Journal of
Mechanics of
Materials and Structures

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Volume 7, No. 4 April 2012

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 7, No. 4, 2012

msp

ELASTIC SOLUTION IN A FUNCTIONALLY GRADED COATING SUBJECTED TO A CONCENTRATED FORCE

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Functionally graded materials are currently being actively explored in coating design to reduce the mismatch of thermomechanical properties at the interface and thus increase the resistance of coatings to fracture mechanisms. Many established and potential applications of graded materials involve contact or impact problems that are primarily load transfer problems; consequently, the goal is to study basic elasticity problems for graded inhomogeneous solids. Here we study the three-dimensional elastic deformation of a graded coating subjected to a point load on the free surface, deposited on a homogeneous elastic half-space. By assuming an isotropic coating for which Young's modulus depends exponentially on the thickness and Poisson's ratio is constant, the elastic solution is obtained using Plevako's representation, which reduces the problem to the construction of a potential function satisfying a linear fourth-order partial differential equation. We explicitly obtain the elastic solution for the coating and the substrate for two different interface conditions: the frictionless case and the perfectly bonded case. A comparative study of FGMs and homogeneous coatings is presented to investigate the effect of the graded coating properties.

1. Introduction

Functionally graded materials (FGMs) are two-phase composites with continuously varying volume fractions. Owing to the importance of the engineering applications of these materials, the thermomechanical behavior of FGM coatings has been investigated by many researchers [Suresh 2001]. The concept of grading the thermomechanical properties of materials provides an important tool to design new materials for certain specific functions. To take full advantage of this new tool, research is needed for developing efficient processing methods and material characterization techniques. If FGMs are used as coatings or in interfacial zones, they can reduce the thermally and mechanically induced stresses resulting from material property mismatches and, consequently, the risk of cracks and debonding of the coating or layer. Most studies of FGM coatings on substrates have focused on their fracture mechanisms, contact and impact response, and vibrational and thermoelastic behavior [Erdogan 1995; Birman and Byrd 2007].

The problem of a concentrated force acting normally to the free surface of a semi-infinite solid is of interest in contact mechanics. For example, Martin et al. [2002] studied it for a functional exponentially graded unbonded elastic solid subjected to a point force by assuming that the Lamé moduli vary exponentially in a given fixed direction, and obtained solutions that allow the development of boundary-integral methods for graded materials.

Various studies have considered the elastic response of a functionally graded coating deposited on a substrate. Kashtalyan and Menshykova [2008] determined a three-dimensional elastic solution of a

Work financed by the Italian Ministry of Education, University, and Research (MIUR), Project No. 2009XWLFKW: "Multiscale modeling of materials and structures".
Keywords: elasticity, coating, functionally graded materials.

functionally graded coating on a homogeneous finite thickness elastic substrate subjected to mechanical loading. By assuming that the elasticity modulus varies exponentially through the thickness of the coating, the solution allows the authors to analyze the effects of the coating type on the stress and displacement fields. [Liu and Wang 2008] studied the problem of a functionally graded coating half-space indented by an axisymmetric smooth rigid punch by using the Hankel integral transform technique to reduce the contact problem to a Cauchy singular equation to be solved numerically.

The aim of this paper is to study the three-dimensional elastic deformation of a functionally graded coating deposited on a homogeneous substrate subjected to a point load. We investigate the interface between the coating and the substrate in order to describe the difference in behavior due to localized damage. For this reason, we consider two ideal interface conditions: the perfectly bonded case and the frictionless contact case. (Real interface conditions lie in between: the contact is neither frictionless nor perfectly bonded.) In the isotropic coating, the Young's modulus depends exponentially on the position along the thickness, while the Poisson ratio is assumed to be constant and equal to the homogeneous substrate. Similar investigations have been performed to describe the elastic response in indentation tests on homogeneous films [Sburlati 2009a].

The solution is obtained in the framework of three-dimensional elasticity using Plevako's representation form [1971], which reduces the problem to determining a potential function satisfying a linear fourth-order partial differential equation. We explicitly obtain the solution by writing the potential function in terms of a Bessel expansion with respect to the radial coordinate [Sneddon 1966]. We investigate the stresses and displacements with respect to the corresponding homogeneous coating, and show that there is an increase of the radial stress on the free surface and a reduction on the interface zone due to the graded properties. Furthermore, the different interface conditions permit us to study the effects of localized interface damage in order to provide useful suggestions for the design of graded coatings.

