Journal of
Mechanics of
Materials and Structures

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Volume 7, No. 2 February 2012

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 7, No. 2, 2012

FINITE ELEMENT ANALYSIS OF BENDING-STIFF COMPOSITE CONICAL SHELLS WITH MULTIPLE DELAMINATION

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This paper presents a finite element method to investigate the effects of multiple delaminations on the free-vibration characteristics of graphite-epoxy bending-stiff composite pretwisted shallow conical shells. (We call *bending-stiff* a laminate configuration having maximum stiffness for the spanwise first bending mode.) The generalized dynamic equilibrium equation is derived from Lagrange's equation of motion neglecting the Coriolis effect for moderate rotational speeds. An eight-noded isoparametric plate bending element is employed in the finite element formulation incorporating rotary inertia and the effects of transverse shear deformation based on Mindlin theory. A multipoint constraint algorithm is utilized to ensure the compatibility of deformation and equilibrium of resultant forces and moments at the delamination crack front. The standard eigenvalue problem is solved by applying the QR iteration algorithm. Finite element codes are developed to obtain numerical results concerning the effects of twist angle and rotational speed on the natural frequencies of multiple delaminated bending-stiff composite conical shells. The mode shapes are also depicted for a typical laminate configuration. The numerical results obtained for comparison of single and multiple delaminated bending-stiff composite laminates are the first known nondimensional natural frequencies under the combined effect of rotation and twist for the type of analyses carried out here.

1. Introduction

Composite structures are extensively used in aerospace, automobile, civil, and other various high performance applications. Composite materials are immensely popular in weight-sensitive applications because of their high specific stiffness, low weight, and high strength-to-weight ratio, but one of the major causes of failure in fiber-reinforced laminate composites is the delamination resulting from interlaminar debonding of constituent laminae. Pretwisted composite conical shells with low aspect ratios can be idealized as turbomachinery blades. Prior knowledge of the resonant characteristics of turbomachinery blades is of utmost importance in ensuring a reliably long life for turbine engines. The presence of invisible delamination in such a structural element made of composite materials can be detected with the help of prior knowledge of the natural frequencies if delamination exists. Moreover, the initial stress system in a rotating shell due to centrifugal body forces has appreciable cascading affects on the natural frequency. The vibratory characteristics are thus of critical influence on the performance and safety of such composite structures. In realistic situations, pretwisted conical shell structures have geometrical complexities arising due to their specific applications in various service environments. Hence certain dynamic parameters are to be considered when these structural elements are in rotation, and the finite

Keywords: delamination, finite element, vibration, conical shell, bending-stiff.

element method is an efficient tool for the dynamic analysis of such types of applications. Multiple delaminated composite laminated structures exhibit new vibration frequencies depending on the size and location of the delamination. In order to ensure safety of operation, a profound understanding of the dynamic characteristics of composite pretwisted conical shells is essential for designers.

The first established work on pretwisted composite plates [Qatu and Leissa 1991b] determined the natural frequencies of stationary plates using laminated shallow shell theory and the Ritz method. Liew et al. [1994] investigated a pretwisted conical shell to find out the vibratory characteristics of a stationary conical shell by using the Ritz procedure. By using the same method, the first known three-dimensional continuum vibration analysis including full geometric nonlinearities and centrifugal accelerations in composite blades was carried out in [McGee and Chu 1994].

The two important investigations on delamination model were by Shen and Grady [1992] and Krawczuk et al. [1997]. The first dealt with the analytical and experimental determination of natural frequencies of delaminated composite beam while the second one undertook a finite element free vibration analysis of the delaminated composite cantilever beam and plate. Rebière and Gamby [2004] employed a variational approach to model the behavior of composite cross-ply laminates damaged by transverse and longitudinal cracking and delamination, while Aydogdu and Timarci [2003] and Tripathi et al. [2007] studied the free-vibration behavior of a delaminated composite employing the finite element method. Lee et al. [2002] carried out the vibration analysis of a twisted cantilevered conical composite shell using a finite element method based on the Hellinger–Reissner principle, again for single and multiple delaminations. Parhi et al. [2001] and Aymerich et al. [2009] have demonstrated the effects of multiple delamination on laminated composites using FEM. The first dealt with failure analysis of a composite plate due to bending and impact, while the second simulated cross-ply laminates subject to impact based on cohesive interface elements.

