}\times)\boldsymbol{\Phi}. \tag{3}$$

Here, $\boldsymbol{t} = \boldsymbol{x}' - \boldsymbol{\Phi}\boldsymbol{x}$ is called the *translation vector*. This vector is different from the displacement $\boldsymbol{u} = \boldsymbol{x}' - \boldsymbol{x} = \boldsymbol{t} + (\boldsymbol{\Phi} - \boldsymbol{I})\boldsymbol{x}$, when the particle rotates; the choice of the term *translation* for $\boldsymbol{t}$ is motivated in [Borri et al. 2000; Merlini and Morandini 2004a].

As orthogonal tensors, both the rotation and the rototranslation can be represented as exponential maps of skew-symmetric tensors:

$$\boldsymbol{\Phi} = \exp(\boldsymbol{\varphi}\times) = \sum_{k=0}^{\infty} \frac{\boldsymbol{\varphi}\times^k}{k!}, \tag{4}$$

$$\boldsymbol{H} = \exp(\boldsymbol{\eta}\times) = \sum_{k=0}^{\infty} \frac{\boldsymbol{\eta}\times^k}{k!}, \tag{5}$$

where $\boldsymbol{\varphi}$ is the rotation vector, and $\boldsymbol{\eta} = \boldsymbol{\varphi} + \varepsilon\,\boldsymbol{\rho}$ is called the *helix* of the rototranslation. Also, $\boldsymbol{T} = \exp(\varepsilon\,\boldsymbol{t}\times)$.

It is worth noting that the rototranslation inherits all the properties of the rotation. As the latter belongs to the special orthogonal Lie group SO(3), so the former can be shown to belong to a six-dimensional extension of such Lie group [Borri et al. 2000]. In particular, an important issue concerning rotations and rototranslations is their differentiation. It is well known that a *differential rotation vector* $\boldsymbol{\varphi}_{\mathrm{d}}$ characterizes the differentiation of a rotation tensor, that is, $\mathrm{d}\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{\varphi}_{\mathrm{d}}\times$. Analogously, a *differential helix* $\boldsymbol{\eta}_{\mathrm{d}} = \boldsymbol{\varphi}_{\mathrm{d}} + \varepsilon\,\boldsymbol{\rho}_{\mathrm{d}}$ characterizes the differentiation of the rototranslation tensor, $\mathrm{d}\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}} = \boldsymbol{\eta}_{\mathrm{d}}\times$. These differential vectors are connected to the differentiations $\mathrm{d}\boldsymbol{\varphi}$ and $\mathrm{d}\boldsymbol{\eta}$ by means of the differential maps associated to the relevant exponential maps (see [Borri et al. 2000]):

$$\boldsymbol{\varphi}_{\mathrm{d}} = \boldsymbol{\Gamma}\,\mathrm{d}\boldsymbol{\varphi}, \qquad \boldsymbol{\Gamma} = \mathrm{dexp}(\boldsymbol{\varphi}\times) = \sum_{k=0}^{\infty} \frac{\boldsymbol{\varphi}\times^k}{(k+1)!}, \tag{6}$$

$$\boldsymbol{\eta}_{\mathrm{d}} = \boldsymbol{\Lambda}\,\mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\Lambda} = \mathrm{dexp}(\boldsymbol{\eta}\times) = \sum_{k=0}^{\infty} \frac{\boldsymbol{\eta}\times^k}{(k+1)!}. \tag{7}$$

Incidentally, the differential mapping tensor $\boldsymbol{\Gamma}$ provides a link between the linear part of the helix and the translation vector, $\boldsymbol{t} = \boldsymbol{\Gamma}\boldsymbol{\rho}$ [Merlini and Morandini 2004b].

Now, let's bring the rototranslation to the infinitesimal limit. The differential of the orientation is described by means of a differential rotation vector, that is, $\mathrm{d}\boldsymbol{\alpha} = \boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\alpha}$; analogously, the differential of the orientoposition is described by means of a differential helix, $\mathrm{d}\boldsymbol{A} = \boldsymbol{\eta}_{\mathrm{d}} \times \boldsymbol{A}$. Recalling the factorized form in (1) and resorting to the property $\boldsymbol{B}\boldsymbol{c}\times = (\boldsymbol{B}\boldsymbol{c}) \times \boldsymbol{B}$, which holds true for any orthogonal tensor $\boldsymbol{B}$, it is easily seen that the angular part of $\boldsymbol{\eta}_{\mathrm{d}}$ is just $\boldsymbol{\varphi}_{\mathrm{d}}$ whereas the linear part is given by $\boldsymbol{\rho}_{\mathrm{d}} = \mathrm{d}\boldsymbol{x} + \boldsymbol{x} \times \boldsymbol{\varphi}_{\mathrm{d}} = \boldsymbol{\alpha}\mathrm{d}(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x})$. This is a remarkable issue, that clearly points out that the tangent space of this rototranslatory motion is controlled by a vectorial pair $(\boldsymbol{\varphi}_{\mathrm{d}}, \boldsymbol{\rho}_{\mathrm{d}})$ far different from the pair $(\mathrm{d}\boldsymbol{x}, \boldsymbol{\varphi}_{\mathrm{d}})$ usually considered in classical mechanics. The differential helix $\boldsymbol{\eta}_{\mathrm{d}}$ couples intimately the infinitesimal rotations and displacements, with a significant effect in a helicoidally modeled variational context [Merlini and Morandini 2004a], where forces and moments will work for $\boldsymbol{\rho}_{\delta}$ and $\boldsymbol{\varphi}_{\delta}$, respectively. Conversely, this coupling feature can also be understood by observing that $\mathrm{d}\boldsymbol{x} = \boldsymbol{\rho}_{\mathrm{d}} - \boldsymbol{x} \times \boldsymbol{\varphi}_{\mathrm{d}}$, so both the linear and the angular parts of $\boldsymbol{\eta}_{\mathrm{d}}$ contribute to the differential of the position. In Figure 2, a finite motion of a particle is shown on two different paths, both with a constant tangent space: on path (a), where displacement is uncoupled from rotation, $\mathrm{d}\boldsymbol{x}$ and $\boldsymbol{\varphi}_{\mathrm{d}}$ are constant, whereas on the seemingly more natural helicoidal path (b), $\boldsymbol{\varphi}_{\mathrm{d}}$ and $\boldsymbol{\rho}_{\mathrm{d}}$ are constant. These aspects of the helicoidal parameterization of motion become quite important in computational mechanics, where the choice of an effective discretization of motion may be crucial.

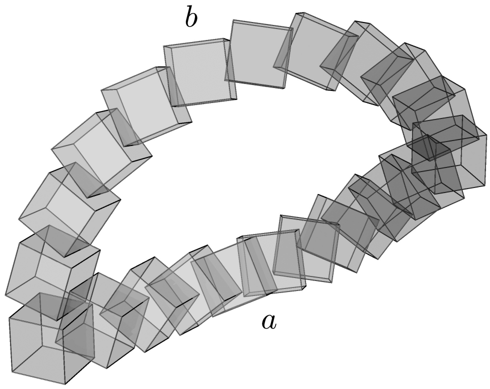

Figure 2. Motion of a particle on two different paths: (a) classical (displacement and rotation are uncoupled); (b) helicoidal.

2.2. *Helicoidal modeling.* Helicoidal modeling is the attempt to translate the helicoidal parameterization of motion into the parameterization of the geometry itself of a deformable body at any instant of motion; as a matter of fact, the orientopositions of any two material particles within a body "differ" by a rototranslation. To understand this rewriting, we ought to convert the concept of differentation into the concept of gradient on a three-dimensional field. We shall see that at the limit of an infinitesimal region, the position gradient itself is influenced by the mutual orientations of the neighbor particles. So, the tangent space of the local micropolar geometry can be far different from the classical tangent space proper of a Euclidean modeling.

With reference to a system of convective curvilinear coordinates ξ^j $(j = 1, 2, 3)$ traced within the body, consider the partial derivative $\boldsymbol{A},_j = \partial\boldsymbol{A}/\partial\xi^j$ of the orientoposition field along one coordinate. According to the differentiation formula of orthogonal tensors, $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A},_j = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{k}_j)\times$, the derivative $\boldsymbol{A},_j$ is characterized by a dual vector $\boldsymbol{k}_j$ that can be referred to as the (generalized) curvature along that coordinate. The dyadic composition with the metric contravariant base vectors $\boldsymbol{g}^j$ yields the tensorial relation

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}_{/\otimes} = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{k})^{\times}, \tag{8}$$

where $(\,)_{/\otimes} = (\,),_j \otimes \boldsymbol{g}^j$ denotes the gradient, $(\,)^{\times} = (\,)_j \times \otimes \boldsymbol{g}^j$ denotes the tensor-cross operator, and $\boldsymbol{k} = \boldsymbol{k}_j \otimes \boldsymbol{g}^j$ is the *curvature* dual tensor. Equation (8) states that the gradient of the orientoposition field, a third-order dual tensor, is characterized by a second-order dual tensor field, the curvature $\boldsymbol{k}$. The tensor-cross operator produces a third-order tensor of skew-symmetric nature; see [Merlini and Morandini 2004a]. Note that, in (8), $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{k}_j$ are the axial vectors of tensors $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A},_j$, formally $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{k}_j = \operatorname{ax}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A},_j)$, and analogously $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{k}$ is called the axial tensor of third-order tensor $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}_{/\otimes}$, formally $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{k} = \operatorname{ax}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}_{/\otimes})$.

Using (1) in (8), the dual explicit form of the curvature is easily obtained:

$$\boldsymbol{k} = \boldsymbol{k}_{\mathrm{a}} + \varepsilon\boldsymbol{k}_{\mathrm{l}} = \boldsymbol{\alpha}\operatorname{ax}(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}_{/\otimes}) + \varepsilon\boldsymbol{\alpha}(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x})_{/\otimes}. \tag{9}$$

The angular part, referred to as the angular curvature, characterizes the gradient of the orientation, $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}_{/\otimes} = (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_{\mathrm{a}})^{\times}$, analogously to (8). The linear part is the corotational gradient of the position vector, also given by $\boldsymbol{k}_{\mathrm{l}} = \boldsymbol{x}_{/\otimes} + \boldsymbol{x} \times \boldsymbol{k}_{\mathrm{a}}$. Equation (9) is very important and represents the core of the *helicoidal* *modeling*; it allows us to see that $\boldsymbol{k}$, which controls the tangent space of the orientoposition field $\boldsymbol{A}$, couples intimately the spatial derivatives of the orientation with those of the position.

The curvature $\boldsymbol{k}$ is a pole-based dual vector. Changing the pole from O to the point P itself, by means of the arm operator $\boldsymbol{X} = \boldsymbol{I} + \varepsilon\boldsymbol{x}\times$, yields the self-based version of the curvature,

$$\boldsymbol{X}^{\mathrm{T}}\boldsymbol{k} = \boldsymbol{k}_{\mathrm{a}} + \varepsilon(\boldsymbol{k}_{\mathrm{l}} - \boldsymbol{x} \times \boldsymbol{k}_{\mathrm{a}}) = \boldsymbol{\alpha}\,\mathrm{ax}\,(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}_{/\otimes}) + \varepsilon\boldsymbol{x}_{/\otimes}. \tag{10}$$

Here, the spatial derivatives of orientation and position are disjoint from each other: $\boldsymbol{k}_{\mathrm{a}}$ controls the tangent space of the orientation field whereas $\boldsymbol{x}_{/\otimes}$ represents the tangent space of the classical Euclidean position vector field. It can be seen from (10) that the position vector tangent space is controlled by either part of $\boldsymbol{k}$; in other words, following the concept of a rototranslation-based micropolar description, the orientation field strongly affects the evaluation of neighboring positions.

3. Material surface kinematics

What we mean by *material surface* is a single layer of continuous material particles (infinitesimal yet three-dimensional) lying on a generally curved smooth geometric surface: no particle is allowed to stay out of the surface. Each particle is identified on a two-coordinates domain, spanned by two families of material curvilinear coordinates ξ^α $(\alpha = 1, 2)$ traced on the surface. There is no way to define a transversal material coordinate to the surface, nor a material director. The status of the material surface is defined by the two-dimensional field of the positions and orientations of its particles. The particles of a deformable material surface may change their relative distances and orientations, yet they keep on lying on a deformed geometric surface, proper of the current configuration. Since the orientations can change independently from the positions, the mechanism of transverse shear strains is implicitly allowed for. Such an infinitely thin body, also referred to as a Cosserat surface (in [Sansour and Bednarczyk 1995], for example), is the image of a shell-like material solid devoid of thickness.

In this section, we establish the geometry and kinematics of the material surface, using a micropolar description approach in the context of helicoidal modeling (details are found in [Merlini 2008]). In the next section, we will start from the kinematics of the material surface to build the solid shell model, and in Part II we will address the mechanics of the material surface in view of a finite element approximation of the shell.

3.1. ***Reference configuration.*** The reference configuration is identified by the orientoposition dual tensor field $\boldsymbol{A}(\xi^\alpha)$, as defined in (1). Two curvature dual vectors $\boldsymbol{k}_\alpha$ characterize the derivatives of $\boldsymbol{A}$ with respect to the surface coordinates,

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}_{,\alpha} = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{k}_\alpha)\times. \tag{11}$$

Their explicit dual forms are $\boldsymbol{k}_\alpha = \boldsymbol{k}_{\mathrm{a}\alpha} + \varepsilon\boldsymbol{k}_{\mathrm{l}\alpha} = \boldsymbol{k}_{\mathrm{a}\alpha} + \varepsilon(\boldsymbol{x}_{,\alpha} + \boldsymbol{x} \times \boldsymbol{k}_{\mathrm{a}\alpha})$, where vectors $\boldsymbol{k}_{\mathrm{a}\beta}$ characterize the derivatives of the orientation, $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}_{,\beta} = (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_{\mathrm{a}\beta})\times$. The relevant self-based versions are $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{k}_\alpha = \boldsymbol{k}_{\mathrm{a}\alpha} + \varepsilon(\boldsymbol{k}_{\mathrm{l}\alpha} - \boldsymbol{x} \times \boldsymbol{k}_{\mathrm{a}\alpha}) = \boldsymbol{k}_{\mathrm{a}\alpha} + \varepsilon\boldsymbol{x}_{,\alpha}$.

Two covariant base vectors are obtained as the dual parts of the self-based curvature vectors,

$$\boldsymbol{g}_\alpha = \boldsymbol{x}_{,\alpha} = \mathrm{dual}\,(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{k}_\alpha). \tag{12}$$

We like to think of the coordinates ξ^α as dimensionless measures, so the $\boldsymbol{g}_\alpha$ take physical dimensions of length here. Since we want a solid local triad of base vectors, we conveniently borrow the geometric unit normal $\boldsymbol{n} = \boldsymbol{g}_1 \times \boldsymbol{g}_2 / |\boldsymbol{g}_1 \times \boldsymbol{g}_2|$ to arbitrarily build a third base vector

$$\boldsymbol{g}_3 = h\boldsymbol{n}, \tag{13}$$

where h is an arbitrary length, referred to as the *characteristic length* of the material surface. The dyadic composition of vectors $\boldsymbol{g}_j$ $(j = 1, 2, 3)$ with the absolute unit vectors gives the invertible base frame $\boldsymbol{G} = \boldsymbol{g}_j \otimes \boldsymbol{i}^j$. The reciprocal frame $\boldsymbol{G}^{-\mathrm{T}} = \boldsymbol{g}^j \otimes \boldsymbol{i}_j$ is made of the contravariant base vectors $\boldsymbol{g}^\alpha$ tangent to the surface and of the normal vector $\boldsymbol{g}^3 = h^{-1}\boldsymbol{n}$. Keep in mind, however, that $\boldsymbol{g}_3$ and $\boldsymbol{g}^3$ have nothing to do with material base vectors; they are purely geometric supplementary vectors.

By dyadic composition of (11) with the contravariant base vectors, the definition of the *surface curvature* dual tensor $\boldsymbol{k} = \boldsymbol{k}_\alpha \otimes \boldsymbol{g}^\alpha$ is obtained:

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}_{/\otimes} = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{k})^{\times}, \tag{14}$$

where now $(\ )_{/\otimes} = (\ )_{,\alpha} \otimes \boldsymbol{g}^\alpha$ and $(\ )^{\times} = (\ )_\alpha \times \otimes \boldsymbol{g}^\alpha$ are the surface gradient and tensor-cross operators. In the explicit dual form of the surface curvature,

$$\boldsymbol{k} = \boldsymbol{\alpha}\,\mathrm{ax}\,(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}_{/\otimes}) + \varepsilon\boldsymbol{\alpha}(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x})_{/\otimes} = \boldsymbol{k}_{\mathrm{a}} + \varepsilon\boldsymbol{k}_{\mathrm{l}} = \boldsymbol{k}_{\mathrm{a}} + \varepsilon(\boldsymbol{x}_{/\otimes} + \boldsymbol{x} \times \boldsymbol{k}_{\mathrm{a}}),$$

it is seen that the angular part is the angular curvature as defined by $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}_{/\otimes} = (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k}_{\mathrm{a}})^{\times}$, whereas in the linear part there appear the derivatives of both the position and the orientation. Note that, in the self-based version $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{k} = \boldsymbol{k}_{\mathrm{a}} + \varepsilon\boldsymbol{x}_{/\otimes}$, the linear part is the surface position gradient $\boldsymbol{x}_{/\otimes} = \boldsymbol{g}_\alpha \otimes \boldsymbol{g}^\alpha$, which coincides with the projector on the tangent plane, $-\boldsymbol{n} \times \boldsymbol{n} \times = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}$.

3.2. ***Current configuration.*** During the deformation process, the current configuration shall be described in exactly the same way as the reference configuration: the variables pertaining to the current configuration will be denoted by the same symbols as above, followed by an appended prime, $(\ )'$.