2. Formulation of the problem

We consider a functionally graded material coating of thickness h deposited on a homogeneous substrate subjected to a point load on the upper face (see Figure 1). We introduce a cylindrical coordinate system

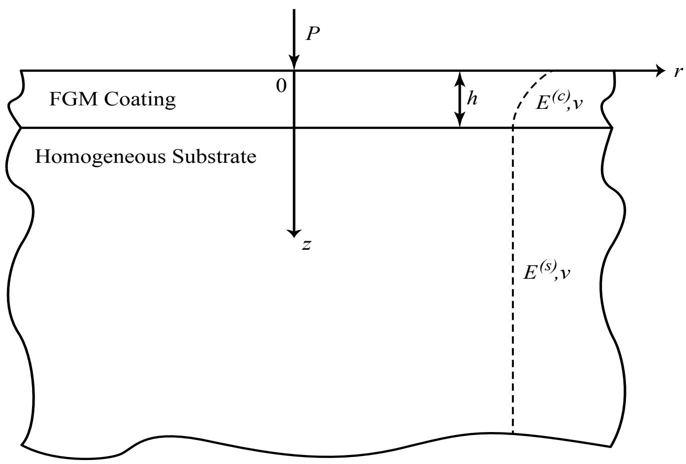

Figure 1. Sketch of the problem.

and assume that in the coating, the Young's modulus varies exponentially in the z-direction and that it is homogeneous and isotropic in the substrate:

$$E^{(c)}(z) = E_0\, e^{2kz}, \qquad E^{(s)}(z) = E_h = E_0\, e^{2kh}, \tag{2-1}$$

while the Poisson ratios $\nu_c = \nu_s = \nu$ are uniform.

We denote the quantities in the coating by $(i) = (c)$ and the quantities in the substrate by $(i) = (s)$. Assuming a null body force, the elasticity equations are [Love 1927]:

$$\nabla^2 u^{(i)} - \frac{u^{(i)}}{r^2} + \frac{1}{1-2\nu}\frac{\partial \Theta^{(i)}}{\partial r} + \left(\frac{\partial u^{(i)}}{\partial z} + \frac{\partial w^{(i)}}{\partial r}\right)\frac{1}{E^{(i)}(z)}\frac{d}{dz}E^{(i)}(z) = 0, \tag{2-2}$$

$$\nabla^2 w^{(i)} + \frac{1}{1-2\nu}\frac{\partial \Theta}{\partial z} + \left(\frac{\partial w^{(i)}}{\partial z} + \frac{\nu}{1-2\nu}\Theta^{(i)}\right)\frac{2}{E^{(i)}(z)}\frac{d}{dz}E^{(i)}(z) = 0,$$

where

$$\Theta^{(i)} = \frac{\partial u^{(i)}}{\partial r} + \frac{u^{(i)}}{r} + \frac{\partial w^{(i)}}{\partial z}.$$

The load conditions on the free surface of the coating are

$$\sigma_z^{(c)}(r, 0) = -P\delta(\boldsymbol{r}) = -\frac{P\delta(r)}{\pi r}, \qquad \tau_{rz}^{(c)}(r, 0) = 0, \tag{2-3}$$

where δ denotes the Dirac function. We study the problem for *perfectly bonded* and *frictionless contact* interface conditions.

In the perfectly bonded case we assume continuity of the displacement and stress components:

$$\begin{aligned} w^{(c)}(r, h) &= w^{(s)}(r, h), & u^{(c)}(r, h) &= u^{(s)}(r, h), \\ \sigma_z^{(c)}(r, h) &= \sigma_z^{(s)}(r, h), & \tau_{rz}^{(c)}(r, h) &= \tau_{rz}^{(s)}(r, h). \end{aligned} \tag{2-4}$$

In the frictionless contact case, we assume

$$w^{(c)}(r, h) = w^{(s)}(r, h), \qquad \sigma_z^{(c)}(r, h) = \sigma_z^{(s)}(r, h), \qquad \tau_{rz}^{(c)}(r, h) = \tau_{rz}^{(s)}(r, h) = 0. \tag{2-5}$$

In both cases, in the substrate, we set

$$\lim_{z\to\infty} \sigma_z^{(s)}(r, z) = 0, \qquad \lim_{z\to\infty} \tau_{rz}^{(s)}(r, z) = 0. \tag{2-6}$$

3. Solution technique

In order to find the elastic solution we adopt Plevako's approach [1971] for an axisymmetric problem. To this end, the displacement field is expressed in terms of a potential function $L(r, z)$, in the form

$$u^{(i)}(r, z) = -\frac{1+\nu}{E^{(i)}(z)}\frac{\partial}{\partial r}\left(\nu\nabla_r^2 L^{(i)} - (1-\nu)\frac{\partial^2}{\partial z^2}L^{(i)}\right), \tag{3-1}$$

$$w^{(i)}(r, z) = -\frac{2(1+\nu)}{E^{(i)}(z)}\frac{\partial}{\partial z}\nabla_r^2 L^{(i)} + (1+\nu)\frac{\partial}{\partial z}\left[\frac{1}{E^{(i)}(z)}\left(\nu\nabla_r^2 L^{(i)} - (1-\nu)\frac{\partial^2}{\partial z^2}L^{(i)}\right)\right], \tag{3-2}$$

where ∇_r^2 is the radial Laplace operator. The function $L^{(i)}(r,z)$ is required to satisfy the Plevako equation

$$\nabla^2\left(\frac{1}{E^{(i)}(z)}\nabla^2 L^{(i)}\right)-\frac{1}{1-\nu}\nabla_r^2 L^{(i)}\,\frac{d^2}{dz^2}\,\frac{1}{E^{(i)}(z)}=0. \tag{3-3}$$