As far as the authors are aware, there is no work available in the literature which deals with rotating multiple delaminated composite pretwisted cantilever conical shells by a finite element method considering the combined effect of rotation and twist on the vibration characteristics of the *bending-stiff* configuration $[0^\circ{}_2/\pm30^\circ]$s [Crawley 1979]. (We call *bending-stiff* a laminate configuration having maximum stiffness for the spanwise first bending mode in terms of design compliance.) Turbomachinery blades may flutter due to high-speed rotation which may lead to fouling of the blades in the cantilevered arrangements. Although the free ends of the blades are restrained by lacing wire, in resonant condition, excessive vibration may lead to severe damage of the vibratory blades. This can be prevented to a great extent provided the blades have high stiffness against spanwise bending. In this paper, the multipoint constraint algorithm [Gim 1994] is incorporated which leads to asymmetric element stiffness matrices. The QR iteration algorithm [Bathe 1990] is utilized to solve the standard eigenvalue problem. The present analysis is aimed at obtaining the nondimensional fundamental frequency (NDFF) and nondimensional second natural frequencies (NDSF) of pretwisted bending-stiff composite shallow conical shells having delamination without taking care of the effect of dynamic contact between delaminated layers.

In the present study, the shell surface is considered as a shallow conical shell with length L, reference width b_0, thickness h, vertex angle θ_v, and base subtended angle of cone θ_0 as depicted in Figure 1. Since the conical shell is shallow, it may be assumed that the cross section is elliptical. The component of radius of curvature in the chord-wise direction $R_y(x, y)$ is a parameter varying both in the x and y-directions. The variation in the x-direction is linear. There is no curvature along the spanwise direction ($R_x = \infty$).

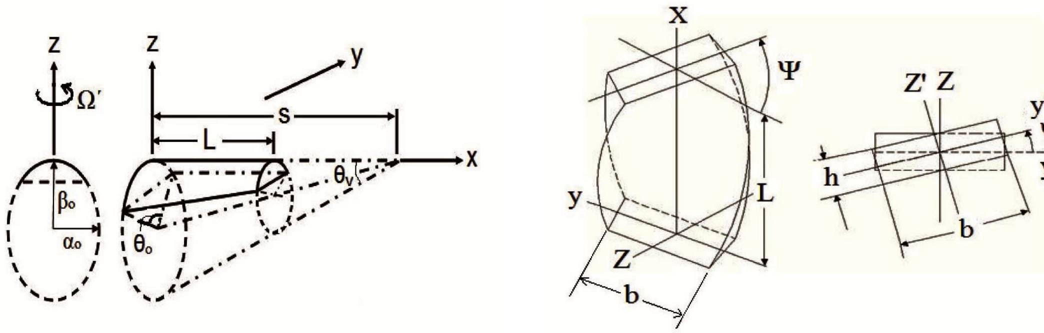

Figure 1. Geometry of an untwisted conical shell (left) and a twisted plate (right).

The cantilever shell, clamped along $x = 0$, is pretwisted with radius of twist R_{xy} as shown in the figure. Thus a pretwisted shallow conical shell of uniform thickness, made of laminated composite is considered.

2. Mathematical formulation

A shallow shell is characterized by its middle surface which is defined by the equation [Leissa et al. 1984]

$$z = -\tfrac{1}{2}\left[\frac{x^2}{R_x} + 2\frac{xy}{R_{xy}} + \frac{y^2}{R_y}\right], \tag{1}$$

where R_x and R_y denote the radii of curvature in the x and y-directions, respectively. The radius of twist (R_{xy}), length (L) of shell, and twist angle (ψ) are related as

$$\tan\psi = -\frac{L}{R_{xy}}. \tag{2}$$

The dynamic equilibrium equation for moderate rotational speeds neglecting the Coriolis effect is derived employing Lagrange's equation of motion. The equation in global form is expressed as [Karmakar and Sinha 2001]

$$[M]\{\ddot{\delta}\} + ([K] + [K_\sigma])\{\delta\} = \{F(\Omega^2)\}, \tag{3}$$

where $[M]$, $[K]$, and $[K_\sigma]$ are the global mass, elastic stiffness, and geometric stiffness matrices, respectively. $\{F(\Omega^2)\}$ is the nodal equivalent centrifugal force and $\{\delta\}$ is the global displacement vector. $[K_\sigma]$ depends on the initial stress distribution and is obtained by the iterative procedure upon solving

$$([K] + [K_\sigma])\{\delta\} = \{F(\Omega^2)\}. \tag{4}$$

The natural frequencies (ω_n) are determined from the standard eigenvalue problem [Bathe 1990], which is represented below and is solved by the QR iteration algorithm:

$$[A]\{\delta\} = \lambda\{\delta\}, \tag{5}$$

where

$$[A] = ([K] + [K_\sigma])^{-1}[M] \quad \text{and} \quad \lambda = 1/\omega_n^2. \tag{6}$$

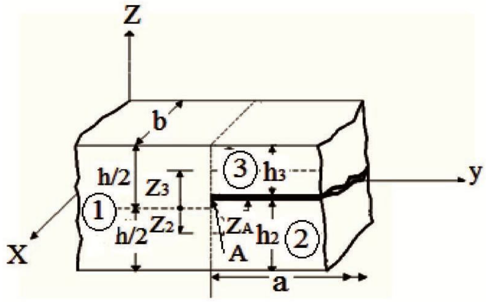

Figure 2. Plate elements at a delamination crack tip.