So, $\boldsymbol{A}' = \boldsymbol{X}'\boldsymbol{\alpha}' = (\boldsymbol{I} + \varepsilon\boldsymbol{x}'\times)\boldsymbol{\alpha}'$ denotes the current orientoposition; see Figure 3. The derivative relation

$$\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{A}'_{/\otimes} = (\boldsymbol{A}'^{\mathrm{T}}\boldsymbol{k}')^{\times} \tag{15}$$

defines the *current surface curvature* dual tensor $\boldsymbol{k}' = \boldsymbol{k}'_\alpha \otimes \boldsymbol{g}^\alpha = \boldsymbol{k}'_{\mathrm{a}} + \varepsilon\boldsymbol{k}'_{\mathrm{l}} = \boldsymbol{k}'_{\mathrm{a}} + \varepsilon(\boldsymbol{x}'_{/\otimes} + \boldsymbol{x}' \times \boldsymbol{k}'_{\mathrm{a}})$. Vectors $\boldsymbol{g}'_\alpha = \boldsymbol{x}'_{,\alpha} = \mathrm{dual}\,(\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{k}'_\alpha)$ are the first two current covariant base vectors and $\boldsymbol{g}'_3 = h\boldsymbol{n}'$ (with h the material surface characteristic length) is assumed to be the third one. Vectors $\boldsymbol{g}'_j$ $(j = 1, 2, 3)$ form the current base frame $\boldsymbol{G}' = \boldsymbol{g}'_j \otimes \boldsymbol{i}^j$. In the self-based version of the current curvature, that is, $\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{k}' = \boldsymbol{k}'_{\mathrm{a}} + \varepsilon\boldsymbol{x}'_{/\otimes}$, the linear part is the current surface position gradient $\boldsymbol{x}'_{/\otimes} = \boldsymbol{g}'_\alpha \otimes \boldsymbol{g}^\alpha$. Tensor $\boldsymbol{x}'_{/\otimes}$ differs from the three-dimensional deformation gradient $\boldsymbol{F} = \boldsymbol{g}'_j \otimes \boldsymbol{g}^j$ (the invertible tensor that transforms $\boldsymbol{G}$ to $\boldsymbol{G}' = \boldsymbol{F}\boldsymbol{G}$); in fact, $\boldsymbol{F} = \boldsymbol{x}'_{/\otimes} + \boldsymbol{n}' \otimes \boldsymbol{n}$.

The rototranslation field $\boldsymbol{H}(\xi^\alpha)$ from the reference configuration represents a meaningful alternative choice of the unknowns that define the current configuration,

$$\boldsymbol{A}' = \boldsymbol{H}\boldsymbol{A}. \tag{16}$$

Expressions for the tensor $\boldsymbol{H}$ are found in (2) and (3).

It is known that when a body moves rigidly, all its particles undergo the same rototranslation $\boldsymbol{H}$ [Borri et al. 2000]: this unique rototranslation characterizes the rigid body motion. So, let's examine

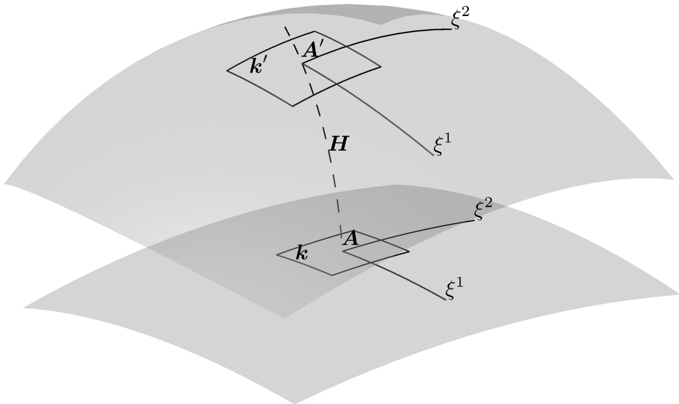

Figure 3. Material surface in the reference configuration and in the current configuration.

the derivatives $\partial \boldsymbol{H}/\partial \xi^\alpha$ within a moving surface. As for the curvatures, two surface strain dual vectors $\boldsymbol{\omega}_\alpha$ characterize such derivatives, $\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}_{,\alpha} = (\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha)\times$. A dyadic composition with the reference contravariant base vectors gives the tensorial relation

$$\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}_{/\otimes} = (\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega})^{\times}, \tag{17}$$

which actually defines the *surface strain* dual tensor $\boldsymbol{\omega} = \boldsymbol{\omega}_\alpha \otimes \boldsymbol{g}^\alpha$. The explicit dual form of the surface strain is

$$\boldsymbol{\omega} = \boldsymbol{\Phi}\,\mathrm{ax}\,(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}_{/\otimes}) + \varepsilon\,\boldsymbol{\Phi}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{t})_{/\otimes} = \boldsymbol{\omega}_{\mathrm{a}} + \varepsilon\,\boldsymbol{\omega}_{\mathrm{l}} = \boldsymbol{\omega}_{\mathrm{a}} + \varepsilon\,(\boldsymbol{t}_{/\otimes} + \boldsymbol{t}\times\boldsymbol{\omega}_{\mathrm{a}}),$$

where $\boldsymbol{t} = \boldsymbol{x}' - \boldsymbol{\Phi}\boldsymbol{x}$ is the translation vector, see (3). The angular part characterizes the surface gradient of the rotation, $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}_{/\otimes} = (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}_{\mathrm{a}})^{\times}$. The self-based version of the surface strain is $\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathrm{a}} + \varepsilon\,(\boldsymbol{x}'_{/\otimes} - \boldsymbol{\Phi}\boldsymbol{x}_{/\otimes})$.

By taking the gradient of (16) and recalling (15), (17), and (14), a relation between the curvatures in the reference and in the current configuration, involving the surface strain tensor, is easily obtained: $\boldsymbol{k}' = \boldsymbol{\omega} + \boldsymbol{H}\boldsymbol{k}$. When solved for $\boldsymbol{\omega}$, this equation leads to a meaningful expression for the surface strain tensor,

$$\boldsymbol{\omega} = \boldsymbol{k}' - \boldsymbol{H}\boldsymbol{k}, \tag{18}$$

as the difference between the current curvature and the reference curvature rototranslated forward by $\boldsymbol{H}$. In fact, when the motion is rigid, the curvature field $\boldsymbol{k}$ also undergoes the same unique rototranslation and becomes $\boldsymbol{k}' = \boldsymbol{H}\boldsymbol{k}$. Therefore, the difference in (18) is a good strain measure and $\boldsymbol{\omega}$ will be referred to as the *kinematical strain measure* of the material surface. Equation (18) also provides a profitable means to compute $\boldsymbol{\omega}$ in a numerical context, once $\boldsymbol{A}$, $\boldsymbol{k}$ and $\boldsymbol{A}'$, $\boldsymbol{k}'$ become available, for instance after an interpolation process.

The angular and linear parts of the self-based version of $\boldsymbol{\omega}$ represent the surface angular strain tensor $\boldsymbol{\omega}_{\mathrm{a}} = \boldsymbol{\omega}_{\mathrm{a}\alpha} \otimes \boldsymbol{g}^\alpha = \boldsymbol{k}'_{\mathrm{a}} - \boldsymbol{\Phi}\boldsymbol{k}_{\mathrm{a}}$ and the surface linear strain tensor $\boldsymbol{\chi} = \boldsymbol{\chi}_\alpha \otimes \boldsymbol{g}^\alpha = \boldsymbol{x}'_{/\otimes} - \boldsymbol{\Phi}\boldsymbol{x}_{/\otimes}$, respectively. It is interesting to note that the latter is different from the three-dimensional linear strain tensor $\boldsymbol{F} - \boldsymbol{\Phi}\boldsymbol{I}$ of the Biot kind; in fact, it can be seen that $\boldsymbol{\chi} = \boldsymbol{F} - \boldsymbol{\Phi}\boldsymbol{I} - (\boldsymbol{n}' - \boldsymbol{\Phi}\boldsymbol{n}) \otimes \boldsymbol{n}$. Moreover, the expression $\boldsymbol{\chi} = \boldsymbol{x}'_{/\otimes} - \boldsymbol{\Phi}\boldsymbol{x}_{/\otimes}$ can be used to prove that the present description of the material surface allows for transverse shear strains. For example, consider a change of configuration where a flat surface keeps flat, but the material particles rotate around an in-plane direction; in such a case, $\boldsymbol{x}'_{/\otimes} = \boldsymbol{x}_{/\otimes} = \boldsymbol{g}_\alpha \otimes \boldsymbol{g}^\alpha$, but a

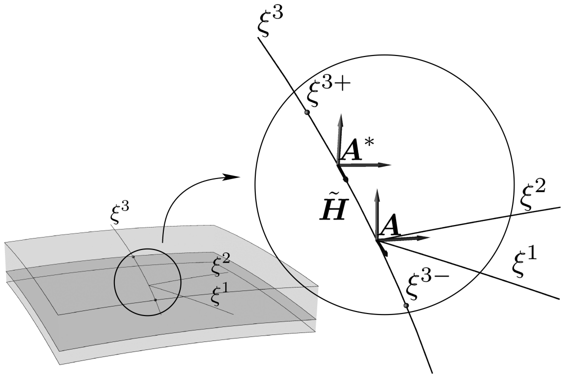

Figure 4. The helicoidal shell model.

nonnull linear strain tensor $\boldsymbol{\chi} = (\boldsymbol{g}_\alpha - \boldsymbol{\Phi}\boldsymbol{g}_\alpha) \otimes \boldsymbol{g}^\alpha$ arises, endowed with out-of-plane component vectors $\boldsymbol{\chi}_\alpha = -(\boldsymbol{\Phi} - \boldsymbol{I})\boldsymbol{g}_\alpha$.

4. Shell kinematical model

A solid shell is a three-dimensional body, lying on a smooth curved surface and thin in the direction locally normal to the surface. For numerical purposes, a solid shell is approximated by a *shell model*; here by shell model we mean a substitute body made of the layers generated by the particles of a material surface that sweep along a transverse curvilinear coordinate ξ^3, from ξ^{3-} to ξ^{3+}. For further details on the shell kinematical model discussed below, refer to [Merlini and Morandini 2008].

4.1. ***The helicoidal shell model.*** The proposed solid shell model is based on the most simple and natural rototranslation of the particles of a reference layer located at $\xi^3 = 0$. This reference layer is understood to coincide with the parent material surface. The above rototranslation is characterized by a curvature dual vector $\boldsymbol{k}_3$ constant along ξ^3 — actually, this is a helicoidal motion, whence the name *helicoidal shell model*. Moreover, a particular dual vector field $\boldsymbol{k}_3(\xi^\alpha)$ is assumed, so that also the curvature dual vectors $\boldsymbol{k}_\alpha$ of each layer hold constant along ξ^3.

The orientoposition $\boldsymbol{A}^*(\xi^\alpha, \xi^3)$ of a material particle of the solid shell model can always be described by a rototranslation $\tilde{\boldsymbol{H}}(\xi^\alpha, \xi^3)$ from the orientoposition $\boldsymbol{A}(\xi^\alpha)$ of the parent particle on the reference material surface (Figure 4),

$$\boldsymbol{A}^* = \tilde{\boldsymbol{H}}\boldsymbol{A}. \tag{19}$$

Equation (19) establishes a multiplicative decomposition of the local orientoposition $\boldsymbol{A}^* = \boldsymbol{X}^*\boldsymbol{\alpha}^* = (\boldsymbol{I} + \varepsilon\boldsymbol{x}^*\times)\boldsymbol{\alpha}^*$. The relative rototranslation can also be expressed as $\tilde{\boldsymbol{H}} = \boldsymbol{X}^*\tilde{\boldsymbol{\Phi}}\boldsymbol{X}^{\mathrm{T}}$, see (2), where $\tilde{\boldsymbol{\Phi}}(\xi^\alpha, \xi^3)$ is the rotation tensor from the orientation $\boldsymbol{\alpha}$ to the orientation $\boldsymbol{\alpha}^* = \tilde{\boldsymbol{\Phi}}\boldsymbol{\alpha}$. Note that we must have $\tilde{\boldsymbol{H}}(\xi^\alpha, 0) = \boldsymbol{I}$ and $\tilde{\boldsymbol{\Phi}}(\xi^\alpha, 0) = \boldsymbol{I}$, in order to have $\boldsymbol{A}^*(\xi^\alpha, 0) \equiv \boldsymbol{A}(\xi^\alpha)$ and $\boldsymbol{\alpha}^*(\xi^\alpha, 0) \equiv \boldsymbol{\alpha}(\xi^\alpha)$.

The helicoidal shell model is characterized as follows. First, $\tilde{\boldsymbol{H}} = \exp(\tilde{\boldsymbol{\eta}}\times)$ is specified as a *constant-curvature* rototranslation across the shell; to this end, the function $\tilde{\boldsymbol{\eta}}(\xi^\alpha, \xi^3)$ is assumed to have the factorized form

$$\tilde{\boldsymbol{\eta}} = \xi^3\boldsymbol{k}_3, \tag{20}$$

with $\boldsymbol{k}_3(\xi^\alpha)$ a dual vector independent of ξ^3. This assumption provides a constant curvature along ξ^3, as shown below. Next, the dual vector field $\boldsymbol{k}_3(\xi^\alpha)$ is specified by connecting it to the material surface orientoposition field $\boldsymbol{A}(\xi^\alpha)$. Formally, it is assumed that

$$\boldsymbol{k}_3 = \boldsymbol{A}\boldsymbol{\theta}, \tag{21}$$

with $\boldsymbol{\theta}$ a unique dual vector independent of ξ^α. We refer to $\boldsymbol{\theta}$ as the *dual director* of the helicoidal shell model. The helicoidal shell model, as described by (19)–(21), is based on the orthogonal dual tensor field $\boldsymbol{A}(\xi^\alpha)$ and on the unique dual parameter $\boldsymbol{\theta}$. The transverse curvature $\boldsymbol{k}_3$ is restricted to rototranslate on the material surface as the orientoposition $\boldsymbol{A}$, hence it is anchored to each material surface particle. It can be claimed that the last assumption, (21), is too strong; indeed, this assumption can be safely relaxed, as it will be shown later on.

Let's examine the curvatures within the helicoidal shell model, both along the layer's coordinates ξ^α and along the transverse coordinate ξ^3. Altogether, the three curvature dual vectors $\boldsymbol{k}^*_j$ $(j = 1, 2, 3)$ are defined by $\boldsymbol{A}^{*\mathrm{T}}\boldsymbol{A}^*{,}_j = (\boldsymbol{A}^{*\mathrm{T}}\boldsymbol{k}^*_j)\times$. Within this formula, (19)–(21) are exploited in order to derive explicit expressions for $\boldsymbol{k}^*_j$. First it is recognized that $\tilde{\boldsymbol{H}}^{\mathrm{T}}\tilde{\boldsymbol{H}}{,}_j = (\tilde{\boldsymbol{H}}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{\eta}}{,}_j)\times$, recall (7); then the derivatives of the transverse curvature are expanded as $\boldsymbol{k}_3{,}_\alpha = \boldsymbol{A}{,}_\alpha\boldsymbol{\theta} = \boldsymbol{k}_\alpha\times\boldsymbol{A}\boldsymbol{\theta} = \boldsymbol{k}_\alpha\times\boldsymbol{k}_3$, whence $\tilde{\boldsymbol{\eta}}{,}_\alpha = -\tilde{\boldsymbol{\eta}}\times\boldsymbol{k}_\alpha$; finally the identity $\tilde{\boldsymbol{H}} = \boldsymbol{I} + \tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{\eta}}\times$ is recalled, see (5) and (7). Using the property $\tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\eta}}$ (whence $\tilde{\boldsymbol{\Lambda}}\boldsymbol{k}_3 = \boldsymbol{k}_3$), it is found that $\boldsymbol{k}^*_3 = \boldsymbol{k}_3$ and $\boldsymbol{k}^*_\alpha = \boldsymbol{k}_\alpha$, or

$$\boldsymbol{k}^*_j = \boldsymbol{k}_j. \tag{22}$$

Since $\boldsymbol{k}_\alpha$ and $\boldsymbol{k}_3$ are independent of ξ^3, it follows that all three curvature dual vectors $\boldsymbol{k}^*_j$ are *constant* across the shell model thickness.

The explicit dual forms $\boldsymbol{k}^*_j = \boldsymbol{k}^*_{\mathrm{a}j} + \varepsilon\boldsymbol{k}^*_{1j} = \boldsymbol{k}^*_{\mathrm{a}j} + \varepsilon(\boldsymbol{x}^*{,}_j + \boldsymbol{x}^*\times\boldsymbol{k}^*_{\mathrm{a}j})$, and their self-based version $\boldsymbol{X}^{*\mathrm{T}}\boldsymbol{k}^*_j = \boldsymbol{k}^*_{\mathrm{a}j} + \varepsilon\boldsymbol{x}^*{,}_j$, allow us to derive the metric of the shell model. The covariant base vectors are obtained from (22) as

$$\boldsymbol{g}^*_j = \boldsymbol{x}^*{,}_j = \mathrm{dual}\,(\boldsymbol{X}^{*\mathrm{T}}\boldsymbol{k}_j).$$

They form the base frame $\boldsymbol{G}^* = \boldsymbol{g}^*_j\otimes\boldsymbol{i}^j$. Inversion of tensor $\boldsymbol{G}^*$ yields the reciprocal frame $\boldsymbol{G}^{*-\mathrm{T}} = \boldsymbol{g}^{*j}\otimes\boldsymbol{i}_j$ of the contravariant base vectors such that $\boldsymbol{g}^{*j}\cdot\boldsymbol{g}^*_k = \delta^j_k$, the Kronecker symbol. It is worth comparing the values assumed by $\boldsymbol{g}^*_j(\xi^\alpha, \xi^3)$ at the reference layer with the material surface $\boldsymbol{g}_j(\xi^\alpha)$, (12) and (13): the base vectors tangent to the layers do reduce to the material surface base vectors, $\boldsymbol{g}^*_\beta(\xi^\alpha, 0)\equiv\boldsymbol{g}_\beta(\xi^\alpha)$, whereas the transverse one becomes $\boldsymbol{g}^*_3(\xi^\alpha, 0) = \mathrm{dual}\,(\boldsymbol{A}\boldsymbol{\theta})$ and may differ from $\boldsymbol{g}_3(\xi^\alpha) = h\boldsymbol{n}$, which was introduced by a purely geometric construction. Note that, in the proposed solid shell model, $\boldsymbol{g}^*_3$ is relieved from being normal to the layers and, as a consequence, the contravariant base vectors $\boldsymbol{g}^{*\alpha}$ are no more tangent to the layers. Instead, $\boldsymbol{g}^{*3}$ keeps normal to the layers.

The model base frame $\boldsymbol{G}^*(\xi^\alpha, \xi^3)$ can be related to the material surface base frame $\boldsymbol{G}(\xi^\alpha)$ by means of a sort of modeling gradient $\tilde{\boldsymbol{F}}(\xi^\alpha, \xi^3)$,

$$\boldsymbol{G}^* = \tilde{\boldsymbol{F}}\boldsymbol{G}.$$

An interesting interpretation of the so-called shifter tensor $\tilde{\boldsymbol{F}} = \boldsymbol{g}^*_j\otimes\boldsymbol{g}^j$ as a deformation gradient and its implication in classical shell theories can be found in [Schlebusch and Zastrau 2005]. Note that in the proposed model, it may be $\boldsymbol{G}^*(\xi^\alpha, 0)\neq\boldsymbol{G}(\xi^\alpha)$, whence $\tilde{\boldsymbol{F}}(\xi^\alpha, 0)\neq\boldsymbol{I}$. The base vectors allow writing tensors in dyadic form with reference to the model base frame. Moreover, they provide useful expressions for the model gradient operator, $(\ )_{/\otimes *} = (\ )_{,j} \otimes \boldsymbol{g}^{*j}$, and for its trace, the model divergence operator, $(\ )_{/\bullet *} = (\ )_{,j} \cdot \boldsymbol{g}^{*j}$. Then, the modeling gradient $\tilde{\boldsymbol{F}}$ allows us to relate dyads in the model base with dyads in the material surface base.