3. Multipoint constraint

Figure 2 represents the cross-sectional view of a typical delamination crack tip [Gim 1994] where nodes of three plate elements meet together to form a common node. The undelaminated region is modeled by plate element 1 of thickness h, and the delaminated region is modeled by plate elements 2 and 3 whose interface contains the delamination (h_2 and h_3 are the thicknesses of elements 2 and 3, respectively). Elements 1, 2, and 3 are freely allowed to deform prior to imposition of the constraint conditions. The plate elements at a delamination crack front are shown in Figure 3. The nodal displacements of elements 2 and 3 at the crack tip are expressed as

$$U_j = U'_j - (Z - Z'_j)\theta_{xj}, \quad V_j = V'_j - (Z - Z'_j)\theta_{yj}, \quad W_j = W'_j \quad (j = 2, 3), \tag{7}$$

where U'_j, V'_j, and W'_j are the midplane displacements, Z'_j is the z-coordinate of the midplane of element j, and θ_x and θ_y are the rotations about x and y-axes, respectively. The above equation also holds good for element 1 and Z'_1 equal to zero. The transverse displacements and rotations at a common node have values expressed as

$$W_1 = W_2 = W_3 = W, \quad \theta_{x1} = \theta_{x2} = \theta_{x3} = \theta_x, \quad \theta_{y1} = \theta_{y2} = \theta_{y3} = \theta_y. \tag{8}$$

(See [Gim 1994] for Equations (8)–(13).) The in-plane displacements of all three elements at the crack tip are equal and they are related as

$$U'_2 = U'_1 - Z'_2\theta_x, \quad \text{and} \quad V'_2 = V'_1 - Z'_2\theta_y, \tag{9}$$

$$U'_3 = U'_1 - Z'_3\theta_x, \quad \text{and} \quad V'_3 = V'_1 - Z'_3\theta_y, \tag{10}$$

where U'_1 is the midplane displacement of element 1. Equations (8)–(10) relating the nodal displacements and rotations of elements 1, 2, and 3 at the delamination crack tip are the multipoint constraint equations used in the finite element formulation to satisfy the compatibility of the displacements and rotations. The midplane strains between elements 2 and 3 are related as

$$\{\epsilon'\}_j = \{\epsilon'\}_j + Z'_j\{k\}, \tag{11}$$

where $\{\epsilon'\}$ represents the strain vector and $\{k\}$ is the curvature vector, which is identical at the crack tip for elements 1, 2, and 3. This equation can be considered as a special case for element 1 and Z'_1 equal

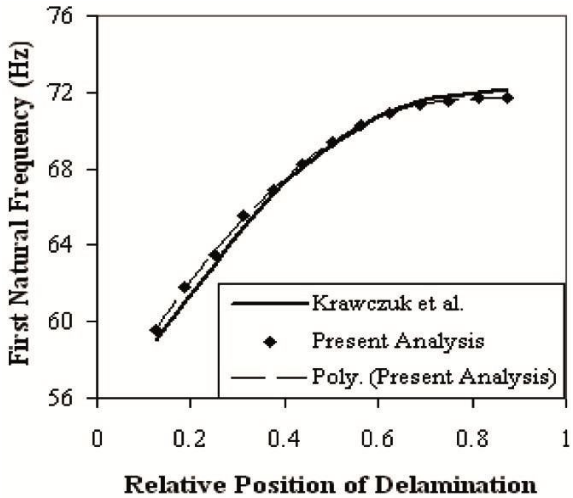

Figure 3. Influence of the relative position of delamination on the first natural frequency of the composite cantilever beam [Krawczuk et al. 1997].

to zero. The in-plane stress-resultants, $\{N\}$, and moment resultants, $\{M\}$, of elements 2 and 3 can be expressed as

$$\{N\}_j = [A]_j\{\epsilon'\}_1 + (Z'_j[A]_j + [B]_j)\{k\}, \tag{12}$$

$$\{M\}_j = [B]_j\{\epsilon'\}_1 + (Z'_j[B]_j + [D]_j)\{k\}. \tag{13}$$

The resultant forces and moments at the delamination front for elements 1, 2, and 3 satisfy the following equilibrium conditions:

$$\{N\} = \{N\}_1 = \{N\}_2 + \{N\}_3, \tag{14}$$

$$\{M\} = \{M\}_1 = \{M\}_2 + \{M\}_3 + Z'_2\{N\}_2 + Z'_3\{N\}_3, \tag{15}$$

$$\{Q\} = \{Q\}_1 = \{Q\}_2 + \{Q\}_3, \tag{16}$$

where $\{Q\}$ denotes the transverse shear resultants.

In the finite element analysis the structure has to be discretized into a number of elements connected at the nodal points. The element shall be such that it can properly define the behavior of the structure. In the present analysis an eight-noded quadratic isoparametric element with five degrees of freedom at each node (three translational and two rotational) is employed. The quadrilateral element has four corner nodes and four midside nodes. The isoparametric plate bending element shall be oriented in the natural coordinate system wherein the shape functions are as follows [Bathe 1990]:

$$N_i = (1+\epsilon\epsilon_i)(1+\eta\eta_i)(\epsilon\epsilon_i+\eta\eta_i-1)/4 \qquad \text{(for } i = 1, \dots, 4\text{)}, \tag{17}$$

$$N_i = (1-\epsilon^2)(1+\eta\eta_i)/2 \qquad \text{(for } i = 5, 6\text{)}, \tag{18}$$

$$N_i = (1-\eta^2)(1+\epsilon\epsilon_i)/2 \qquad \text{(for } i = 6, 8\text{)}, \tag{19}$$

where η and ϵ are local natural coordinates of the element.

ψ	Aspect ratio (L/s)	Present FEM (8×8)	Present FEM (6×6)	[Liew et al. 1994]
	0.6	0.3524	0.3552	0.3599
0°	0.7	0.2991	0.3013	0.3060
	0.8	0.2715	0.2741	0.2783
	0.6	0.2805	0.2834	0.2882
30°	0.7	0.2507	0.2528	0.2575
	0.8	0.2364	0.2389	0.2417

Table 1. Convergence study for NDFF ($\omega = \omega_n b_0^2 \sqrt{(\rho h/D)}$, where $D = Eh^3/12(1-\nu^2)$) of the pretwisted shallow conical shells, considering $\nu = 0.3$, $s/h = 1000$, $\theta_v = 15°$, and $\theta_0 = 30°$.

4. Results and discussion

The nondimensional fundamental frequencies (NDFF) and nondimensional second natural frequencies (NDSF) for conical shells having a curvature ratio (b_0/R_y) of 0.5 and a thickness ratio (s/h) of 1000 are obtained corresponding to different nondimensional speeds of rotation, $\Omega = 0.0$, 0.5, and 1.0 (where $\Omega = \Omega'/\omega_0$), with relative distance $d/L = 0.5$; the parameters n_d, n, a, Ω', ω_0, ρ, and d represent the number of delaminations, number of layers, crack length, actual angular speed of rotation, fundamental natural frequency of a nonrotating shell, and density, respectively. The material properties of the graphite-epoxy composite [Qatu and Leissa 1991a] are considered as $E_1 = 138.0\,\text{GPa}$, $E_2 = 8.96\,\text{GPa}$, $\nu = 0.3$, $G_{12} = 7.1\,\text{GPa}$, $G_{13} = 7.1\,\text{GPa}$, and $G_{23} = 2.84\,\text{GPa}$. Convergence studies are also performed to determine the converged mesh size (Table 1). It is observed from the convergence study that uniform mesh divisions of 6×6 and 8×8 considering the complete planform of the shell provide nearly equal results, the difference being around one percent; the results also corroborate monotonic downward convergence. The slight differences between the values of the present solution and those of [Liew et al. 1994] can be attributed to consideration of transverse shear deformation and rotary inertia in the present FEM and also to the fact that the Ritz method always overestimates the structural stiffness. Moreover, increasing the size of the matrix because of higher mesh size increases the ill-conditioning of the numerical eigenvalue problem [Qatu and Leissa 1991b]. Hence, the lower mesh size (6×6), consisting of 36 elements and 133 nodes, has been used for the analysis due to computational efficiency. The total number of degrees of freedom involved in the computation is 665 as each node of the isoparametric plate bending element has five degrees of freedom, three translational and two rotational.

4.1. ***Validation of results.*** Computer codes were developed based on the present finite element method. The numerical results obtained are compared and validated with the results of [Liew et al. 1994; Krawczuk et al. 1997; Karmakar et al. 2005] as furnished in Table 1, Figure 3, and Table 2, respectively. The comparative study shows excellent agreement with the previously published results and hence demonstrates the capability of the code developed and proves the accuracy of the analyses.

4.2. ***Effect of stacking sequence and twist angle.*** A parametric study is conducted to obtain the nondimensional natural frequencies of eight-layered graphite-epoxy bending-stiff composite shallow conical

	Two delaminations		Three delaminations	
Ω	Present FEM	[Karmakar et al. 2005]	Present FEM	[Karmakar et al. 2005]
0.0	1.9316	1.9332	1.9048	1.8994
0.5	2.1080	2.1921	2.0469	2.1495
1.0	2.6735	2.7475	2.5996	2.6874

Table 2. NDFF ($\omega = \omega_n L^2 \sqrt{(\rho/E_1 h^2)}$) of eight-layered graphite-epoxy composite [0°/0°/30°/−30°]s rotating cylindrical shells with 25% delaminations located at several positions (for 2 delaminations: 0°/0°/30°//−30°/−30°/30°/0°//0° and for 3 delaminations: 0°//0°/30°/−30°//−30°/30°//0°/0°, where // indicates the location of delamination) across the thickness, $a/b = 1$, $b/h = 100$, and $b_0/R_y = 0.5$.