The dyadic composition of the curvature dual vectors with the contravariant base vectors leads to the definition of the *model curvature* dual tensor $\boldsymbol{k}^* = \boldsymbol{k}_j^* \otimes \boldsymbol{g}^{*j}$,

$$\boldsymbol{A}^{*\mathrm{T}} \boldsymbol{A}^*_{/\otimes *} = (\boldsymbol{A}^{*\mathrm{T}} \boldsymbol{k}^*)^\times, \tag{23}$$

where in this context $(\ )^\times = (\ )_j \times \otimes \boldsymbol{g}^{*j}$. Note that the curvature tensor may be *not* constant across the shell thickness, yet it is made of three constant curvature vectors, $\boldsymbol{k}^* = \boldsymbol{k}_j \otimes \boldsymbol{g}^{*j}$, see (22). We may find it convenient to include the transverse curvature $\boldsymbol{k}_3$ in the dyadic definition of the *surface curvature* dual tensor $\boldsymbol{k} = \boldsymbol{k}_\alpha \otimes \boldsymbol{g}^\alpha$, and bring the latter to the three-dimensional form $\boldsymbol{k} = \boldsymbol{k}_j \otimes \boldsymbol{g}^j$. So, it can be easily seen that

$$\boldsymbol{k}^* = \boldsymbol{k} \tilde{\boldsymbol{F}}^{-1}, \tag{24}$$

and, again, it may be $\boldsymbol{k}^*(\xi^\alpha, 0) \neq \boldsymbol{k}(\xi^\alpha)$.

It is worth pointing out that the rototranslation that generates the solid shell model from a material surface is by no means a rigid motion. In fact, $\tilde{\boldsymbol{H}}$ is a function of the coordinates ξ^α and is not unique for the whole surface; also, each layer has its own curvature dual tensor, which is different from the curvature dual tensor of the parent material surface rototranslated by $\tilde{\boldsymbol{H}}$. Rather, this is like an inflating motion that brings the reference curved layer into a new, and in a sense parallel, curved layer. It is also worth noting that, whereas the tangent coordinate lines ξ^α generated by the shell model reflect those chosen on the parent material surface, the transverse coordinate lines ξ^3 are by construction helices with constant curvature.

4.2. *Relaxation of a kinematical hypothesis.* The shell dual director is a kinematical variable of the solid shell model and is extraneous to the material surface kinematics, so it should be advantageous to leave it out of the problem of the material surface mechanics. This observation indicates the opportunity to solve $\boldsymbol{\theta}$ locally. Therefore, we now propose to *relax* the hypothesis of a unique director for the material surface and adopt instead a *piecewise constant director field*; in other words, we assume that $\boldsymbol{\theta}$ is a locally constant function of ξ^α.

This new assumption does not affect the characteristics of the proposed kinematical model: from (21), we still have $\boldsymbol{k}_{3,\alpha} = \boldsymbol{k}_\alpha \times \boldsymbol{k}_3$, so the curvature dual vectors are still constant across the thickness. However, $\boldsymbol{\theta}$ is now allowed to vary from point to point on the material surface and, as a consequence, the transverse curvature $\boldsymbol{k}_3$ becomes unfastened from the orientation of the surface particle. Thus, $\boldsymbol{\theta}$ becomes a true variable field, but its gradient is left out of the variational formulation; the assumption is that a spatial variation of the dual director does not affect the shell mechanical behavior. In the finite element approximation of the material surface mechanics, the dual director shall be understood as an "internal", local variable; typically it shall be confined within the neighborhood of each quadrature point.

4.3. *Shell model in the current configuration.* Let the helicoidal shell model so far described refer to the undeformed, or *reference*, configuration. The *current* configuration will be described the same way, with an appended prime to distinguish the variables pertaining to the deformed model. The current orientoposition $\boldsymbol{A}'^* = \boldsymbol{X}'^* \boldsymbol{\alpha}'^* = (\boldsymbol{I} + \varepsilon \boldsymbol{x}'^* \times) \boldsymbol{\alpha}'^*$ of a material particle is decomposed multiplicatively into the orientoposition $\boldsymbol{A}'$ of the parent material particle followed by a rototranslation,

$$\boldsymbol{A}'^{*} = \tilde{\boldsymbol{H}}'\boldsymbol{A}', \tag{25}$$

with

$$\tilde{\boldsymbol{H}}' = \exp(\tilde{\boldsymbol{\eta}}'\times) = \exp(\xi^3\boldsymbol{k}_3'\times) = \exp\big(\xi^3(\boldsymbol{A}'\boldsymbol{\theta}')\times\big) \tag{26}$$

and $\boldsymbol{\theta}'$ a piecewise constant dual vector field on the material surface.

With a total-Lagrangian description in mind, the current tensors are based on the reference configuration, that is, their dyadic forms are written with the reference base vectors. So, the current model curvature dual tensor $\boldsymbol{k}'^{*} = \boldsymbol{k}_j'^{*} \otimes \boldsymbol{g}^{*j}$ is defined by

$$\boldsymbol{A}'^{*\mathrm{T}}\boldsymbol{A}'^{*}_{/\otimes *} = (\boldsymbol{A}'^{*\mathrm{T}}\boldsymbol{k}'^{*})^{\times}. \tag{27}$$

Across the shell thickness, $\boldsymbol{k}'^{*}$ will not be constant, though its dual component vectors $\boldsymbol{k}'^{*} \cdot \boldsymbol{g}_j^{*}$ will. In fact, as in (22), $\boldsymbol{k}_j'^{*} = \boldsymbol{k}_j'$. Thus, we can also write $\boldsymbol{k}'^{*} = \boldsymbol{k}_j' \otimes \boldsymbol{g}^{*j}$, and relate $\boldsymbol{k}'^{*}$ to the current surface curvature three-dimensional dual tensor $\boldsymbol{k}' = \boldsymbol{k}_j' \otimes \boldsymbol{g}^{j}$ by

$$\boldsymbol{k}'^{*} = \boldsymbol{k}'\tilde{\boldsymbol{F}}^{-1}. \tag{28}$$

The current covariant base vectors are given by $\boldsymbol{g}_j'^{*} = \boldsymbol{x}'^{*}{}_{,j} = \mathrm{dual}\,(\boldsymbol{X}'^{*\mathrm{T}}\boldsymbol{k}_j')$, and form the current base frame $\boldsymbol{G}'^{*} = \boldsymbol{g}_j'^{*} \otimes \boldsymbol{i}^{j}$. The model deformation gradient $\boldsymbol{F}^{*} = \boldsymbol{x}'^{*}_{/\otimes *} = \boldsymbol{g}_j'^{*} \otimes \boldsymbol{g}^{*j} = \boldsymbol{G}'^{*}\boldsymbol{G}^{*-1}$ can be obtained as $\boldsymbol{F}^{*} = \mathrm{dual}\,(\boldsymbol{X}'^{*\mathrm{T}}\boldsymbol{k}'^{*}) = \mathrm{dual}\,(\boldsymbol{X}'^{*\mathrm{T}}\boldsymbol{k}'\tilde{\boldsymbol{F}}^{-1})$. The modeling gradient $\tilde{\boldsymbol{F}}$ and the model deformation gradient $\boldsymbol{F}^{*}$ allow us to recover the current model base frame from the reference material surface base frame, $\boldsymbol{G}'^{*} = \boldsymbol{F}^{*}\tilde{\boldsymbol{F}}\boldsymbol{G}$.

4.4. ***Model deformation.*** Let's address now the *model rototranslation* that brings the reference orientoposition to the current orientoposition,

$$\boldsymbol{A}'^{*} = \boldsymbol{H}^{*}\boldsymbol{A}^{*}. \tag{29}$$

Useful expressions for $\boldsymbol{H}^{*}$ are obtained from (2) and (3):

$$\boldsymbol{H}^{*} = \boldsymbol{A}'^{*}\boldsymbol{A}^{*\mathrm{T}} = \boldsymbol{X}'^{*}\boldsymbol{\Phi}^{*}\boldsymbol{X}^{*\mathrm{T}} = \big(\boldsymbol{I} + \varepsilon\,(\boldsymbol{x}'^{*} - \boldsymbol{\Phi}^{*}\boldsymbol{x}^{*})\times\big)\boldsymbol{\Phi}^{*}, \tag{30}$$

where $\boldsymbol{\Phi}^{*}$ is the rotation from the reference to the current orientation, $\boldsymbol{\alpha}'^{*} = \boldsymbol{\Phi}^{*}\boldsymbol{\alpha}^{*}$.

The gradient of the rototranslation $\boldsymbol{H}^{*}$ allows us to define the *model strain* dual tensor $\boldsymbol{\omega}^{*} = \boldsymbol{\omega}_j^{*} \otimes \boldsymbol{g}^{*j}$,

$$\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{H}^{*}_{/\otimes *} = (\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{\omega}^{*})^{\times}. \tag{31}$$

Taking the gradient of (29) and using (27), (31), and (23), a relation between the reference and current model curvatures, involving the model strain, is obtained: $\boldsymbol{k}'^{*} = \boldsymbol{\omega}^{*} + \boldsymbol{H}^{*}\boldsymbol{k}^{*}$. Solving this equation for $\boldsymbol{\omega}^{*}$,

$$\boldsymbol{\omega}^{*} = \boldsymbol{k}'^{*} - \boldsymbol{H}^{*}\boldsymbol{k}^{*}, \tag{32}$$

gives a meaningful expression for the model strain, as the difference between the current curvature and the reference curvature rototranslated forward by $\boldsymbol{H}^{*}$. We shall refer to $\boldsymbol{\omega}^{*}$ as the *kinematical strain measure* of the helicoidal shell model. It can be seen that the angular and linear parts of the self-based version of $\boldsymbol{\omega}^{*}$ represent the model angular-strain tensor $\boldsymbol{\omega}_{\mathrm{a}}^{*} = \boldsymbol{k}_{\mathrm{a}}'^{*} - \boldsymbol{\Phi}^{*}\boldsymbol{k}_{\mathrm{a}}^{*}$ and the model linear-strain tensor $\boldsymbol{\chi}^{*} = \boldsymbol{F}^{*} - \boldsymbol{\Phi}^{*}\boldsymbol{I}$, respectively.

Last, let's investigate the relationship between the model rototranslation and strain tensors, $\boldsymbol{H}^*$ and $\boldsymbol{\omega}^*$ respectively, and the corresponding surface rototranslation and strain tensors, $\boldsymbol{H}$ and $\boldsymbol{\omega}$. Using (19) and (25), the model rototranslation $\boldsymbol{H}^*(\xi^\alpha, \xi^3)$ from (30) can be related to the surface rototranslation $\boldsymbol{H}(\xi^\alpha)$ of (16), by

$$\boldsymbol{H}^* = \tilde{\boldsymbol{H}}' \boldsymbol{H} \tilde{\boldsymbol{H}}^{\mathrm{T}}. \tag{33}$$

Note that, as expected, $\boldsymbol{H}^*(\xi^\alpha, 0) \equiv \boldsymbol{H}(\xi^\alpha)$, whereas, in general, $\boldsymbol{H}^*$ differs from $\boldsymbol{H}$ when $\xi^3 \neq 0$. However, a special case occurs when $\boldsymbol{k}'_3 = \boldsymbol{H}\boldsymbol{k}_3$ (that is, when the dual director stays unchanged, $\boldsymbol{\theta}' = \boldsymbol{\theta}$). In this case $\tilde{\boldsymbol{H}}' = \boldsymbol{H}\tilde{\boldsymbol{H}}\boldsymbol{H}^{\mathrm{T}}$ and from (33) it follows that $\boldsymbol{H}^* = \boldsymbol{H}$ independently from ξ^3: the rototranslation is unique for the whole material line lying along ξ^3, that is, that material line rototranslates rigidly.

Therefore, the difference $\boldsymbol{k}'_3 - \boldsymbol{H}\boldsymbol{k}_3 = \boldsymbol{A}'(\boldsymbol{\theta}' - \boldsymbol{\theta})$, which is independent of ξ^3, represents a unique strain measure across the thickness. We refer to such strain measure as the *through-the-thickness strain*,

$$\boldsymbol{\omega}_3 = \boldsymbol{k}'_3 - \boldsymbol{H}\boldsymbol{k}_3. \tag{34}$$

We may also find it convenient to include $\boldsymbol{\omega}_3(\xi^\alpha)$ in the dyadic definition of the surface strain dual tensor $\boldsymbol{\omega} = \boldsymbol{\omega}_\alpha \otimes \boldsymbol{g}^\alpha$, and bring the latter to the three-dimensional form $\boldsymbol{\omega} = \boldsymbol{\omega}_j \otimes \boldsymbol{g}^j$. Thus, (18) is expanded to the form $\boldsymbol{\omega} = \boldsymbol{k}' - \boldsymbol{H}\boldsymbol{k}$, where $\boldsymbol{k}'$, $\boldsymbol{k}$ and $\boldsymbol{\omega}$ are now full three-dimensional dual tensors on the surface.

Using (28) and (24), the model strain tensor in (32) can be related to the surface strain tensor,

$$\boldsymbol{\omega}^* = (\boldsymbol{k}' - \boldsymbol{H}^*\boldsymbol{k})\tilde{\boldsymbol{F}}^{-1} = \boldsymbol{\omega}\tilde{\boldsymbol{F}}^{-1} - (\boldsymbol{H}^* - \boldsymbol{H})\boldsymbol{k}\tilde{\boldsymbol{F}}^{-1},$$

where $\boldsymbol{\omega}$ is now the full three-dimensional dual tensor. This expression highlights the separate contributions to the local strain, coming from the material surface strains $\boldsymbol{\omega}_\alpha$ and from the through-the-thickness strain $\boldsymbol{\omega}_3$ (in fact, when $\boldsymbol{\omega}_3$ is null, $\boldsymbol{H}^* - \boldsymbol{H}$ is also null).

4.5. ***Geometric invariance.*** The shell model so far described is endowed with a distinctive property that is becoming very important in modern computational mechanics: it is *geometrically invariant*, in the sense pointed out in [Bottasso et al. 2002]. A shell kinematical model can be seen as a mathematical scheme to reduce three-dimensional kinematics to those pertaining to a two-coordinate material surface; being independent of the choice of the reference material surface is a characteristic of the proposed kinematical model. To be more precise, the physical value of the kinematical variables (orientopositions, curvatures, and strains) is not affected by the placement of the reference material surface with respect to the shell thickness. It is only required that the reference surface be identifiable as a layer of the shell model, but which layer is chosen is arbitrary.

The geometric invariance of the proposed shell model is inherent in helicoidal modeling itself across the thickness. To prove the geometric invariance, consider a given solid shell and set up a first kinematical model (A) as follows. Choose the mid surface as the reference surface and trace the coordinate lines ξ^α on it; the orientopositions $\boldsymbol{A}_{\mathrm{A}}$ and the curvatures $\boldsymbol{k}_{\mathrm{A}}$ are known; assume a dual director $\boldsymbol{\theta}_{\mathrm{A}}$ and build the transverse coordinate lines ξ^3 with a linear scale from $\xi^{3-} = -1$ to $\xi^{3+} = +1$. Next, set up a second kinematical model (B) of the same solid shell as follows. Choose the lower surface of model (A) as the reference surface for model (B); hold the coordinate lines ξ^α traced on it, and keep the same coordinate lines ξ^3 across the thickness as model (A), with a linear scale, however, ranging from $\xi^{3-} = 0$ to $\xi^{3+} = +1$; the orientopositions $\boldsymbol{A}_{\mathrm{B}}$, the curvatures $\boldsymbol{k}_{\mathrm{B}}$, and the dual director $\boldsymbol{\theta}_{\mathrm{B}}$ are determined from

$\boldsymbol{A}_{\mathrm{A}}$, $\boldsymbol{k}_{\mathrm{A}}$, and $\boldsymbol{\theta}_{\mathrm{A}}$. Note that $\boldsymbol{\theta}_{\mathrm{A}}$ and $\boldsymbol{\theta}_{\mathrm{B}} = 2\boldsymbol{\theta}_{\mathrm{A}}$ are parallel, however they are scaled due to the different measures of ξ^3 from ξ^{3-} to ξ^{3+}. So far you have built two different helicoidal models for the same shell body. As an exercise, compute the model orientoposition $\boldsymbol{A}^*$ and curvature $\boldsymbol{k}^*$ at a generic placement: you will find values coincident for either model. Now consider a deformed configuration of both models: again, $\boldsymbol{A}'^*$ and $\boldsymbol{k}'^*$ will coincide for either model. So both the rototranslation $\boldsymbol{H}^*$ and the strain $\boldsymbol{\omega}^*$ coincide for either model. This proves the geometric invariance of the helicoidal shell model.

5. Shell model mechanics

The starting point for the present formulation in shell mechanics is the principle of virtual work; this can be stated in the form $\Pi_{\mathrm{int}\delta} + \Pi_{\mathrm{ext}\delta} + \Pi_{\mathrm{bc}\delta} = 0$, where the contributions to the virtual functional Π_δ from the stresses, the external loads and the boundary constraints are kept separate. The terms $\Pi_{\mathrm{ext}\delta}$ and $\Pi_{\mathrm{bc}\delta}$ will be discussed in Part II, when dealing with the material surface mechanics. The internal work contribution, written as

$$\Pi_{\mathrm{int}\delta} = \int_S \pi_{\mathrm{int}\delta}\,\mathrm{d}S, \tag{35}$$

is the integral, over the shell surface S in the reference configuration, of a surface density of the internal work virtual functional of the shell model. The present section concerns the evaluation of $\pi_{\mathrm{int}\delta}$; refer to [Merlini and Morandini 2008] for deeper discussion.

5.1. *Internal work virtual functional.* The expression of $\pi_{\mathrm{int}\delta}$ is taken from the internally constrained form of the principle of virtual work [Merlini and Morandini 2005],

$$\pi_{\mathrm{int}\delta} = \int_{\xi^{3-}}^{\xi^{3+}} \delta\big(w^* + \langle \hat{\boldsymbol{\tau}}^*, \mathrm{dual}\, 2\,\mathrm{ax}\,(\boldsymbol{\Phi}^{*\mathrm{T}} \boldsymbol{X}'^{*\mathrm{T}} \boldsymbol{\omega}^*)\rangle\big)\, \tilde{f} h\, \mathrm{d}\xi^3, \tag{36}$$

by considering that $\mathrm{d}V^* = g^*\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3$ and $\mathrm{d}S = h^{-1} g\,\mathrm{d}\xi^1\mathrm{d}\xi^2$, with $g = \det \boldsymbol{G}$, $\tilde{f} = \det \tilde{\boldsymbol{F}}$ and $g^* = \det \boldsymbol{G}^* = \tilde{f} g$. Equation (36) is formulated in a micropolar description context and refers to a hyperelastic nonpolar medium: $w^*(\boldsymbol{\varepsilon}^{*\mathrm{S}})$ is the shell model strain-energy density per unit initial volume, a function of the symmetric strain parameter $\boldsymbol{\varepsilon}^{*\mathrm{S}}$, and $\hat{\boldsymbol{\tau}}^*$ is the axial vector of the Biot stress tensor. The angle brackets $\langle\ ,\ \rangle$ denote the scalar product between two vectors (or tensors), one of which belongs to the kinematic vector space and the other to the cokinematic vector space.

The constitutive characterization of the nonpolar medium, see [Merlini and Morandini 2005], passes through the identification of the back rotated kinematical linear-strain tensor

$$\boldsymbol{\Phi}^{*\mathrm{T}}\,\mathrm{dual}\,(\boldsymbol{X}'^{*\mathrm{T}}\boldsymbol{\omega}^*) = \boldsymbol{\Phi}^{*\mathrm{T}}\boldsymbol{\chi}^* = \boldsymbol{\Phi}^{*\mathrm{T}}\boldsymbol{F}^* - \boldsymbol{I}$$

as a strain measure; tensor $\boldsymbol{\Phi}^{*\mathrm{T}}\boldsymbol{F}^*$ is usually referred to as the Cosserat deformation tensor in micropolar elasticity [Ramezani and Naghdabadi 2007]. For a nonpolar medium, the strain energy is restricted to be a function of a symmetric strain parameter $\boldsymbol{\varepsilon}^{*\mathrm{S}}$, which is connected to the symmetric part of the above strain measure by the *strain-displacement relation*

$$\boldsymbol{\varepsilon}^{*\mathrm{S}} = \mathrm{dual}\,(\boldsymbol{\Phi}^{*\mathrm{T}} X'^{*\mathrm{T}}\boldsymbol{\omega}^*)^{\mathrm{S}}. \tag{37}$$

The superscript $(\ )^{\mathrm{S}}$ denotes the symmetric part of a tensor. Correspondingly, the skew-symmetric part of the above strain measure must vanish, whence the *kinematical constraint equation*

$$\operatorname{ax}\operatorname{dual}(\boldsymbol{\Phi}^{*\mathrm{T}}\boldsymbol{X}'^{*\mathrm{T}}\boldsymbol{\omega}^{*}) = \boldsymbol{0}, \tag{38}$$

which represents a statement of the polar decomposition theorem of the deformation gradient.

The stress parameter work-conjugate to the strain parameter is defined by the *constitutive equation*

$$\hat{\boldsymbol{T}}^{*\mathrm{S}} = w^{*}_{/\boldsymbol{\varepsilon}^{*\mathrm{S}}}, \tag{39}$$

and is itself a function, in general nonlinear, of the strain parameter, that is, $\hat{\boldsymbol{T}}^{*\mathrm{S}}(\boldsymbol{\varepsilon}^{*\mathrm{S}})$. Linearization of (39) gives the incremental *stress-strain elastic law*

$$\partial\hat{\boldsymbol{T}}^{*\mathrm{S}} = \hat{\mathbb{E}}^{*\mathrm{SS}} : \partial\boldsymbol{\varepsilon}^{*\mathrm{S}}, \tag{40}$$

where

$$\hat{\mathbb{E}}^{*\mathrm{SS}} = \hat{\boldsymbol{T}}^{*\mathrm{S}}_{/\boldsymbol{\varepsilon}^{*\mathrm{S}}} = w^{*}_{/\boldsymbol{\varepsilon}^{*\mathrm{S}}\boldsymbol{\varepsilon}^{*\mathrm{S}}} \tag{41}$$

is the fourth-order *elastic tensor* mapping strain-parameter variations onto stress-parameter variations. Of course, the elastic tensor is itself a function of the strain parameter for nonlinear constitutive models.

The internal balance requires the stress parameter to coincide with the symmetric part of the Biot stress tensor $\hat{\boldsymbol{T}}^{*}$, defined as

$$\hat{\boldsymbol{T}}^{*} = \boldsymbol{\Phi}^{*\mathrm{T}}\acute{\boldsymbol{T}}^{*}, \tag{42}$$

with $\acute{\boldsymbol{T}}^{*}$ the first Piola–Kirchhoff stress tensor. Then it clearly appears, from the Euclidean decomposition $\hat{\boldsymbol{T}}^{*} = \hat{\boldsymbol{T}}^{*\mathrm{S}} + \hat{\boldsymbol{\tau}}^{*}\times$, that a workless stress field, the Biot-axial $\hat{\boldsymbol{\tau}}^{*}$, must exist as an independent unknown field. Thus, the mechanics of the nonpolar continuum, when formulated via a micropolar description, rely on the displacement, the rotation *and* the Biot-axial as the three primary unknown fields; the relevant governing equations are the linear and angular balances (where (42) and (39) are understood and (37) is assumed as fulfilled) and the kinematical constraint (38); in weak form, the irreducible variational principle is the internally constrained principle of virtual work, with the internal work virtual functional as in (36), where the kinematical constraint equation and the role of the Biot-axial as a Lagrange multiplier are evident.

Our approach to the three-dimensional finite elasticity belongs to a line of variational formulations developed in the nineties and best represented by [Simo et al. 1992; Bufler 1995]. These works either introduce the polar decomposition of the deformation gradient like an appended constraint requiring a Lagrange multiplier or try to give a constitutive characterization for the full Biot stress tensor to circumvent the introduction of Lagrange multipliers. Our approach, however, distinguishes itself as descending from a sound constitutive characterization of the hyperelastic nonpolar medium as proposed in [Merlini 1997] and later explained in [Merlini and Morandini 2005]. It is based on a strain energy function of a symmetric strain (6 parameters) and leads naturally to a variational principle that holds the definition of the rotation field and allows for a workless stress vector field, which identifies with the axial of the skew-symmetric part of the Biot stress tensor. Note that in all the formulations of this kind the angular balance is forced in a weak sense, hence the symmetry of the second Piola–Kirchhoff stress tensor is relaxed; consistently, in our opinion, the polar decomposition of the deformation gradient (that actually defines the rotation within a nonpolar medium) must be forced in a weak sense as well. That means that even in the case of isotropic media, for which a symmetrical Biot-stress is expected [Bufler 1985], the symmetry of the Biot stress tensor ought to be relaxed.

5.2. ***The Biot-axial model.*** The approximation of the Biot-axial primary unknown field across the thickness is part of the proposed shell model. On the strength of experience gained with the three-dimensional helicoidal finite elements [Merlini and Morandini 2005], we assume that the Biot-axial covariant components $\hat{\boldsymbol{\tau}}^* \cdot \boldsymbol{g}^*_j$ are linear functions of the transverse coordinate ξ^3. The six coefficients are in turn understood as the covariant components, on the material surface base, of two parameter vectors, $\hat{\boldsymbol{\tau}}$ and $\hat{\boldsymbol{\mu}}$, which are functions only of the coordinates ξ^α. The general expression of the proposed Biot-axial model can then be cast as follows:

$$\hat{\boldsymbol{\tau}}^* = h^{-1}(\hat{\boldsymbol{\tau}} + \xi^3 h^{-1} \hat{\boldsymbol{\mu}}) \cdot \tilde{\boldsymbol{F}}^{-1}, \tag{43}$$

where h is the shell characteristic length. Vectors $\hat{\boldsymbol{\tau}}(\xi^\alpha)$ and $\hat{\boldsymbol{\mu}}(\xi^\alpha)$ have physical dimensions of force per unit length and couple per unit length, respectively; they can also be thought of as the linear and angular parts, respectively, of a dual Biot-axial parameter $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\tau}} + \varepsilon \hat{\boldsymbol{\mu}}$.

5.3. ***Linearization of the internal work virtual functional.*** The variational principle ought to be linearized in view of its use in a numerical context. Linearization of the internal work virtual functional means a truncated Taylor expansion of $\pi_{\mathrm{int}\delta}$ and yields the *incremental form* $\pi_{\mathrm{int}\delta} + \partial\pi_{\mathrm{int}\delta}$. The first term is simply called the virtual functional, the second term the virtual tangent functional: in a Newton–Raphson solution process, they will generate respectively the residual column and the tangent matrix.

The following notes can help developing (36) and its increment $\partial\pi_{\mathrm{int}\delta}$.

(1) In scalar-valued dot products, symmetric and skew-symmetric tensors are uncoupled. This property also holds in a scalar product: $\langle \boldsymbol{A}^{\mathrm{S}} + \boldsymbol{a}\times, \boldsymbol{B}^{\mathrm{S}} + \boldsymbol{b}\times\rangle = \langle \boldsymbol{A}^{\mathrm{S}}, \boldsymbol{B}^{\mathrm{S}}\rangle + \langle \boldsymbol{a}\times, \boldsymbol{b}\times\rangle = \langle \boldsymbol{A}^{\mathrm{S}}, \boldsymbol{B}^{\mathrm{S}}\rangle + 2\langle \boldsymbol{a}, \boldsymbol{b}\rangle$.

(2) Assuming (37) and (39) as fulfilled, the virtual variation of the strain-energy function $w^*(\boldsymbol{\varepsilon}^{*\mathrm{S}})$ in (36) can be written $\delta w^* = \langle \delta\boldsymbol{\varepsilon}^{*\mathrm{S}}, w^*_{/\boldsymbol{\varepsilon}^{*\mathrm{S}}}\rangle = \langle \delta\,\mathrm{dual}\,(\boldsymbol{\Phi}^{*\mathrm{T}} X'^{*\mathrm{T}} \boldsymbol{\omega}^*)^{\mathrm{S}}, \hat{\boldsymbol{T}}^{*\mathrm{S}}\rangle$. Then, recalling (40), the linearized virtual variation becomes

$$\partial\delta w^* = \delta\,\mathrm{dual}\,(\boldsymbol{\Phi}^{*\mathrm{T}} X'^{*\mathrm{T}} \boldsymbol{\omega}^*)^{\mathrm{S}} : \hat{\mathbb{E}}^{*\mathrm{SS}} : \partial\,\mathrm{dual}\,(\boldsymbol{\Phi}^{*\mathrm{T}} X'^{*\mathrm{T}} \boldsymbol{\omega}^*)^{\mathrm{S}} + \langle \partial\delta\,\mathrm{dual}\,(\boldsymbol{\Phi}^{*\mathrm{T}} X'^{*\mathrm{T}} \boldsymbol{\omega}^*)^{\mathrm{S}}, \hat{\boldsymbol{T}}^{*\mathrm{S}}\rangle.$$

(3) The mixed virtual-incremental variation variables are retained here in consideration of a possible nonlinear dependence of the local variables on the ultimate problem unknowns.

(4) The transformations $\boldsymbol{H}^*\boldsymbol{X}^*$ and $\boldsymbol{X}'^*\boldsymbol{\Phi}^*$ are interchangeable, see (30). For convenience, a short notation for such orthogonal transformations is introduced:

$$\boldsymbol{\Psi}^* = \boldsymbol{H}^*\boldsymbol{X}^* = \boldsymbol{X}'^*\boldsymbol{\Phi}^*. \tag{44}$$

The terms of the incremental form of the internal work virtual functional are then obtained:

$$\begin{aligned}
\pi_{\mathrm{int}\delta} &= \int_{\xi^{3-}}^{\xi^{3+}} \big(\mathrm{dual}\,\delta(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) : \hat{\boldsymbol{T}}^* + \delta\hat{\boldsymbol{\tau}}^*\times : \mathrm{dual}\,(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*)\big)\,\tilde{f} h\,\mathrm{d}\xi^3, \\
\partial\pi_{\mathrm{int}\delta} &= \int_{\xi^{3-}}^{\xi^{3+}} \big(\mathrm{dual}\,\delta(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) : \hat{\mathbb{E}}^{*\mathrm{SS}} : \mathrm{dual}\,\partial(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) + \mathrm{dual}\,\delta(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) : \partial\hat{\boldsymbol{\tau}}^*\times \\
&\qquad + \delta\hat{\boldsymbol{\tau}}^*\times : \mathrm{dual}\,\partial(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) + \partial\delta\hat{\boldsymbol{\tau}}^*\times : \mathrm{dual}\,(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) + \mathrm{dual}\,\partial\delta(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) : \hat{\boldsymbol{T}}^*\big)\,\tilde{f} h\,\mathrm{d}\xi^3.
\end{aligned} \tag{45}$$

Here the scalar products have been converted into dot products inside one vector space. The virtual functionals $\pi_{\mathrm{int}\delta}$ and $\partial\pi_{\mathrm{int}\delta}$ are nonlinear functions of the kinematic and cokinematic variables of the shell model and linear functions of the relevant variation variables, specifically the variations of the linear-strain dual $(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^{*})$ and of the Biot-axial $\hat{\boldsymbol{\tau}}^{*}$. Such variations are virtual variations (δ) for the virtual functional $\pi_{\mathrm{int}\delta}$ and virtual (δ), incremental (∂) and mixed virtual-incremental ($\partial\delta$) variations for the virtual tangent functional $\partial\pi_{\mathrm{int}\delta}$. In (45)$_2$, three distinct contributions to $\partial\pi_{\mathrm{int}\delta}$ can be observed: the *elastic* contribution related to $\hat{\mathbb{E}}^{*\mathrm{SS}}$, the *kinematical constraint* contribution made of three terms involving the variations of $\hat{\boldsymbol{\tau}}^{*}$, and the *geometric* contribution related to the stress state $\hat{\boldsymbol{T}}^{*}$.

5.4. ***Adaptation of the local constitutive law.*** It can be shown that evaluating the transverse strain component $\varepsilon^{*}_{\underline{33}} = \boldsymbol{g}^{*}_{3}\cdot\boldsymbol{\varepsilon}^{*\mathrm{S}}\cdot\boldsymbol{g}^{*}_{3}$ according to (37) yields $\varepsilon^{*}_{\underline{33}} = \boldsymbol{g}^{*}_{3}\cdot\boldsymbol{\alpha}^{*}\,\mathrm{dual}\,(\boldsymbol{\theta}'-\boldsymbol{\theta})$. With the most natural choice for the initial value of $\boldsymbol{k}_3$ (that is, a pure dual vector of length h, normal to the surface), it would be $\boldsymbol{g}^{*}_{3} = \boldsymbol{g}_3$ and $\boldsymbol{\alpha}^{*} = \boldsymbol{\alpha}$, whence $\varepsilon^{*}_{\underline{33}} = \boldsymbol{g}_3\cdot\boldsymbol{\alpha}\,\mathrm{dual}\,(\boldsymbol{\theta}'-\boldsymbol{\theta})$ would be always independent of the transverse coordinate ξ^3. This proves that $\varepsilon^{*}_{\underline{33}}$ is likely to be anyway uniform across the shell thickness.

This kinematical behavior is inherent in the helicoidal shell model itself (just like a classical 6-parameter model) and is, of course, inconsistent with the change of sign of the transverse normal strain across the shell thickness due to Poisson's effect in shell bending. The strong prevention of a variable transverse strain across the shell thickness induces severe locking in bending problems — a well known phenomenon in shell mechanics, usually referred to as Poisson thickness locking [Bischoff and Ramm 2000]. The classical remedy is to adopt local constitutive laws specifically adjusted for the so-called plane stress state; in nonlinear or complex constitutive models, however, such an adjustment may be quite difficult, so formulations that allow the use of unmodified three-dimensional constitutive laws have been developed. The most popular one is the method of enhanced assumed strains, originated in [Simo and Rifai 1990] and widely exploited since then (for example, in [Brank 2008]): full three-dimensional constitutive laws are allowed at the expense of some more strain variables across the thickness. Alternatively, a plane stress state can be forced dynamically on the three-dimensional constitutive law at the place where the latter is used, that is, at each quadrature point; see the techniques developed in [de Borst 1991; Klinkel and Govindjee 2002].

The method proposed here belongs to the last class of remedies and features a formulation deeply integrated with the incremental variational context. The transverse normal strain is disjoint from the shell model kinematics and allocated at each quadrature point across the thickness as a local scalar variable. The incremental form of the nonlinear constitutive law provides the way to solve locally the increment of the local variable for the increments of the other kinematically related strain variables. The nonlinear constitutive law can then be condensed and a reduced incremental form is obtained. Lastly, the incremental constitutive law is expanded again by means of an artificial transverse stiffness that provides an elastic restraint to the kinematically related transverse normal strain. A full three-dimensional incremental constitutive law, in terms of the whole kinematically related strain of (37), is obtained; this constitutive law is dynamically provided by the shell model routine during the iterative solution of the elastic problem, and the whole process is completely transparent to the user.

First, let's shorten the kinematical-strain notation: denote with $\boldsymbol{\kappa}^{*\mathrm{S}}$ the symmetric tensor in the right-hand side of (37). It is worth stressing that $\boldsymbol{\kappa}^{*\mathrm{S}}$ must be just viewed as a short name for $\mathrm{dual}\,(\boldsymbol{\Phi}^{*\mathrm{T}}X'^{*\mathrm{T}}\boldsymbol{\omega}^{*})^{\mathrm{S}}$. Next, introduce the components in the local base frame, $\kappa^{*}_{\underline{kl}} = \boldsymbol{g}^{*}_{k}\cdot\boldsymbol{\kappa}^{*\mathrm{S}}\cdot\boldsymbol{g}^{*}_{l} = \boldsymbol{g}^{*}_{k}\otimes\boldsymbol{g}^{*}_{l} : \boldsymbol{\kappa}^{*\mathrm{S}}$, and analogously with $\varepsilon^*_{\underline{kl}}$, $\hat{T}^{*\underline{ij}}$, and $\hat{E}^{*\underline{ij}\underline{kl}}$; here, the underlined couples of indexes reflect the tensor symmetries, and hence are interchangeable. Then, note that the internal *compatibility condition* (37) states that the strain parameter $\boldsymbol{\varepsilon}^{*\mathrm{S}}$ must equal the kinematically related strain $\boldsymbol{\kappa}^{*\mathrm{S}}$. The present formulation is based on the principle of virtual work, and this implies that the internal compatibility is assumed as fulfilled and $\boldsymbol{\varepsilon}^{*\mathrm{S}}$ ought to coincide with $\boldsymbol{\kappa}^{*\mathrm{S}}$. In this adaptation of the local constitutive law, however, we *relax* this assumption and disjoin some components of $\boldsymbol{\varepsilon}^{*\mathrm{S}}$ from the corresponding components of $\boldsymbol{\kappa}^{*\mathrm{S}}$. Specifically, we assume $\varepsilon^*_{\underline{\alpha\beta}} \equiv \kappa^*_{\underline{\alpha\beta}}$ and $\varepsilon^*_{\underline{\alpha 3}} \equiv \kappa^*_{\underline{\alpha 3}}$, but we keep the transverse normal component $\varepsilon^*_{\underline{33}}$ disjoint from $\kappa^*_{\underline{33}}$. That leads us to reason as in a more general three-field context (displacement-stress-strain as unknowns) just for this transverse normal variable; so, disjoining $\varepsilon^*_{\underline{33}}$ from $\kappa^*_{\underline{33}}$ is not, in a sense, an assumption, rather the release of an assumption — the fulfillment of a compatibility condition.