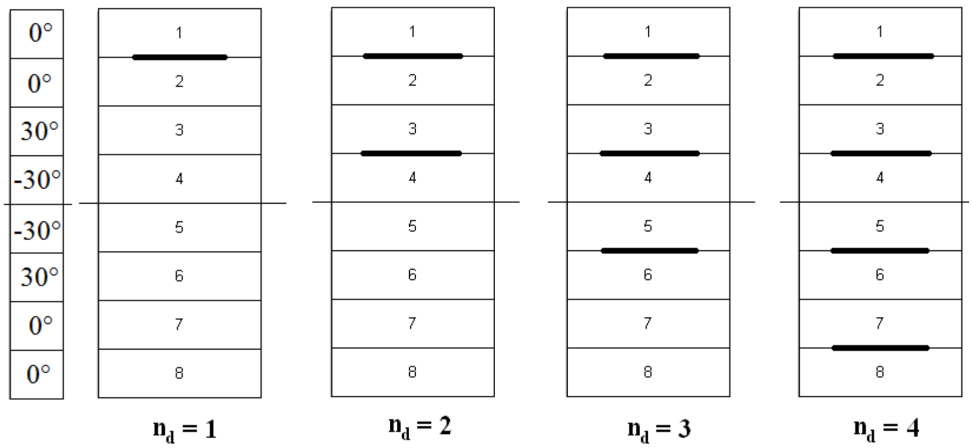

Figure 4. Arrangement of layers of eight-layered laminated composite with delamination.

shells under single, double, triple, and quadruple delaminations with different twist angles, as furnished in Table 3. The arrangement of layers with delamination is shown in Figure 4. At a stationary condition for a particular number of delaminations, nondimensional fundamental natural frequencies are identified to attain a maximum value for twist angle $\psi = 0°$ and gradually decrease to a minimum value for twist angle $\psi = 45°$. It is also noted that at a stationary condition for a particular angle of twist, the nondimensional fundamental frequencies are found to reduce with the increase of the number of delamination. This can be attributed to the fact that the delamination leads to reduction in the elastic stiffness. The centrifugal stiffening effect (that is, the increase of structural stiffness with increase of rotational speed) is predominantly found with reference to nondimensional fundamental and second natural frequencies of bending-stiff composites irrespective of twist angle. Under a rotating condition, the nondimensional fundamental frequencies of delaminated composite laminates found a drooping trend with an increase of twist angle.

4.3. ***Effect of relative frequency.*** The trends of the relative frequencies (the ratio of rotating natural frequency and stationary natural frequency) at $\Omega = 0.5$ and $\Omega = 1.0$ for a bending stiff configuration corresponding to NDFF are furnished in Figures 5a and 5b, respectively. The percentage difference

n_d	ψ	NDFF $\Omega = 0.0$	NDFF $\Omega = 0.5$	NDFF $\Omega = 1.0$	NDSF $\Omega = 0.0$	NDSF $\Omega = 0.5$	NDSF $\Omega = 1.0$
1	0°	0.3925	0.5231	0.6015	0.4953	0.6281	0.6279
	15°	0.2378	0.4552	0.5584	0.4946	0.5669	0.5644
	30°	0.1434	0.2617	0.4863	0.4935	0.5042	0.4983
	45°	0.0947	0.2056	0.3641	0.4932	0.5041	0.4550
2	0°	0.3190	0.5130	0.5529	0.5337	0.7227	0.6175
	15°	0.2092	0.3884	0.4353	0.4677	0.6555	0.6982
	30°	0.1295	0.1457	0.1401	0.4197	0.5249	0.5731
	45°	0.0882	0.1505	0.2388	0.4196	0.4867	0.5289
3	0°	0.2449	0.4967	0.4430	0.3825	0.5551	0.8587
	15°	0.1831	0.3316	0.4004	0.3339	0.3729	0.4402
	30°	0.1174	0.0664	0.4125	0.3017	0.2992	0.6361
	45°	0.0828	0.1134	0.2469	0.2965	0.5030	0.4840
4	0°	0.2565	0.4880	0.4075	0.2727	0.4670	0.6622
	15°	0.1841	0.3235	0.3667	0.2343	0.4136	0.5655
	30°	0.1162	0.2815	0.2168	0.2087	0.3711	0.6706
	45°	0.0816	0.0983	0.4494	0.2010	0.3549	0.7109

Table 3. NDFF and NDSF ($\omega = \omega_n L^2 \sqrt{(\rho/E_1 h^2)}$) of delaminated bending-stiff composite conical shells for various twist angles, considering $n = 8$, $h = 0.0004$, $s/h = 1000$, $a/L = 0.33$, $d/L = 0.5$, $L/s = 0.7$, $\theta_0 = 45°$, and $\theta_v = 20°$.