The incremental form of the strain-energy virtual variation, written in terms of components, reads

$$
\begin{aligned}
&\delta w^* + \partial\delta w^* \\
&= \begin{Bmatrix} \delta\kappa^*_{\underline{\alpha\beta}} \\ 2\delta\kappa^*_{\underline{\alpha 3}} \\ \delta\varepsilon^*_{\underline{33}} \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} \hat{T}^{*\underline{\alpha\beta}} \\ \hat{T}^{*\underline{\alpha 3}} \\ \hat{T}^{*\underline{33}} \end{Bmatrix} + \begin{bmatrix} \hat{E}^{*\underline{\alpha\beta}\underline{\gamma\delta}} & \hat{E}^{*\underline{\alpha\beta}\underline{\gamma 3}} & \hat{E}^{*\underline{\alpha\beta}\underline{33}} \\ \hat{E}^{*\underline{\alpha 3}\underline{\gamma\delta}} & \hat{E}^{*\underline{\alpha 3}\underline{\gamma 3}} & \hat{E}^{*\underline{\alpha 3}\underline{33}} \\ \hat{E}^{*\underline{33}\underline{\gamma\delta}} & \hat{E}^{*\underline{33}\underline{\gamma 3}} & \hat{E}^{*\underline{33}\underline{33}} \end{bmatrix} \begin{Bmatrix} \partial\kappa^*_{\underline{\gamma\delta}} \\ 2\partial\kappa^*_{\underline{\gamma 3}} \\ \partial\varepsilon^*_{\underline{33}} \end{Bmatrix} \right) + \begin{Bmatrix} \partial\delta\kappa^*_{\underline{\alpha\beta}} \\ 2\partial\delta\kappa^*_{\underline{\alpha 3}} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \hat{T}^{*\underline{\alpha\beta}} \\ \hat{T}^{*\underline{\alpha 3}} \end{Bmatrix}.
\end{aligned} \tag{46}
$$

Here, $\kappa^*_{\underline{\alpha\beta}}$ and $\kappa^*_{\underline{\alpha 3}}$ have been substituted for $\varepsilon^*_{\underline{\alpha\beta}}$ and $\varepsilon^*_{\underline{\alpha 3}}$, respectively, but $\varepsilon^*_{\underline{33}}$ has been retained; the term with $\partial\delta\varepsilon^*_{\underline{33}}$ is of course lacking, since $\varepsilon^*_{\underline{33}}$ is now understood as a local, independent variable. Equation (46) contributes to the integrand of $\pi_{\mathrm{int}\delta} + \partial\pi_{\mathrm{int}\delta}$, however the term in $\delta\varepsilon^*_{\underline{33}}$ keeps disjoint from the overall principle and originates the local incremental equation

$$
\hat{T}^{*\underline{33}} + \begin{bmatrix} \hat{E}^{*\underline{33}\underline{\gamma\delta}} & \hat{E}^{*\underline{33}\underline{\gamma 3}} \end{bmatrix} \begin{Bmatrix} \partial\kappa^*_{\underline{\gamma\delta}} \\ 2\partial\kappa^*_{\underline{\gamma 3}} \end{Bmatrix} + \hat{E}^{*\underline{33}\underline{33}}\partial\varepsilon^*_{\underline{33}} = 0. \tag{47}
$$

It is worth noting that (47) is the incremental form of the nonlinear equation $\hat{T}^{*\underline{33}} = \partial w^*/\partial\varepsilon^*_{\underline{33}} = 0$, stating the independence of the strain-energy function from the transverse normal strain. Note that the condition of transverse normal stress — $\hat{T}^{*\underline{33}}$ in the present Biot-type parameterization of the strain energy w^* — identically zero is the classical assumption introduced to reduce the constitutive law to the so-called plane-stress state in shell analyses; see, for example, [Bischoff et al. 2004] (though the popular notion "plane-stress" is strictly speaking not correct when transverse shear stresses are allowed).

Equation (47) is now solved for $\partial\varepsilon^*_{\underline{33}}$ and the result is substituted within (46), yielding the condensed form

$$
\delta w^* + \partial\delta w^* = \begin{Bmatrix} \delta\kappa^*_{\underline{\alpha\beta}} \\ 2\delta\kappa^*_{\underline{\alpha 3}} \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} \overset{\wedge}{\hat{T}}{}^{*\underline{\alpha\beta}} \\ \overset{\wedge}{\hat{T}}{}^{*\underline{\alpha 3}} \end{Bmatrix} + \begin{bmatrix} \overset{\wedge}{\hat{E}}{}^{*\underline{\alpha\beta}\underline{\gamma\delta}} & \overset{\wedge}{\hat{E}}{}^{*\underline{\alpha\beta}\underline{\gamma 3}} \\ \overset{\wedge}{\hat{E}}{}^{*\underline{\alpha 3}\underline{\gamma\delta}} & \overset{\wedge}{\hat{E}}{}^{*\underline{\alpha 3}\underline{\gamma 3}} \end{bmatrix} \begin{Bmatrix} \partial\kappa^*_{\underline{\gamma\delta}} \\ 2\partial\kappa^*_{\underline{\gamma 3}} \end{Bmatrix} \right) + \begin{Bmatrix} \partial\delta\kappa^*_{\underline{\alpha\beta}} \\ 2\partial\delta\kappa^*_{\underline{\alpha 3}} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \hat{T}^{*\underline{\alpha\beta}} \\ \hat{T}^{*\underline{\alpha 3}} \end{Bmatrix}, \tag{48}
$$

where

$$
\begin{aligned}
\begin{Bmatrix} \overset{\wedge}{\hat{T}}{}^{*\underline{\alpha\beta}} \\ \overset{\wedge}{\hat{T}}{}^{*\underline{\alpha 3}} \end{Bmatrix} &= \begin{Bmatrix} \hat{T}^{*\underline{\alpha\beta}} \\ \hat{T}^{*\underline{\alpha 3}} \end{Bmatrix} - \begin{bmatrix} \hat{E}^{*\underline{\alpha\beta}\underline{33}} \\ \hat{E}^{*\underline{\alpha 3}\underline{33}} \end{bmatrix} \hat{E}^{*\underline{33}\underline{33}-1}\hat{T}^{*\underline{33}}, \\
\begin{bmatrix} \overset{\wedge}{\hat{E}}{}^{*\underline{\alpha\beta}\underline{\gamma\delta}} & \overset{\wedge}{\hat{E}}{}^{*\underline{\alpha\beta}\underline{\gamma 3}} \\ \overset{\wedge}{\hat{E}}{}^{*\underline{\alpha 3}\underline{\gamma\delta}} & \overset{\wedge}{\hat{E}}{}^{*\underline{\alpha 3}\underline{\gamma 3}} \end{bmatrix} &= \begin{bmatrix} \hat{E}^{*\underline{\alpha\beta}\underline{\gamma\delta}} & \hat{E}^{*\underline{\alpha\beta}\underline{\gamma 3}} \\ \hat{E}^{*\underline{\alpha 3}\underline{\gamma\delta}} & \hat{E}^{*\underline{\alpha 3}\underline{\gamma 3}} \end{bmatrix} - \begin{bmatrix} \hat{E}^{*\underline{\alpha\beta}\underline{33}} \\ \hat{E}^{*\underline{\alpha 3}\underline{33}} \end{bmatrix} \hat{E}^{*\underline{33}\underline{33}-1} \begin{bmatrix} \hat{E}^{*\underline{33}\underline{\gamma\delta}} & \hat{E}^{*\underline{33}\underline{\gamma 3}} \end{bmatrix},
\end{aligned} \tag{49}
$$

collect the reduced components of the stress parameter and elastic tensor.

The incremental form (48) deserves further development. It is apparent that, in (48),

$$\delta w^* = \delta\kappa^*_{\underline{\alpha\beta}}\hat{\bar{T}}^{*\underline{\alpha\beta}} + 2\delta\kappa^*_{\underline{\alpha 3}}\hat{\bar{T}}^{*\underline{\alpha 3}},$$

so

$$\partial\delta w^* = \delta\kappa^*_{\underline{\alpha\beta}}\partial\hat{\bar{T}}^{*\underline{\alpha\beta}} + 2\delta\kappa^*_{\underline{\alpha 3}}\partial\hat{\bar{T}}^{*\underline{\alpha 3}} + \partial\delta\kappa^*_{\underline{\alpha\beta}}\hat{\bar{T}}^{*\underline{\alpha\beta}} + 2\partial\delta\kappa^*_{\underline{\alpha 3}}\hat{\bar{T}}^{*\underline{\alpha 3}}$$

is also identified as the remaining part of $\delta w^* + \partial\delta w^*$. Therefore, using $(49)_1$, it follows that

$$\begin{Bmatrix}\delta\kappa^*_{\underline{\alpha\beta}}\\ 2\delta\kappa^*_{\underline{\alpha 3}}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial\hat{\bar{T}}^{*\underline{\alpha\beta}}\\ \partial\hat{\bar{T}}^{*\underline{\alpha 3}}\end{Bmatrix}H = \begin{Bmatrix}\delta\kappa^*_{\underline{\alpha\beta}}\\ 2\delta\kappa^*_{\underline{\alpha 3}}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\hat{\bar{E}}^{*\underline{\alpha\beta\gamma\delta}} & \hat{\bar{E}}^{*\underline{\alpha\beta\gamma 3}}\\ \hat{\bar{E}}^{*\underline{\alpha 3\gamma\delta}} & \hat{\bar{E}}^{*\underline{\alpha 3\gamma 3}}\end{bmatrix}\begin{Bmatrix}\partial\kappa^*_{\underline{\gamma\delta}}\\ 2\partial\kappa^*_{\underline{\gamma 3}}\end{Bmatrix} + \begin{Bmatrix}\partial\delta\kappa^*_{\underline{\alpha\beta}}\\ 2\partial\delta\kappa^*_{\underline{\alpha 3}}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\hat{E}^{*\underline{\alpha\beta 33}}\\ \hat{E}^{*\underline{\alpha 333}}\end{bmatrix}\hat{E}^{*\underline{3333}-1}\hat{T}^{*\underline{33}}. \tag{50}$$

Now, in the context of a discrete approximation, $\partial\delta\boldsymbol{\kappa}^{*\mathrm{S}}$ will be always reduced to a linear function of the ultimate virtual variation unknowns (δ) and of the ultimate incremental variation unknowns (∂) of the discrete problem [Merlini and Morandini 2005]; thus, the second term in the right-hand side of (50) is always logically reducible to a form like the first term and thus will add to the first term. When doing so, the matrix elements $\hat{\bar{E}}^*$ of the first term will be corrected by terms proportional to $\hat{T}^{*\underline{33}}$ and dependent on the whole shell model. However, in the present adaptation of a local constitutive law, we find it convenient to miss this correction of the reduced stress-strain tangent map and retain only that part coming from the first term. Note that the term omitted is almost negligible, since $\hat{T}^{*\underline{33}}$ will vanish at convergence, see (47). Moreover, the error is on the problem tangent matrix, but not on the residual, so it will affect only the convergence rate, not the final result.

The foregoing discussion allows us to correct (48) and write the separate terms of the incremental form of the strain-energy virtual variation as

$$\begin{aligned}\delta w^* &= \begin{Bmatrix}\delta\kappa^*_{\underline{\alpha\beta}}\\ 2\delta\kappa^*_{\underline{\alpha 3}}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\hat{\bar{T}}^{*\underline{\alpha\beta}}\\ \hat{\bar{T}}^{*\underline{\alpha 3}}\end{Bmatrix},\\ \partial\delta w^* &= \begin{Bmatrix}\delta\kappa^*_{\underline{\alpha\beta}}\\ 2\delta\kappa^*_{\underline{\alpha 3}}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\hat{\bar{E}}^{*\underline{\alpha\beta\gamma\delta}} & \hat{\bar{E}}^{*\underline{\alpha\beta\gamma 3}}\\ \hat{\bar{E}}^{*\underline{\alpha 3\gamma\delta}} & \hat{\bar{E}}^{*\underline{\alpha 3\gamma 3}}\end{bmatrix}\begin{Bmatrix}\partial\kappa^*_{\underline{\gamma\delta}}\\ 2\partial\kappa^*_{\underline{\gamma 3}}\end{Bmatrix} + \begin{Bmatrix}\partial\delta\kappa^*_{\underline{\alpha\beta}}\\ 2\partial\delta\kappa^*_{\underline{\alpha 3}}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\hat{\bar{T}}^{*\underline{\alpha\beta}}\\ \hat{\bar{T}}^{*\underline{\alpha 3}}\end{Bmatrix},\end{aligned} \tag{51}$$

where only the reduced components of the stress parameter and elastic tensor appear.

The strain-energy variations in (51) depend implicitly on any strain components $\kappa^*_{\underline{\alpha\beta}}$, $\kappa^*_{\underline{\alpha 3}}$, *and* $\varepsilon^*_{\underline{33}}$, and explicitly on the variations of all the components of the kinematical strain $\boldsymbol{\kappa}^{*\mathrm{S}}$ except $\kappa^*_{\underline{33}}$. It is seen that the component $\kappa^*_{\underline{33}}$ is left elastically unrestrained. The release of the transverse normal strain from the shell model kinematics may overcome the deficiencies of the kinematical model in representing correctly the actual strain state, for example, in shell bending; however this expedient leaves an intrinsic transverse lability. Since no more stiffness is guaranteed against transverse collapse — a deformation consistent with a uniform transverse normal strain $\kappa^*_{\underline{33}}$ — the present constitutive adaptation infers a lability in membrane stretching.

A convenient remedy for this lability is to modify again the constitutive model by introducing an artificial transverse stiffness. This can be accomplished by enhancing the strain energy with a term quadratic in $\kappa^*_{\underline{33}}$, say

$$w^*_{33} = \tfrac{1}{2}\,\hat{E}^{*\underline{3333}}\kappa^{*2}_{\underline{33}}, \tag{52}$$

where the elastic constant is conveniently borrowed from the proper component of the original elastic tensor in (46). In order not to affect the mechanical response of the overall shell model, this contribution must be independent of the other kinematically related strain components, so the current value of $\hat{E}^{*\underline{3333}}$ must hold constant in (52). Taking the derivatives of $w^*_{33}(\kappa^*_{\underline{33}})$, the expressions of a transverse constitutive equation and the relevant tangent map are obtained:

$$\hat{\bar{T}}^{*\underline{33}} = \hat{E}^{*\underline{3333}}\kappa^*_{\underline{33}}, \qquad \hat{\bar{E}}^{*\underline{3333}} = \hat{E}^{*\underline{3333}}. \tag{53}$$

The virtual variation and the linearized virtual variation of the energy term (52), $\delta w^*_{33} = \delta\kappa^*_{\underline{33}}\hat{\bar{T}}^{*\underline{33}}$ and $\partial\delta w^*_{33} = \delta\kappa^*_{\underline{33}}\hat{\bar{E}}^{*\underline{3333}}\partial\kappa^*_{\underline{33}} + \partial\delta\kappa^*_{\underline{33}}\hat{\bar{T}}^{*\underline{33}}$, are now added to (51). The latter can be finally written in tensor form as

$$\delta w^* = \delta\boldsymbol{\kappa}^{*\mathrm{S}} : \hat{\bar{\boldsymbol{T}}}^{*\mathrm{S}}, \qquad \partial\delta w^* = \delta\boldsymbol{\kappa}^{*\mathrm{S}} : \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} : \partial\boldsymbol{\kappa}^{*\mathrm{S}} + \partial\delta\boldsymbol{\kappa}^{*\mathrm{S}} : \hat{\bar{\boldsymbol{T}}}^{*\mathrm{S}}, \tag{54}$$

where

$$\hat{\bar{\boldsymbol{T}}}^{*\mathrm{S}} = \hat{\bar{T}}^{*\underline{ij}}\,\boldsymbol{g}^*_i \otimes \boldsymbol{g}^*_j, \qquad \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} = \hat{\bar{E}}^{*\underline{ijkl}}\,\boldsymbol{g}^*_i \otimes \boldsymbol{g}^*_j \otimes \boldsymbol{g}^*_k \otimes \boldsymbol{g}^*_l, \tag{55}$$

are referred to as the *full reduced stress parameter* and the *full reduced elastic tensor*, respectively. Equation (54) holds the elements of a full three-dimensional constitutive law, written in incremental form, with $\hat{\bar{\boldsymbol{T}}}^{*\mathrm{S}}$ the stress parameter and $\hat{\bar{\mathbb{E}}}^{*\mathrm{SS}}$ the relevant elastic tensor. $\hat{\bar{\boldsymbol{T}}}^{*\mathrm{S}}$ should replace $\hat{\boldsymbol{T}}^{*\mathrm{S}}$ in the computation of the Biot stress tensor, $\hat{\boldsymbol{T}}^* = \hat{\bar{\boldsymbol{T}}}^{*\mathrm{S}} + \hat{\boldsymbol{\tau}}^*\times$, to be used within (45). In $(45)_2$, $\hat{\bar{\mathbb{E}}}^{*\mathrm{SS}}$ should replace $\hat{\mathbb{E}}^{*\mathrm{SS}}$.

The three-dimensional incremental constitutive law is built dynamically during the solution process. Every component in (55) is computed as in (49) and (53), using the current components of the stress parameter and elastic tensor of the original three-dimensional constitutive law. These computations are performed within the Newton–Raphson iterations themselves of the overall problem solution, as in [de Borst 1991]. At the end of each iteration, once the increments $\partial\boldsymbol{\kappa}^{*\mathrm{S}}$ are known from the overall solution, $\partial\varepsilon^*_{\underline{33}}$ is computed locally from (47) and the locally stored transverse normal strain component is updated additively, $\varepsilon^*_{\underline{33}} \leftarrow \varepsilon^*_{\underline{33}} + \partial\varepsilon^*_{\underline{33}}$. Of course, at that stage the coefficients in (47) must be consistent with those in (49), so they must be saved in the quadrature point storage area.