between the maximum and minimum relative frequencies with respect to NDFF at lower rotational speeds are found as 38.6%, 39.4%, 72.1%, and 50.3% corresponding to $n_d = 1$, 2, 3, and 4, respectively. On the other hand, the same at higher rotational speeds are found as 60.1%, 60.0%, 48.5% and 71.2% corresponding to $n_d = 1$, 2, 3, and 4, respectively. Hence this also proves the fact that for higher rotational speeds, the relative frequencies have a pronounced effect. Considering the twist of the laminate at lower rotational speeds, the percentage difference between the maximum and minimum relative frequencies with respect to NDFF are found as 34.3%, 8.2%, 76.7%, and 44.5% corresponding to $\psi = 0°$, 15°, 30°, and 45°, respectively. On the other hand, the same at higher rotational speeds are found as 15.3%, 15.2%, 69.2%, and 50.8% corresponding to $\psi = 0°$, 15°, 30°, and 45°, respectively. Hence it is observed that under a rotating condition, the relative frequencies (NDFF) have a pronounced effect corresponding to twist angle $\psi = 30°$.

In contrast, at lower rotational speeds, the percentage difference between the maximum and minimum relative frequencies with respect to NDSF are found as 19.4%, 17.2%, 41.5%, and 3.7% corresponding to $n_d = 1$, 2, 3, and 4, respectively, while the same at higher rotational speeds are found as 15.3%, 15.2%, 69.2%, and 50.8% corresponding to $n_d = 1$, 2, 3, and 4, respectively. On the other hand, the percentage difference between the maximum and minimum relative frequencies with respect to NDSF are found as 25.9%, 36.7%, 44.2%, and 42.1% corresponding to $\psi = 0°$, 15°, 30°, and 45°, respectively, while the same at higher rotational speeds are obtained as 52.4%, 52.7%, 68.6%, and 73.9% corresponding

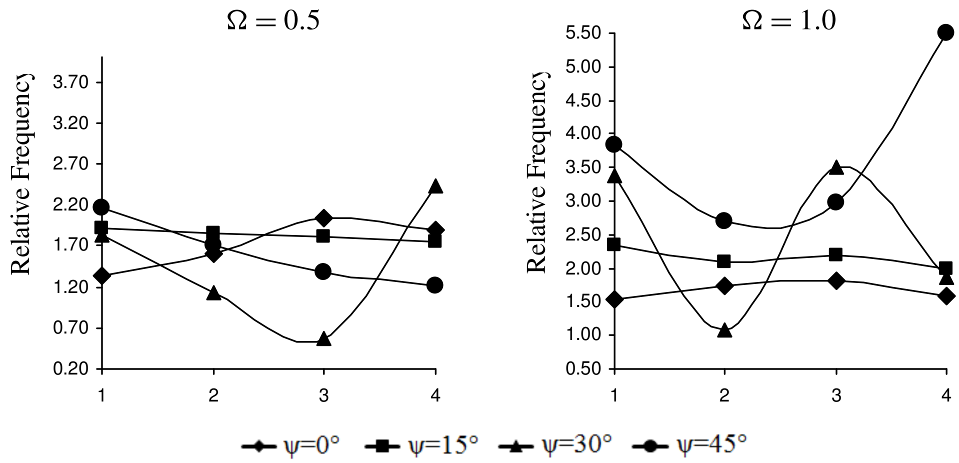

Figure 5. Variation of relative frequencies (NDFF) of graphite-epoxy bending stiff composite conical shells with delamination at rotation speeds $\Omega = 0.5$ (left) and $\Omega = 1.0$ (right) for different twist angles Ψ and different numbers of delaminations. Other parameters are $n = 8$, $h = 0.0004$, $a/L = 0.33$, $d/L = 0.5$, $s/h = 1000$, $L/s = 0.7$, $\theta_0 = 45°$, and $\theta_v = 20°$.

to $\psi = 0°$, $15°$, $30°$, and $45°$, respectively. Hence it could be inferred that the relative frequencies corresponding to NDSF have a greater effect only for higher rotational speeds.

5. Mode shapes

The mode shapes corresponding to NDFF and NDSF are shown in Figures 6 and 7, respectively, for various twist angles at the stationary condition ($\Omega = 0.0$) and a number of delaminations, for eight-layered graphite-epoxy symmetric bending-stiff composite shallow conical shells. The fundamental mode corresponds to the first torsion. The symmetry modes are absent when the twist angle is nonzero and the nodal lines indicate zero displacement amplitude. The first spanwise bending is observed for an untwisted conical shell at the stationary condition corresponding to the second natural frequencies for single, double, and triple delamination cases ($n_d = 1, 2, 3$), but the dominance of the first torsional mode is identified for the twisted cases corresponding to the second natural frequency.

6. Conclusions

The following conclusions are drawn from the present study:

(1) The finite element formulation presented in this paper can be successfully applied to analyze the natural frequencies of multiple delaminated conical shells for any particular laminate configuration.

(2) In general, at a stationary condition, the nondimensional fundamental frequency parameter decreases with increase in the twist angle. Under a rotating condition, nondimensional fundamental frequencies of the delaminated composite laminates with bending-stiff configuration are found to decrease with an increase of the twist angle.

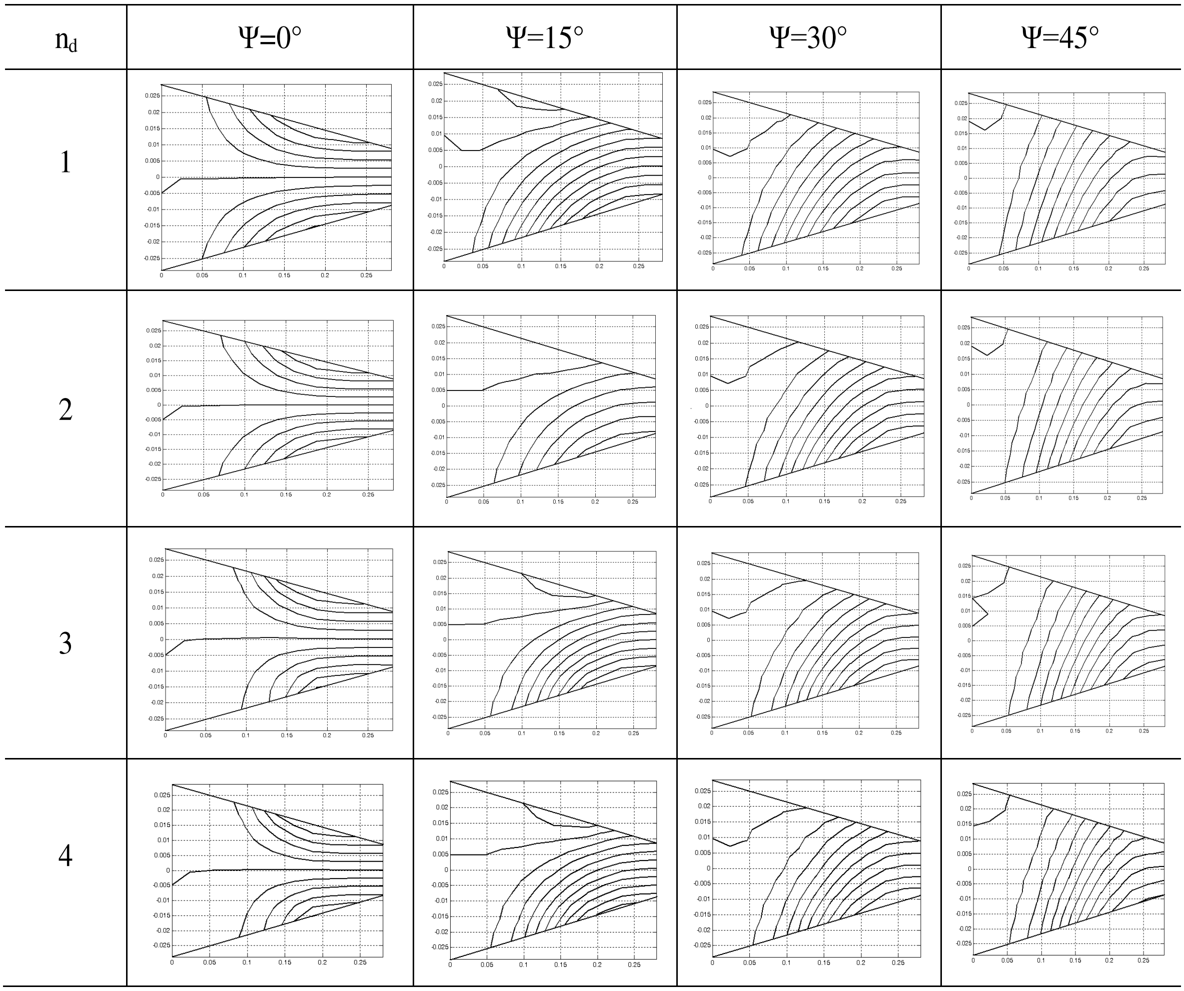

Figure 6. Effect of twist and number of delaminations on mode shapes for NDFF of graphite-epoxy bending-stiff delaminated composite conical shells, considering $n = 8$, $h = 0.0004$, $s/h = 1000$, $a/L = 0.33$, $d/L = 0.5$, $L/s = 0.7$, $\theta_0 = 45°$, and $\theta_v = 20°$.

(3) An increase in the number of delaminations leads to a reduction in elastic stiffness irrespective of the twist angles.

(4) The relative frequencies corresponding to the nondimensional second natural frequencies have a pronounced effect for higher rotational speeds.

(5) The fundamental mode corresponds to the first torsion. The first spanwise bending is observed for an untwisted conical shell at a stationary condition corresponding to the second natural frequencies for single, double, and triple delamination cases.

(6) The nondimensional frequencies obtained are the first known results which can serve as reference solutions for future investigations.