5.5. ***Case of linear elasticity.*** The purpose of this section is to show how the adaptation discussed above can accommodate the classical plane-stress state in linear elasticity. This section is included for the sake of completeness, and not reading it does not impair understanding of the formulation.

The adaptation of the local constitutive law is somewhat simpler in linear elasticity, when

$$w^*(\boldsymbol{\varepsilon}^{*\mathrm{S}}) = \tfrac{1}{2}\boldsymbol{\varepsilon}^{*\mathrm{S}} : \hat{\mathbb{E}}^{*\mathrm{SS}} : \boldsymbol{\varepsilon}^{*\mathrm{S}} \quad \text{and} \quad \hat{\boldsymbol{T}}^{*\mathrm{S}}(\boldsymbol{\varepsilon}^{*\mathrm{S}}) = \hat{\mathbb{E}}^{*\mathrm{SS}} : \boldsymbol{\varepsilon}^{*\mathrm{S}},$$

with $\hat{\mathbb{E}}^{*\mathrm{SS}}$ a constant fourth-order tensor. In this case, study of the simple strain-energy virtual variation $\delta w^* = \delta\boldsymbol{\varepsilon}^{*\mathrm{S}} : \hat{\boldsymbol{T}}^{*\mathrm{S}} = \delta\boldsymbol{\varepsilon}^{*\mathrm{S}} : \hat{\mathbb{E}}^{*\mathrm{SS}} : \boldsymbol{\varepsilon}^{*\mathrm{S}}$ suffices, and study of the incremental form is not required. In component form, with $\kappa^*_{\underline{\alpha\beta}}$ and $\kappa^*_{\underline{\alpha 3}}$ substituted for $\varepsilon^*_{\underline{\alpha\beta}}$ and $\varepsilon^*_{\underline{\alpha 3}}$, δw^* reads

$$\delta w^* = \begin{Bmatrix} \delta\kappa^*_{\underline{\alpha\beta}} \\ 2\delta\kappa^*_{\underline{\alpha 3}} \\ \delta\varepsilon^*_{\underline{33}} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \hat{T}^{*\underline{\alpha\beta}} \\ \hat{T}^{*\underline{\alpha 3}} \\ \hat{T}^{*\underline{33}} \end{Bmatrix} = \begin{Bmatrix} \delta\kappa^*_{\underline{\alpha\beta}} \\ 2\delta\kappa^*_{\underline{\alpha 3}} \\ \delta\varepsilon^*_{\underline{33}} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \hat{E}^{*\underline{\alpha\beta\gamma\delta}} & \hat{E}^{*\underline{\alpha\beta\gamma 3}} & \hat{E}^{*\underline{\alpha\beta 33}} \\ \hat{E}^{*\underline{\alpha 3\gamma\delta}} & \hat{E}^{*\underline{\alpha 3\gamma 3}} & \hat{E}^{*\underline{\alpha 333}} \\ \hat{E}^{*\underline{33\gamma\delta}} & \hat{E}^{*\underline{33\gamma 3}} & \hat{E}^{*\underline{3333}} \end{bmatrix} \begin{Bmatrix} \kappa^*_{\underline{\gamma\delta}} \\ 2\kappa^*_{\underline{\gamma 3}} \\ \varepsilon^*_{\underline{33}} \end{Bmatrix}. \tag{56}$$

The term in $\delta\varepsilon^*_{\underline{33}}$ of this contribution to the overall principle yields now the local finite equation

$$\hat{T}^{*\underline{33}} = \begin{bmatrix} \hat{E}^{*\underline{33\gamma\delta}} & \hat{E}^{*\underline{33\gamma 3}} \end{bmatrix} \begin{Bmatrix} \kappa^*_{\underline{\gamma\delta}} \\ 2\kappa^*_{\underline{\gamma 3}} \end{Bmatrix} + \hat{E}^{*\underline{3333}}\varepsilon^*_{\underline{33}} = 0, \tag{57}$$

which again states the independence of the strain-energy function from the transverse normal strain. Solving (57) for $\varepsilon^*_{\underline{33}}$ and substituting the result into (56) yields a condensed form where the reduced components of the elastic tensor are those found in $(49)_2$. Finally, a transverse stiffness is introduced as for (52) and (53), and (56) takes the final tensor form

$$\delta w^* = \delta\boldsymbol{\kappa}^{*\mathrm{S}} : \hat{\bar{\boldsymbol{T}}}^{*\mathrm{S}} = \delta\boldsymbol{\kappa}^{*\mathrm{S}} : \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} : \boldsymbol{\kappa}^{*\mathrm{S}}, \tag{58}$$

with

$$\hat{\bar{\boldsymbol{T}}}^{*\mathrm{S}} = \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} : \boldsymbol{\kappa}^{*\mathrm{S}} \tag{59}$$

the full three-dimensional linear constitutive equation. The components of $\hat{\bar{\mathbb{E}}}^{*\mathrm{SS}}$ are found in $(49)_2$ and $(53)_2$. Since such components are constant, they can be computed once and for all, and there is no need to allocate the variable $\varepsilon^*_{\underline{33}}$ in linear elasticity.

6. Reduction to the shell surface mechanics

The next step is to integrate the internal work virtual functional across the shell model thickness and to obtain the shell constitutive equations.

6.1. ***Integration across the thickness.*** The local variation variables in (45) must be solved for the relevant surface variation variables before addressing the integration across the thickness. The kinematical variation variables are quite complicated to develop; they are discussed in the Appendix and yield the final result

$$\begin{aligned}
\delta(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) &= \big(\boldsymbol{\Psi}^{*\mathrm{T}} \cdot \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) + \xi^3\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{k}'_\alpha \times \tilde{\boldsymbol{\Lambda}}' \cdot \boldsymbol{A}'\delta\boldsymbol{\theta}'\big) \otimes \boldsymbol{g}^{*\alpha} + \boldsymbol{\Psi}^{*\mathrm{T}}\tilde{\boldsymbol{H}}' \cdot \boldsymbol{A}'\delta\boldsymbol{\theta}' \otimes \boldsymbol{g}^{*3}, \\
\partial(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) &= \big(\boldsymbol{\Psi}^{*\mathrm{T}} \cdot \boldsymbol{H}\partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) + \xi^3\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{k}'_\alpha \times \tilde{\boldsymbol{\Lambda}}' \cdot \boldsymbol{A}'\partial\boldsymbol{\theta}'\big) \otimes \boldsymbol{g}^{*\alpha} + \boldsymbol{\Psi}^{*\mathrm{T}}\tilde{\boldsymbol{H}}' \cdot \boldsymbol{A}'\partial\boldsymbol{\theta}' \otimes \boldsymbol{g}^{*3}, \\
\partial\delta(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) &= \Big(\boldsymbol{\Psi}^{*\mathrm{T}} \cdot \boldsymbol{H}\partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) + \xi^3\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{k}'_\alpha \times \tilde{\boldsymbol{\Lambda}}' \cdot \boldsymbol{A}'\partial\delta\boldsymbol{\theta}' \\
&\qquad - \xi^3\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{I}^\times\tilde{\boldsymbol{\Lambda}}' : \big(\boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \otimes \boldsymbol{A}'\partial\boldsymbol{\theta}' + \boldsymbol{H}\partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \otimes \boldsymbol{A}'\delta\boldsymbol{\theta}'\big) \\
&\qquad + (\xi^3)^2\boldsymbol{\Psi}^{*\mathrm{T}} \cdot \big(\boldsymbol{k}'_\alpha \times \tilde{\boldsymbol{\Lambda}}'^{\underline{123}}_{\mathrm{III}} + (\boldsymbol{I}^\times\boldsymbol{k}'_\alpha \times \tilde{\boldsymbol{\Lambda}}')^{\mathrm{T}132}\tilde{\boldsymbol{\Lambda}}'\big)^{\mathrm{S}\underline{123}} : \boldsymbol{A}'\delta\boldsymbol{\theta}' \otimes \boldsymbol{A}'\partial\boldsymbol{\theta}'\Big) \otimes \boldsymbol{g}^{*\alpha} \\
&\quad + \boldsymbol{\Psi}^{*\mathrm{T}}\tilde{\boldsymbol{H}}' \cdot \boldsymbol{A}'\partial\delta\boldsymbol{\theta}' \otimes \boldsymbol{g}^{*3}.
\end{aligned} \tag{60}$$

Obtaining the Biot-axial variation variables from (43), instead, is immediate:

$$\delta\hat{\boldsymbol{\tau}}^* = h^{-1}\tilde{\boldsymbol{F}}^{-\mathrm{T}}(\delta\hat{\boldsymbol{\tau}} + \xi^3 h^{-1}\delta\hat{\boldsymbol{\mu}}), \quad \partial\hat{\boldsymbol{\tau}}^* = h^{-1}\tilde{\boldsymbol{F}}^{-\mathrm{T}}(\partial\hat{\boldsymbol{\tau}} + \xi^3 h^{-1}\partial\hat{\boldsymbol{\mu}}),$$
$$\partial\delta\hat{\boldsymbol{\tau}}^* = h^{-1}\tilde{\boldsymbol{F}}^{-\mathrm{T}}(\partial\delta\hat{\boldsymbol{\tau}} + \xi^3 h^{-1}\partial\delta\hat{\boldsymbol{\mu}}). \tag{61}$$

Substituting (60) and (61) into (45) yields

$$\pi_{\mathrm{int}\delta} = \begin{Bmatrix} \delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \\ \delta\boldsymbol{\theta}' \\ \delta\hat{\boldsymbol{\mu}} \\ \delta\hat{\boldsymbol{\tau}} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{H}^{\mathrm{T}}\boldsymbol{R}_{\boldsymbol{\omega}}^{\alpha} \\ \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{R}_{\boldsymbol{\theta}} \\ \boldsymbol{R}_{\mu} \\ \boldsymbol{R}_{\tau} \end{Bmatrix},$$

$$\partial\pi_{\mathrm{int}\delta} = \begin{Bmatrix} \delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \\ \delta\boldsymbol{\theta}' \\ \delta\hat{\boldsymbol{\mu}} \\ \delta\hat{\boldsymbol{\tau}} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} \boldsymbol{H}^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{\omega\omega}}^{\alpha\beta}\boldsymbol{H} & \boldsymbol{H}^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{\omega\theta}}^{\alpha}\boldsymbol{A}' & \boldsymbol{H}^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{\omega}\mu}^{\alpha} & \boldsymbol{H}^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{\omega}\tau}^{\alpha} \\ \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{\omega\theta}}^{\beta\mathrm{T}}\boldsymbol{H} & \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{\theta\theta}}\boldsymbol{A}' & \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{\theta}\mu} & \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{\theta}\tau} \\ \boldsymbol{D}_{\boldsymbol{\omega}\mu}^{\beta\mathrm{T}}\boldsymbol{H} & \boldsymbol{D}_{\boldsymbol{\theta}\mu}^{\mathrm{T}}\boldsymbol{A}' & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{D}_{\boldsymbol{\omega}\tau}^{\beta\mathrm{T}}\boldsymbol{H} & \boldsymbol{D}_{\boldsymbol{\theta}\tau}^{\mathrm{T}}\boldsymbol{A}' & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \cdot \begin{Bmatrix} \partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\beta) \\ \partial\boldsymbol{\theta}' \\ \partial\hat{\boldsymbol{\mu}} \\ \partial\hat{\boldsymbol{\tau}} \end{Bmatrix} + \begin{Bmatrix} \partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \\ \partial\delta\boldsymbol{\theta}' \\ \partial\delta\hat{\boldsymbol{\mu}} \\ \partial\delta\hat{\boldsymbol{\tau}} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{H}^{\mathrm{T}}\boldsymbol{R}_{\boldsymbol{\omega}}^{\alpha} \\ \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{R}_{\boldsymbol{\theta}} \\ \boldsymbol{R}_{\mu} \\ \boldsymbol{R}_{\tau} \end{Bmatrix}, \tag{62}$$

where the sequence angular–linear is understood while writing dual vectors and tensors in matrix notation. After some involved manipulations, detailed in [Merlini and Morandini 2008], the vectors $\boldsymbol{R}$ and tensors $\boldsymbol{D}$ in (62) can be written as the integrals

$$\begin{aligned} \{\boldsymbol{R}_{\boldsymbol{\omega}}^{\alpha}\} &= \int_{\xi^{3-}}^{\xi^{3+}} \begin{Bmatrix} \text{dual } \acute{\boldsymbol{S}}^{*\alpha} \\ \text{primal } \acute{\boldsymbol{S}}^{*\alpha} \end{Bmatrix} \tilde{f} h \, \mathrm{d}\xi^3, \\ \{\boldsymbol{R}_{\boldsymbol{\theta}}\} &= \int_{\xi^{3-}}^{\xi^{3+}} \begin{Bmatrix} \text{dual } (\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \acute{\boldsymbol{S}}^{*\gamma} \times \boldsymbol{k}'_\gamma + \tilde{\boldsymbol{H}}'^{\mathrm{T}} \acute{\boldsymbol{S}}^{*3}) \\ \text{primal } (\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \acute{\boldsymbol{S}}^{*\gamma} \times \boldsymbol{k}'_\gamma + \tilde{\boldsymbol{H}}'^{\mathrm{T}} \acute{\boldsymbol{S}}^{*3}) \end{Bmatrix} \tilde{f} h \, \mathrm{d}\xi^3, \\ \boldsymbol{R}_{\mu} &= \int_{\xi^{3-}}^{\xi^{3+}} \xi^3 h^{-2} \tilde{\boldsymbol{F}}^{-1} \, \text{dual} \, 2 \, \text{ax} \, (\boldsymbol{\Psi}^{*\mathrm{T}} \boldsymbol{\omega}^*) \, \tilde{f} h \, \mathrm{d}\xi^3, \\ \boldsymbol{R}_{\tau} &= \int_{\xi^{3-}}^{\xi^{3+}} h^{-1} \tilde{\boldsymbol{F}}^{-1} \, \text{dual} \, 2 \, \text{ax} \, (\boldsymbol{\Psi}^{*\mathrm{T}} \boldsymbol{\omega}^*) \, \tilde{f} h \, \mathrm{d}\xi^3, \end{aligned} \tag{63}$$

and

$$[\boldsymbol{D}_{\boldsymbol{\omega\omega}}^{\alpha\beta}] = \int_{\xi^{3-}}^{\xi^{3+}} \begin{bmatrix} \text{dual } \boldsymbol{\Psi}^* \\ \text{primal } \boldsymbol{\Psi}^* \end{bmatrix} (\boldsymbol{g}^{*\alpha} \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} \boldsymbol{g}^{*\beta}) \begin{bmatrix} \text{dual } \boldsymbol{\Psi}^{*\mathrm{T}} & \text{primal } \boldsymbol{\Psi}^{*\mathrm{T}} \end{bmatrix} \tilde{f} h \, \mathrm{d}\xi^3,$$

$$\begin{aligned} [\boldsymbol{D}_{\boldsymbol{\omega\theta}}^{\alpha}] = \int_{\xi^{3-}}^{\xi^{3+}} & \begin{bmatrix} \text{dual } \boldsymbol{\Psi}^* \\ \text{primal } \boldsymbol{\Psi}^* \end{bmatrix} \begin{bmatrix} \boldsymbol{g}^{*\alpha} \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} \boldsymbol{g}^{*\delta} & \boldsymbol{g}^{*\alpha} \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} \boldsymbol{g}^{*3} \end{bmatrix} \\ & \cdot \begin{bmatrix} \text{dual } (-\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \boldsymbol{k}'_\delta \times \boldsymbol{\Psi}^*)^{\mathrm{T}} & \text{primal } (-\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \boldsymbol{k}'_\delta \times \boldsymbol{\Psi}^*)^{\mathrm{T}} \\ \text{dual } (\tilde{\boldsymbol{H}}'^{\mathrm{T}} \boldsymbol{\Psi}^*)^{\mathrm{T}} & \text{primal } (\tilde{\boldsymbol{H}}'^{\mathrm{T}} \boldsymbol{\Psi}^*)^{\mathrm{T}} \end{bmatrix} \tilde{f} h \, \mathrm{d}\xi^3 \\ & + \int_{\xi^{3-}}^{\xi^{3+}} \begin{bmatrix} \text{dual } (-\xi^3 \acute{\boldsymbol{S}}^{*\alpha} \times \tilde{\boldsymbol{\Lambda}}') & \text{primal } (-\xi^3 \acute{\boldsymbol{S}}^{*\alpha} \times \tilde{\boldsymbol{\Lambda}}') \\ \text{primal } (-\xi^3 \acute{\boldsymbol{S}}^{*\alpha} \times \tilde{\boldsymbol{\Lambda}}') & \boldsymbol{0} \end{bmatrix} \tilde{f} h \, \mathrm{d}\xi^3, \end{aligned}$$

$$[\boldsymbol{D}^{\alpha}_{\boldsymbol{\omega\mu}}] = \int_{\xi^{3-}}^{\xi^{3+}} \begin{bmatrix} \text{dual}\,\boldsymbol{\Psi}^* \\ \text{primal}\,\boldsymbol{\Psi}^* \end{bmatrix} (\xi^3 h^{-2} \tilde{\boldsymbol{F}}^{-1} \boldsymbol{g}^{*\alpha}\times)^{\mathrm{T}} \tilde{f} h \, \mathrm{d}\xi^3,$$

$$[\boldsymbol{D}^{\alpha}_{\boldsymbol{\omega\tau}}] = \int_{\xi^{3-}}^{\xi^{3+}} \begin{bmatrix} \text{dual}\,\boldsymbol{\Psi}^* \\ \text{primal}\,\boldsymbol{\Psi}^* \end{bmatrix} (h^{-1} \tilde{\boldsymbol{F}}^{-1} \boldsymbol{g}^{*\alpha}\times)^{\mathrm{T}} \tilde{f} h \, \mathrm{d}\xi^3,$$