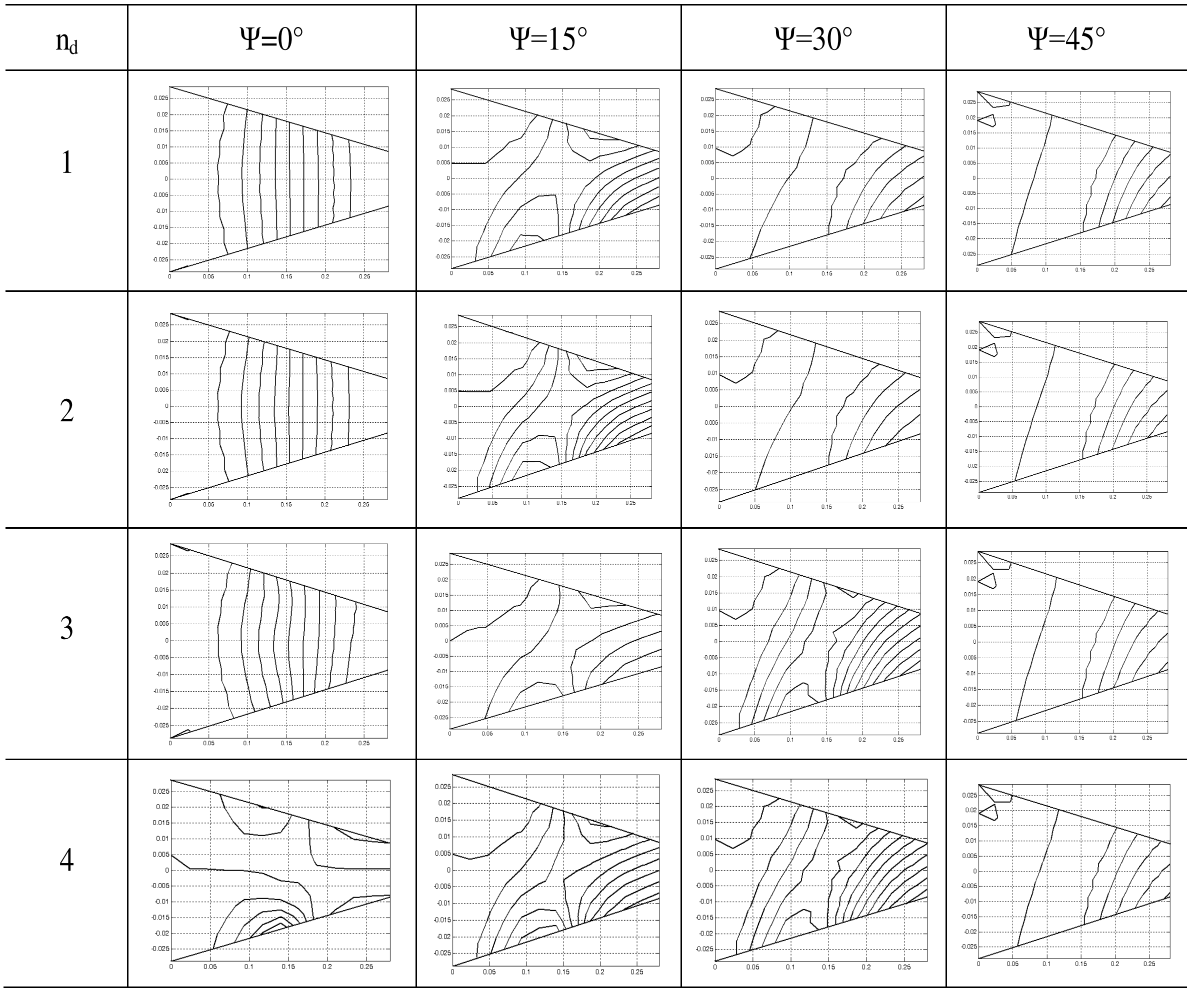

Figure 7. Effect of twist and number of delaminations on mode shapes for NDSF of graphite-epoxy bending-stiff delaminated composite conical shells, considering $n = 8$, $h = 0.0004$, $s/h = 1000$, $a/L = 0.33$, $d/L = 0.5$, $L/s = 0.7$, $\theta_0 = 45°$, and $\theta_v = 20°$.

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Received 26 Oct 2011. Revised 2 Jan 2012. Accepted 3 Jan 2012.

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JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

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Cover design: Alex Scorpan Cover photo: Mando Gomez, www.mandolux.com

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JoMMS (ISSN 1559-3959) is published in 10 issues a year. The subscription price for 2012 is US $555/year for the electronic version, and $735/year (+ $60 shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720–3840.

JoMMS peer-review and production is managed by EditFlow® from Mathematical Sciences Publishers.

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mathematical sciences publishers
http://msp.org/
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Typeset in LaTeX

Journal of Mechanics of Materials and Structures

Volume 7, No. 2 February 2012

1559-3959(2012)7:2;1-C