$$\begin{aligned}[\boldsymbol{D}^{*}_{\boldsymbol{\theta\theta}}] = {}& \int_{\xi^{3-}}^{\xi^{3+}} \begin{bmatrix} \text{dual}\,(-\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \boldsymbol{k}'_\gamma \times \boldsymbol{\Psi}^*) & \text{dual}\,(\tilde{\boldsymbol{H}}'^{\mathrm{T}} \boldsymbol{\Psi}^*) \\ \text{primal}\,(-\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \boldsymbol{k}'_\gamma \times \boldsymbol{\Psi}^*) & \text{primal}\,(\tilde{\boldsymbol{H}}'^{\mathrm{T}} \boldsymbol{\Psi}^*) \end{bmatrix} \begin{bmatrix} \boldsymbol{g}^{*\gamma} \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} \boldsymbol{g}^{*\delta} & \boldsymbol{g}^{*\gamma} \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} \boldsymbol{g}^{*3} \\ \boldsymbol{g}^{*3} \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} \boldsymbol{g}^{*\delta} & \boldsymbol{g}^{*3} \hat{\bar{\mathbb{E}}}^{*\mathrm{SS}} \boldsymbol{g}^{*3} \end{bmatrix} \\ & \cdot \begin{bmatrix} \text{dual}\,(-\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \boldsymbol{k}'_\delta \times \boldsymbol{\Psi}^*)^{\mathrm{T}} & \text{primal}\,(-\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \boldsymbol{k}'_\delta \times \boldsymbol{\Psi}^*)^{\mathrm{T}} \\ \text{dual}\,(\tilde{\boldsymbol{H}}'^{\mathrm{T}} \boldsymbol{\Psi}^*)^{\mathrm{T}} & \text{primal}\,(\tilde{\boldsymbol{H}}'^{\mathrm{T}} \boldsymbol{\Psi}^*)^{\mathrm{T}} \end{bmatrix} \tilde{f} h \, \mathrm{d}\xi^3 \\ & + \int_{\xi^{3-}}^{\xi^{3+}} \begin{bmatrix} \text{dual}\Big((\xi^3)^2 \big(\acute{\boldsymbol{S}}^{*\gamma} \times \boldsymbol{k}'_\gamma \cdot \tilde{\boldsymbol{\Lambda}}'^{\underline{123}}_{\mathrm{III}} + \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} (\acute{\boldsymbol{S}}^{*\gamma} \times \boldsymbol{k}'_\gamma \times)^{\mathrm{S}} \tilde{\boldsymbol{\Lambda}}'\big)\Big) & \text{primal}\Big((\xi^3)^2 \big(\acute{\boldsymbol{S}}^{*\gamma} \times \boldsymbol{k}'_\gamma \cdot \tilde{\boldsymbol{\Lambda}}'^{\underline{123}}_{\mathrm{III}} + \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} (\acute{\boldsymbol{S}}^{*\gamma} \times \boldsymbol{k}'_\gamma \times)^{\mathrm{S}} \tilde{\boldsymbol{\Lambda}}'\big)\Big) \\ \text{primal}\Big((\xi^3)^2 \big(\acute{\boldsymbol{S}}^{*\gamma} \times \boldsymbol{k}'_\gamma \cdot \tilde{\boldsymbol{\Lambda}}'^{\underline{123}}_{\mathrm{III}} + \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} (\acute{\boldsymbol{S}}^{*\gamma} \times \boldsymbol{k}'_\gamma \times)^{\mathrm{S}} \tilde{\boldsymbol{\Lambda}}'\big)\Big) & \boldsymbol{0} \end{bmatrix} \tilde{f} h \, \mathrm{d}\xi^3, \end{aligned} \tag{64}$$

$$[\boldsymbol{D}_{\boldsymbol{\theta\mu}}] = \int_{\xi^{3-}}^{\xi^{3+}} \begin{bmatrix} \text{dual}\,(-\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \boldsymbol{k}'_\gamma \times \boldsymbol{\Psi}^*) & \text{dual}\,(\tilde{\boldsymbol{H}}'^{\mathrm{T}} \boldsymbol{\Psi}^*) \\ \text{primal}\,(-\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \boldsymbol{k}'_\gamma \times \boldsymbol{\Psi}^*) & \text{primal}\,(\tilde{\boldsymbol{H}}'^{\mathrm{T}} \boldsymbol{\Psi}^*) \end{bmatrix} \begin{bmatrix} (\xi^3 h^{-2} \tilde{\boldsymbol{F}}^{-1} \boldsymbol{g}^{*\gamma}\times)^{\mathrm{T}} \\ (\xi^3 h^{-2} \tilde{\boldsymbol{F}}^{-1} \boldsymbol{g}^{*3}\times)^{\mathrm{T}} \end{bmatrix} \tilde{f} h \, \mathrm{d}\xi^3,$$

$$[\boldsymbol{D}_{\boldsymbol{\theta\tau}}] = \int_{\xi^{3-}}^{\xi^{3+}} \begin{bmatrix} \text{dual}\,(-\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \boldsymbol{k}'_\gamma \times \boldsymbol{\Psi}^*) & \text{dual}\,(\tilde{\boldsymbol{H}}'^{\mathrm{T}} \boldsymbol{\Psi}^*) \\ \text{primal}\,(-\xi^3 \tilde{\boldsymbol{\Lambda}}'^{\mathrm{T}} \boldsymbol{k}'_\gamma \times \boldsymbol{\Psi}^*) & \text{primal}\,(\tilde{\boldsymbol{H}}'^{\mathrm{T}} \boldsymbol{\Psi}^*) \end{bmatrix} \begin{bmatrix} (h^{-1} \tilde{\boldsymbol{F}}^{-1} \boldsymbol{g}^{*\gamma}\times)^{\mathrm{T}} \\ (h^{-1} \tilde{\boldsymbol{F}}^{-1} \boldsymbol{g}^{*3}\times)^{\mathrm{T}} \end{bmatrix} \tilde{f} h \, \mathrm{d}\xi^3,$$

where $\acute{\boldsymbol{S}}^*$ must be understood as a short notation for $\boldsymbol{X}'^* \acute{\boldsymbol{T}}^* = \acute{\boldsymbol{T}}^* + \varepsilon \boldsymbol{x}'^* \times \acute{\boldsymbol{T}}^*$, the pole-based dual version of the first Piola–Kirchhoff stress tensor.

Equation (63) collects the vectors work-conjugate to the material surface variation variables in the virtual functional $\pi_{\mathrm{int}\delta}$, whereas (64) gives the tensors of the relevant tangent map in $\partial\pi_{\mathrm{int}\delta}$. Note that the tangent map is symmetrical, see $(62)_2$. Vectors and tensors in (63) and (64) are integrals along ξ^3 of quantities depending ultimately on the current values of the kinematical variables and Biot-axial parameters of the material surface. Thus, the internal-work incremental virtual functional $\pi_{\mathrm{int}\delta} + \partial\pi_{\mathrm{int}\delta}$ becomes two-dimensional: it depends linearly on the virtual, incremental, and mixed virtual-incremental variations of the surface strain component vectors $\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha$, the dual director $\boldsymbol{\theta}'$, and the Biot-axial parameters $\hat{\boldsymbol{\tau}}$ and $\hat{\boldsymbol{\mu}}$. Note that the back-rototranslated dual strains $\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha$ are the pole-based version of the back-rotated self-based dual strains $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{X}'^{\mathrm{T}}\boldsymbol{\omega}_\alpha$.

6.2. ***Shell constitutive equations.*** As stated in Section 4.2, the dual director kinematic field is a gradientless, piecewise constant field on the shell surface. In a finite element context, $\boldsymbol{\theta}$ should be understood as a local variable, and as such, it would be profitably considered for a local condensation process. However, it can be observed that $\boldsymbol{\theta}$ alone cannot be condensed. In fact, the rototranslation generating the shell model, $\tilde{\boldsymbol{H}} = \exp(\xi^3(\boldsymbol{A\theta})\times)$, is endowed ultimately with the contribution of three linear and three angular components. When the shell deforms, the change of the linear part produces strains, which are withstood elastically; the change of the angular part, instead, is a rotation that must fulfill the kinematical constraint (38) and entails a workless Biot-axial field as a multiplier. So, $\boldsymbol{\theta}$ alone is three times indeterminate, and needs three further parameters to be condensed with: the Biot-axial parameter vector $\hat{\boldsymbol{\mu}}$, which controls that part of $\hat{\boldsymbol{\tau}}^*$ variable across the shell thickness, appears a likely candidate to make $\boldsymbol{\theta}$ determinate. As a matter of fact, after a systematic series of numerical tests with several combinations of the components of $\hat{\boldsymbol{\tau}}$ and $\hat{\boldsymbol{\mu}}$, it was confirmed that the set composed of $\boldsymbol{\theta}'$ and the vector $\hat{\boldsymbol{\mu}}$ could be safely solved locally. This choice leaves $\boldsymbol{\omega}_\alpha$ and $\hat{\boldsymbol{\tau}}$ as the *basic parameters* governing the internal work virtual functional of the material surface — a picture consistent with the case of three-dimensional elasticity outlined in Section 5.1.

The condensation process traces the steps carried out in Section 5.4 for the local constitutive law (refer to [Merlini and Morandini 2008] for further details). Starting from (62), the internal-work incremental virtual functional $\pi_{\mathrm{int}\delta} + \partial\pi_{\mathrm{int}\delta}$ is written; since $\boldsymbol{\theta}'$ and $\hat{\boldsymbol{\mu}}$ are local, independent variables, the terms with $\partial\delta\boldsymbol{\theta}'$ and $\partial\delta\hat{\boldsymbol{\mu}}$ are of course omitted, whereas the terms in $\delta\boldsymbol{\theta}'$ and $\delta\hat{\boldsymbol{\mu}}$ keep disjoint from the overall principle and originate the local incremental equation

$$\begin{Bmatrix} \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{R}_\theta \\ \boldsymbol{R}_\mu \end{Bmatrix} + \begin{bmatrix} \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{D}^{\beta\mathrm{T}}_{\omega\theta}\boldsymbol{H} & \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{D}_{\theta\tau} \\ \boldsymbol{D}^{\beta\mathrm{T}}_{\omega\mu}\boldsymbol{H} & \boldsymbol{0} \end{bmatrix} \cdot \begin{Bmatrix} \partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\beta) \\ \partial\hat{\boldsymbol{\tau}} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{D}_{\theta\theta}\boldsymbol{A}' & \boldsymbol{A}'^{\mathrm{T}}\boldsymbol{D}_{\theta\mu} \\ \boldsymbol{D}^{\mathrm{T}}_{\theta\mu}\boldsymbol{A}' & \boldsymbol{0} \end{bmatrix} \cdot \begin{Bmatrix} \partial\boldsymbol{\theta}' \\ \partial\hat{\boldsymbol{\mu}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{Bmatrix}. \tag{65}$$

Solving (65) for $\partial\boldsymbol{\theta}'$ and $\partial\hat{\boldsymbol{\mu}}$ and substituting the result in the expression of $\pi_{\mathrm{int}\delta} + \partial\pi_{\mathrm{int}\delta}$ yields the condensed form

$$\pi_{\mathrm{int}\delta} + \partial\pi_{\mathrm{int}\delta} = \begin{Bmatrix} \delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \\ \delta\hat{\boldsymbol{\tau}} \end{Bmatrix}^{\mathrm{T}} \cdot \left(\begin{Bmatrix} \boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{R}}^\alpha_\omega \\ \bar{\boldsymbol{R}}_\tau \end{Bmatrix} + \begin{bmatrix} \boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{D}}^{\alpha\beta}_{\omega\omega}\boldsymbol{H} & \boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{D}}^{\alpha}_{\omega\tau} \\ \bar{\boldsymbol{D}}^{\beta\mathrm{T}}_{\omega\tau}\boldsymbol{H} & \bar{\boldsymbol{D}}_{\tau\tau} \end{bmatrix} \cdot \begin{Bmatrix} \partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\beta) \\ \partial\hat{\boldsymbol{\tau}} \end{Bmatrix} \right) + \begin{Bmatrix} \partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \\ \partial\delta\hat{\boldsymbol{\tau}} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{H}^{\mathrm{T}}\boldsymbol{R}^\alpha_\omega \\ \boldsymbol{R}_\tau \end{Bmatrix},$$

where

$$\begin{aligned} \begin{Bmatrix} \bar{\boldsymbol{R}}^\alpha_\omega \\ \bar{\boldsymbol{R}}_\tau \end{Bmatrix} &= \begin{Bmatrix} \boldsymbol{R}^\alpha_\omega \\ \boldsymbol{R}_\tau \end{Bmatrix} - \begin{bmatrix} \boldsymbol{D}^\alpha_{\omega\theta} & \boldsymbol{D}^\alpha_{\omega\mu} \\ \boldsymbol{D}^{\mathrm{T}}_{\theta\tau} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_{\theta\theta} & \boldsymbol{D}_{\theta\mu} \\ \boldsymbol{D}^{\mathrm{T}}_{\theta\mu} & \boldsymbol{0} \end{bmatrix}^{-1} \begin{Bmatrix} \boldsymbol{R}_\theta \\ \boldsymbol{R}_\mu \end{Bmatrix}, \\ \begin{bmatrix} \bar{\boldsymbol{D}}^{\alpha\beta}_{\omega\omega} & \bar{\boldsymbol{D}}^\alpha_{\omega\tau} \\ \bar{\boldsymbol{D}}^{\beta\mathrm{T}}_{\omega\tau} & \bar{\boldsymbol{D}}_{\tau\tau} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{D}^{\alpha\beta}_{\omega\omega} & \boldsymbol{D}^\alpha_{\omega\tau} \\ \boldsymbol{D}^{\beta\mathrm{T}}_{\omega\tau} & \boldsymbol{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{D}^\alpha_{\omega\theta} & \boldsymbol{D}^\alpha_{\omega\mu} \\ \boldsymbol{D}^{\mathrm{T}}_{\theta\tau} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_{\theta\theta} & \boldsymbol{D}_{\theta\mu} \\ \boldsymbol{D}^{\mathrm{T}}_{\theta\mu} & \boldsymbol{0} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{D}^{\beta\mathrm{T}}_{\omega\theta} & \boldsymbol{D}_{\theta\tau} \\ \boldsymbol{D}^{\beta\mathrm{T}}_{\omega\mu} & \boldsymbol{0} \end{bmatrix}, \end{aligned} \tag{66}$$

are the *reduced* work-conjugate parameters and tangent map. Resorting to the same arguments as in Section 5.4, the terms $\boldsymbol{R}^\alpha_\omega$ and $\boldsymbol{R}_\tau$ in the expression of $\pi_{\mathrm{int}\delta} + \partial\pi_{\mathrm{int}\delta}$ are corrected for $\bar{\boldsymbol{R}}^\alpha_\omega$ and $\bar{\boldsymbol{R}}_\tau$ and the internal work virtual functional and virtual tangent functional are finally written as

$$\begin{aligned} \pi_{\mathrm{int}\delta} &= \begin{Bmatrix} \delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \\ \delta\hat{\boldsymbol{\tau}} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{R}}^\alpha_\omega \\ \bar{\boldsymbol{R}}_\tau \end{Bmatrix}, \\ \partial\pi_{\mathrm{int}\delta} &= \begin{Bmatrix} \delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \\ \delta\hat{\boldsymbol{\tau}} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} \boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{D}}^{\alpha\beta}_{\omega\omega}\boldsymbol{H} & \boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{D}}^{\alpha}_{\omega\tau} \\ \bar{\boldsymbol{D}}^{\beta\mathrm{T}}_{\omega\tau}\boldsymbol{H} & \bar{\boldsymbol{D}}_{\tau\tau} \end{bmatrix} \cdot \begin{Bmatrix} \partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\beta) \\ \partial\hat{\boldsymbol{\tau}} \end{Bmatrix} + \begin{Bmatrix} \partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \\ \partial\delta\hat{\boldsymbol{\tau}} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{R}}^\alpha_\omega \\ \bar{\boldsymbol{R}}_\tau \end{Bmatrix}. \end{aligned} \tag{67}$$

In (67), the elements of the incremental form of a nonlinear constitutive law of the shell material surface are recognized. $\boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{R}}^\alpha_\omega$ are the (generalized) *stress resultants* work-conjugate to the dual strains

$\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_{\alpha}$, and $\bar{\boldsymbol{R}}_{\boldsymbol{\tau}}$ is the *integral kinematical constraint* work-conjugate to the Biot-axial parameter $\hat{\boldsymbol{\tau}}$. So, (66)$_1$ represents the *shell constitutive equations*. Linearization of the shell constitutive equations yields the *tangent map*

$$\left\{\begin{matrix}\partial(\boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{R}}^{\alpha}_{\boldsymbol{\omega}})\\ \partial\bar{\boldsymbol{R}}_{\boldsymbol{\tau}}\end{matrix}\right\}=\begin{bmatrix}\boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{D}}^{\alpha\beta}_{\boldsymbol{\omega}\boldsymbol{\omega}}\boldsymbol{H} & \boldsymbol{H}^{\mathrm{T}}\bar{\boldsymbol{D}}^{\alpha}_{\boldsymbol{\omega}\boldsymbol{\tau}}\\ \bar{\boldsymbol{D}}^{\beta\mathrm{T}}_{\boldsymbol{\omega}\boldsymbol{\tau}}\boldsymbol{H} & \bar{\boldsymbol{D}}_{\boldsymbol{\tau}\boldsymbol{\tau}}\end{bmatrix}\cdot\left\{\begin{matrix}\partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_{\beta})\\ \partial\hat{\boldsymbol{\tau}}\end{matrix}\right\}, \tag{68}$$

giving the appropriate increments of the resultant vectors from the relevant increments of the surface parameters. The *mapping tensors* $\bar{\boldsymbol{D}}$, which represent the derivatives of the resultant vectors with respect to the surface parameters, are given in (66)$_2$: they comprise altogether every kind of contribution, specifically the elastic contribution, the kinematical constraint contribution, and the geometric contribution due to the stress state.

It is worth comparing (67) with the parent (45), where the interrelation among the primary mechanical variables in three-dimensional elasticity is clear. In order to highlight a particular difference, suppose that $\hat{\boldsymbol{\tau}}$ is a linear function of the free unknowns, so that the terms with $\partial\delta\hat{\boldsymbol{\tau}}^*$ and $\partial\delta\hat{\boldsymbol{\tau}}$ disappear. Equation (45)$_2$ clearly shows that the virtual and the incremental variations of $\hat{\boldsymbol{\tau}}^*$ are locally uncoupled and work directly for the opposite variations of dual $(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*)$ (that is, the incremental and the virtual variations, respectively). In contrast, in (67)$_2$ the variations of $\hat{\boldsymbol{\tau}}$ are coupled together and with the variations of $\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_{\alpha}$ through tensors $\bar{\boldsymbol{D}}_{\boldsymbol{\tau}\boldsymbol{\tau}}$ and $\bar{\boldsymbol{D}}^{\alpha}_{\boldsymbol{\omega}\boldsymbol{\tau}}$. Therefore, the term $\delta\hat{\boldsymbol{\tau}}\cdot\bar{\boldsymbol{R}}_{\boldsymbol{\tau}}$ in (67)$_1$ cannot correspond to a scalar product like $\langle\delta\hat{\boldsymbol{\tau}}\times, \mathrm{dual}\,(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega})\rangle$, as in (45)$_1$, and the virtual functional $\pi_{\mathrm{int}\delta}$ can be hardly given a constrained form as in (36). Rather, the incremental form of the nonlinear shell constitutive law in (67) couples together the surface mechanical variables $\boldsymbol{\omega}_{\alpha}$ and $\hat{\boldsymbol{\tau}}$, and their respective roles become confused. This fact makes it hard to derive the two-dimensional equations governing the material surface mechanics — should this be necessary — from the three-dimensional continuum, with distinct expressions for the balance conditions and for the kinematical constraint. However, though $\hat{\boldsymbol{\tau}}$ cannot be identified as the axial vector of a stress resultant, we keep on referring to $\hat{\boldsymbol{\tau}}$ as the surface Biot-axial stress parameter.

The shell incremental constitutive law is built dynamically during the solution process of the whole shell problem. Vectors and tensors $\bar{\boldsymbol{R}}$ and $\bar{\boldsymbol{D}}$ in (66) are computed from the integrals $\boldsymbol{R}$ and $\boldsymbol{D}$ in (63) and (64); the latter are nonlinear functions of both the surface fields $(\boldsymbol{\omega}_{\alpha}, \hat{\boldsymbol{\tau}})$ and the local variables $(\boldsymbol{\theta}', \hat{\boldsymbol{\mu}})$, and can be computed by a numerical integration across the shell thickness. At the end of each Newton–Raphson iteration, once the increments $\partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_{\beta})$ and $\partial\hat{\boldsymbol{\tau}}$ are known from the overall solution, the local increments $\partial\boldsymbol{\theta}'$ and $\partial\hat{\boldsymbol{\mu}}$ are recovered from (65), and the locally stored variables, which pertain to a Euclidean vector space, are updated additively, $\boldsymbol{\theta}'\leftarrow\boldsymbol{\theta}'+\partial\boldsymbol{\theta}'$ and $\hat{\boldsymbol{\mu}}\leftarrow\hat{\boldsymbol{\mu}}+\partial\hat{\boldsymbol{\mu}}$. At this stage, the coefficients in (65) must be consistent with those used early in (66), so they must be saved for the subsequent recovery. In a finite-element context, the local condensation would be carried out at each quadrature point of a shell element, where the shell director $\boldsymbol{\theta}'$ and the angular Biot-axial parameter $\hat{\boldsymbol{\mu}}$ can be stored as internal variables.

Though the proposed formulation may seem quite unusual in shell mechanics, some similarity with [Wisniewski and Turska 2000] can be observed, however. These authors too aim to release the rotation within the thickness from the reference surface rotation, in particular the drilling rotation, so to account for the in-plane twist. They use the skew-symmetric part of the Biot stress to force the constraint that defines such rotations, both within the thickness (in-plane twist) and on the reference surface (drilling rotation), and arrive at a two-dimensional formulation endowed with a surface Biot-axial stress parameter. The main difference with our formulation lies in the modeling of the continuum: the classical, uncoupled modeling adopted in [Wisniewski and Turska 2000] allows us to examine the physical meaning of every variable component, but it also induces several simplifying hypotheses; instead, the coupled helicoidal modeling used here is concise and stands on only few hypotheses.

7. Conclusion

Let's summarize some noteworthy features of the proposed shell theory.

Micropolar mechanics of the shell material surface. The mechanics of the material surface are based on a thorough micropolar description over a two-coordinate domain; the surface particles have full three-dimensional freedom, including the drilling rotation. The material surface, however, is like a nonpolar medium in its tangent plane, so the drilling rotation is actually a further DOF whose definition entails an extra stress field. In the present formulation, the surface density of the internal work functional comes as a function of the kinematical strain component vectors and a Biot-axial parameter vector. The role of the latter is evident, and this stress parameter must be retained, in our opinion, as a primary unknown field of the material surface mechanics, even in a displacement-based formulation. This strategy could be the answer to the issue raised by [Yu and Hodges 2004] in the closure, and should finally shed full light on the nature of drilling DOFs.

Hypotheses of the solid shell model. The solid shell model is based on the orientoposition of the parent material surface (six DOFs, five of which correspond to those of the so-called 5-parameter shell theory, while the last is the drilling DOF), on an orientoposition field across the thickness, and on a Biot-axial field across the thickness. The inherent approximation of this shell model is related to the hypotheses on which the last two fields are built. Only three hypotheses have been made. (1) The orientoposition field is determined by a *constant-curvature* rototranslation across the thickness; the curvature is governed by the shell director, a dual vector field on the surface (another six DOFs, out of which one corresponds to the sixth DOF of the so-called 6-parameter shell theory, while the others are five additional freedoms of the present theory). (2) The shell director is assumed to belong to a *piecewise constant*, gradientless field on the surface. (3) The Biot-axial field is assumed to be *linear* across the thickness, and is governed by two vector fields on the surface. No more assumptions are made. The helicoidal modeling and its consistent linearization allow us to deal with finite displacements, rotations, and strains of any magnitude, in an easy and natural way.

Computational approach to nonlinear shell mechanics. The proposed formulation is computation oriented in several respects. The mechanics of the solid shell are formulated in weak form, the appropriate variational principle is linearized from the beginning, and the relevant incremental form is made discrete across the thickness by means of an approximation of the finite-element kind. The integrals in the transverse direction are computed numerically at each point of interest on the reference surface. The ensuing two-dimensional problem is stated in a weak incremental form that inherits and exploits the property of geometric invariance of the helicoidal model. The extra parameters of the solid shell model (the shell director and the second Biot-axial parameter) are regarded as local variables and condensed statically. The surface strains and the first Biot-axial parameter are left as the vectorial entities work-conjugate to the stress resultants and the surface kinematical constraint. The latter entities are coupled together in an incremental form, which represents the dynamically built nonlinear constitutive law of the shell. A similar dynamical adaptation of the incremental form of the local constitutive law allows us to prevent Poisson locking related to the low-order kinematical model.

In shell mechanics, a very attractive and often-pursued approach is the so-called direct approach whose object body is the Cosserat surface. From the wide literature on this subject, we quote the early paper [Zhilin 1976], which is still modern in concepts and notation on rotational kinematics; the successful finite-element formulation [Sansour and Bednarczyk 1995]; and the comprehensive survey and deep study [Valid 1995]. When addressing direct approaches, it is worth focusing on the point that, contrary to beams, shells are hybrid structured solids: in fact, as remarked above, a shell can bend and torque but behaves in an essentially nonpolar way in its tangent plane. This particular feature must be carefully accounted for in the formulation of consistent material surface mechanics. Our line, in this respect, is as follows. As in three-dimensional solid mechanics a micropolar rotation unknown is allowed for even in nonpolar media by formulations based on strain and stress parameters of the Biot type and on the Biot-axial as a workless stress unknown, so in the material surface mechanics a full three-parametric rotation unknown can be allowed for by a consistent formulation that retains a surface Biot-axial parameter as a primary unknown. However, though the extension of the direct approach to encompass a surface Biot-axial unknown is feasible [Merlini 2008], its usefulness is questionable: in fact, the interpretation of the surface Biot-axial vector and its relation with the Biot-axial field within the solid give rise to new difficulties that add to the lack of constitutive laws, which is characteristic of direct approaches. In this paper we showed that a sound variational approach to the solid shell leads to the appropriate ingredients to set up the consistent nonlinear two-dimensional mechanics of the material surface. The methodology proposed in this paper is, in our opinion, the right way to tie a reasonably approximate model across the thickness with a full micropolar approach to the material surface mechanics in geometrically nonlinear problems. So, the proposed formulation overcomes the seemingly inconsistent feature of shells of being nonpolar in their tangent plane.

Appendix: Kinematical strain variations

In the development of the simple and mixed variations $\delta(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*)$ and $\partial\delta(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*)$ in (45), it is helpful to focus on the corototranslational variations $\boldsymbol{H}^*\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{\omega}^*)$ and $\boldsymbol{H}^*\partial\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{\omega}^*)$ of the model strain. In fact, the former variations can be always recovered from the latter ones, since for (44) $\boldsymbol{\Psi}^*\delta(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) = \boldsymbol{H}^*\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{\omega}^*)$ and $\boldsymbol{\Psi}^*\partial\delta(\boldsymbol{\Psi}^{*\mathrm{T}}\boldsymbol{\omega}^*) = \boldsymbol{H}^*\partial\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{\omega}^*)$. Using (32), (28), and (24), the following dyadic forms are written:

$$\boldsymbol{H}^*\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{\omega}^*) = \boldsymbol{H}^*\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{k}'_\alpha)\otimes\boldsymbol{g}^{*\alpha} + \boldsymbol{H}^*\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{k}'_3)\otimes\boldsymbol{g}^{*3},$$
$$\boldsymbol{H}^*\partial\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{\omega}^*) = \boldsymbol{H}^*\partial\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{k}'_\alpha)\otimes\boldsymbol{g}^{*\alpha} + \boldsymbol{H}^*\partial\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{k}'_3)\otimes\boldsymbol{g}^{*3}.$$

Recalling (33) and (18), and exploiting the rototranslation differentiation formulae [Merlini and Morandini 2004a]

$$\delta\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}} = \boldsymbol{\eta}_\delta\times, \quad \partial\delta\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}} = \boldsymbol{\eta}_{\partial\delta}\times + \tfrac{1}{2}(\boldsymbol{\eta}_\partial\times\boldsymbol{\eta}_\delta\times + \boldsymbol{\eta}_\delta\times\boldsymbol{\eta}_\partial\times),$$

and

$$\delta\tilde{\boldsymbol{H}}'\tilde{\boldsymbol{H}}'^{\mathrm{T}} = \tilde{\boldsymbol{\eta}}'_\delta\times, \quad \partial\delta\tilde{\boldsymbol{H}}'\tilde{\boldsymbol{H}}'^{\mathrm{T}} = \tilde{\boldsymbol{\eta}}'_{\partial\delta}\times + \tfrac{1}{2}(\tilde{\boldsymbol{\eta}}'_\partial\times\tilde{\boldsymbol{\eta}}'_\delta\times + \tilde{\boldsymbol{\eta}}'_\delta\times\tilde{\boldsymbol{\eta}}'_\partial\times),$$

the surface component vectors can be transformed to

$$
\begin{aligned}
&\boldsymbol{H}^*\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{k}'_\alpha) = \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) - (\tilde{\boldsymbol{\eta}}'_\delta + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\delta) \times \boldsymbol{k}'_\alpha,\\
&\boldsymbol{H}^*\partial\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{k}'_\alpha) = \boldsymbol{H}\partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) - (\tilde{\boldsymbol{\eta}}'_\partial + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\partial) \times \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) - (\tilde{\boldsymbol{\eta}}'_\delta + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\delta) \times \boldsymbol{H}\partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha)\\
&\qquad -\Big(\tilde{\boldsymbol{\eta}}'_{\partial\delta} + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_{\partial\delta} + \tfrac{1}{2}\big((\tilde{\boldsymbol{\eta}}'_\partial + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\partial) \times (\tilde{\boldsymbol{H}}' + \boldsymbol{I})\boldsymbol{\eta}_\delta + \boldsymbol{\eta}_\partial \times (\tilde{\boldsymbol{H}}' + \boldsymbol{I})\boldsymbol{\eta}_\delta\\
&\qquad\qquad + (\tilde{\boldsymbol{\eta}}'_\delta + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\delta) \times (\tilde{\boldsymbol{H}}' + \boldsymbol{I})\boldsymbol{\eta}_\partial + \boldsymbol{\eta}_\delta \times (\tilde{\boldsymbol{H}}' + \boldsymbol{I})\boldsymbol{\eta}_\partial\big)\Big) \times \boldsymbol{k}'_\alpha\\
&\qquad + \tfrac{1}{2}\big((\tilde{\boldsymbol{\eta}}'_\partial + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\partial) \times (\tilde{\boldsymbol{\eta}}'_\delta + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\delta) \times + (\tilde{\boldsymbol{\eta}}'_\delta + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\delta) \times (\tilde{\boldsymbol{\eta}}'_\partial + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\partial)\times\big)\boldsymbol{k}'_\alpha.
\end{aligned}
$$

The differential maps of the rototranslation $\tilde{\boldsymbol{H}}'$ through the thickness,

$$
\tilde{\boldsymbol{\eta}}'_\delta = \tilde{\boldsymbol{\Lambda}}'\delta\tilde{\boldsymbol{\eta}}', \qquad \tilde{\boldsymbol{\eta}}'_{\partial\delta} = \tilde{\boldsymbol{\Lambda}}'\partial\delta\tilde{\boldsymbol{\eta}}' + \tilde{\boldsymbol{\Lambda}}'^{\underline{123}}_{\mathrm{III}} : \delta\tilde{\boldsymbol{\eta}}' \otimes \partial\tilde{\boldsymbol{\eta}}',
$$

are now used to solve the differential helices $\tilde{\boldsymbol{\eta}}'_\delta$ and $\tilde{\boldsymbol{\eta}}'_{\partial\delta}$ for the variations of the helix $\tilde{\boldsymbol{\eta}}'$. The expressions of the mapping tensors $\tilde{\boldsymbol{\Lambda}}'(\tilde{\boldsymbol{\eta}}')$ and $\tilde{\boldsymbol{\Lambda}}'^{\underline{123}}_{\mathrm{III}}(\tilde{\boldsymbol{\eta}}')$ can be found in [Merlini and Morandini 2004b, Appendix B]. Using (26) and (34), the variations $\delta\tilde{\boldsymbol{\eta}}'$ and $\partial\delta\tilde{\boldsymbol{\eta}}'$ are in turn related to the differential helices $\boldsymbol{\eta}_\delta$ and $\boldsymbol{\eta}_{\partial\delta}$ of the surface rototranslation $\boldsymbol{H}$ and to the corototranslational variations of the strain $\boldsymbol{\omega}_3$,

$$
\begin{aligned}
&\delta\tilde{\boldsymbol{\eta}}' = \xi^3\boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3) - \tilde{\boldsymbol{\eta}}' \times \boldsymbol{\eta}_\delta,\\
&\partial\delta\tilde{\boldsymbol{\eta}}' = \xi^3\boldsymbol{H}\partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3) - \tilde{\boldsymbol{\eta}}' \times \boldsymbol{\eta}_{\partial\delta} + \xi^3\boldsymbol{\eta}_\partial \times \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3) + \xi^3\boldsymbol{\eta}_\delta \times \boldsymbol{H}\partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3) + \tfrac{1}{2}(\boldsymbol{\eta}_\partial \times \boldsymbol{\eta}_\delta \times + \boldsymbol{\eta}_\delta \times \boldsymbol{\eta}_\partial \times)\tilde{\boldsymbol{\eta}}'.
\end{aligned}
$$

Putting all together and using the identity $(\tilde{\boldsymbol{\Lambda}}'^{\underline{123}}_{\mathrm{III}}\tilde{\boldsymbol{\eta}}'\times)^{\mathrm{T}132} + \tilde{\boldsymbol{\Lambda}}'\boldsymbol{I}^\times - \boldsymbol{I}^\times\tilde{\boldsymbol{\Lambda}}' + \frac{1}{2}(\boldsymbol{I}^\times\tilde{\boldsymbol{\Lambda}}'\tilde{\boldsymbol{\eta}}'\times)^{\mathrm{T}132}\tilde{\boldsymbol{\Lambda}}' = \boldsymbol{0}$, where $\boldsymbol{I}^\times = \boldsymbol{g}_j \times \otimes \boldsymbol{g}^j$ is the third-order Ricci tensor, after several algebraic manipulations detailed in [Merlini and Morandini 2008], one obtains

$$
\begin{aligned}
&\tilde{\boldsymbol{\eta}}'_\delta + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\delta = \xi^3\tilde{\boldsymbol{\Lambda}}' \cdot \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3),\\
&\tilde{\boldsymbol{\eta}}'_{\partial\delta} + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_{\partial\delta} = \xi^3\tilde{\boldsymbol{\Lambda}}' \cdot \boldsymbol{H}\partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3) + (\xi^3)^2\tilde{\boldsymbol{\Lambda}}'^{\underline{123}}_{\mathrm{III}} : \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3) \otimes \boldsymbol{H}\partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3)\\
&\qquad - \tfrac{1}{2}\big((\tilde{\boldsymbol{\eta}}'_\partial + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\partial) \times (\tilde{\boldsymbol{H}}' + \boldsymbol{I})\boldsymbol{\eta}_\delta + \boldsymbol{\eta}_\partial \times (\tilde{\boldsymbol{H}}' + \boldsymbol{I})\boldsymbol{\eta}_\delta\\
&\qquad\qquad + (\tilde{\boldsymbol{\eta}}'_\delta + (\tilde{\boldsymbol{H}}' - \boldsymbol{I})\boldsymbol{\eta}_\delta) \times (\tilde{\boldsymbol{H}}' + \boldsymbol{I})\boldsymbol{\eta}_\partial + \boldsymbol{\eta}_\delta \times (\tilde{\boldsymbol{H}}' + \boldsymbol{I})\boldsymbol{\eta}_\partial\big).
\end{aligned}
$$

Therefore, the surface component vectors are finally brought to the form

$$
\begin{aligned}
&\boldsymbol{H}^*\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{k}'_\alpha) = \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) + \xi^3\boldsymbol{k}'_\alpha \times \tilde{\boldsymbol{\Lambda}}' \cdot \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3),\\
&\boldsymbol{H}^*\partial\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{k}'_\alpha) = \boldsymbol{H}\partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) + \xi^3\boldsymbol{k}'_\alpha \times \tilde{\boldsymbol{\Lambda}}' \cdot \boldsymbol{H}\partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3)\\
&\qquad - \xi^3\boldsymbol{I}^\times\tilde{\boldsymbol{\Lambda}}' : \big(\boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \otimes \boldsymbol{H}\partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3) + \boldsymbol{H}\partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_\alpha) \otimes \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3)\big)\\
&\qquad + (\xi^3)^2(\boldsymbol{k}'_\alpha \times \tilde{\boldsymbol{\Lambda}}'^{\underline{123}}_{\mathrm{III}} + (\boldsymbol{I}^\times\boldsymbol{k}'_\alpha \times \tilde{\boldsymbol{\Lambda}}')^{\mathrm{T}132}\tilde{\boldsymbol{\Lambda}}')^{\mathrm{S}\underline{123}} : \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3) \otimes \boldsymbol{H}\partial(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3).
\end{aligned}
$$

The development of the transverse component vectors is straightforward:

$$
\boldsymbol{H}^*\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{k}'_3) = \tilde{\boldsymbol{H}}' \cdot \boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3), \qquad \boldsymbol{H}^*\partial\delta(\boldsymbol{H}^{*\mathrm{T}}\boldsymbol{k}'_3) = \tilde{\boldsymbol{H}}' \cdot \boldsymbol{H}\partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3).
$$

At last, recalling (34), (16), and (26), it is seen that $\boldsymbol{H}\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3) = \boldsymbol{A}'\delta\boldsymbol{\theta}'$ and $\boldsymbol{H}\partial\delta(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{\omega}_3) = \boldsymbol{A}'\partial\delta\boldsymbol{\theta}'$, and recalling again (44), (60) is obtained.

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