The stress field components assume the form

$$\begin{aligned}\sigma_r^{(i)}(r,z)&=\frac{\nu}{r}\frac{\partial}{\partial r}\nabla_r^2 L^{(i)}+\frac{\partial^2}{\partial z^2}\nabla_r^2 L^{(i)}-\frac{1-\nu}{r}\frac{\partial^3}{\partial r\partial z^2}L^{(i)},\\ \sigma_\theta^{(i)}(r,z)&=\nu\nabla_r^4 L^{(i)}-\frac{\nu}{r}\frac{\partial}{\partial r}\nabla_r^2 L^{(i)}+\nu\frac{\partial^2}{\partial z^2}\nabla_r^2 L^{(i)}+\frac{1-\nu}{r}\frac{\partial^3}{\partial r\partial z^2}L^{(i)},\\ \tau_{rz}^{(i)}(r,z)&=-\frac{\partial^2}{\partial r\partial z}\nabla_r^2 L^{(i)},\qquad \sigma_z^{(i)}(r,z)=\nabla_r^4 L^{(i)}.\end{aligned} \tag{3-4}$$

By using (2-1), we obtain from (3-3)

$$\nabla_r^4 L^{(c)}+2\left(\frac{\partial^2}{\partial z^2}-2k\frac{\partial}{\partial z}-2k^2\omega^2\right)\nabla_r^2 L^{(c)}+\left(\frac{\partial^2}{\partial z^2}-4k\frac{\partial}{\partial z}+4k^2\right)\frac{\partial^2}{\partial z^2}L^{(c)}=0 \tag{3-5}$$

and

$$\nabla_r^4 L^{(s)}+2\frac{\partial^2}{\partial z^2}\nabla_r^2 L^{(s)}+\frac{\partial^4}{\partial z^4}L^{(s)}=0, \tag{3-6}$$

where $\omega^2=\nu/(1-\nu)$.

In this work, we assume that the elastic deformation is contained within a cylindrical region of some radius $b>h$; in other words, the displacement field and the Plevako function are null for $r\geq b$.

Now, we write the Plevako function with the following Fourier–Bessel expansion [Sneddon 1966; Watson 1922]:

$$L^{(i)}(r,z)=\sum_{j=1}^{\infty}L_j^{(i)}(z)\,J_0(\phi_j r),\qquad \text{where } \phi_j=\frac{z_j^{(0)}}{b} \tag{3-7}$$

and where the $z_j^{(0)}$, for $j=1,2,\dots$, are the positive roots of $J_0(x)$, Bessel function of order zero, and

$$L_j^{(i)}(z)=\frac{2}{b^2 J_1^2(\phi_j b)}\int_0^b L^{(i)}(\rho,z)\,J_0(\phi_j\rho)\,\rho\,d\rho.$$

We remark that this expansion, which assumes $L^{(i)}(b,z)=0$, can be generalized to the case for which this assumption is not made. This requires the introduction of two suitable functions $\beta(z)=L(b,z)$, and $\alpha(z)=\nabla_r^2 L(r,z)|_{r=b}$ which, in turn, give rise to a different componentwise expression of the Plevako equation [Sburlati 2009b; Sburlati and Bardella 2011].

Returning to the present case, we substitute the expansion (3-7) into (3-5) and (3-6), obtaining

$$L_j^{(c)\,''''}(z)-4k\,L_j^{(c)\,'''}(z)+\left(4k^2-2\phi_j^2\right)L_j^{(c)\,''}(z)+4\phi_j^2 k\,L_j^{(c)\,'}(z)+\phi_j^2(4k^2\omega^2+\phi_j^2)L_j^{(c)}(z)=0 \tag{3-8}$$

and

$$L_j^{(s)\,''''}(z)-2\phi_j^2 L_j^{(s)\,''}(z)+\phi_j^4 L_j^{(s)}(z)=0. \tag{3-9}$$

The solution of (3-8) is

$$L_j^{(c)}(z) = e^{kz}\big[e^{\beta_j z}\big(A_j \cos(\alpha_j z) + B_j \sin(\alpha_j z)\big) + e^{-\beta_j z}\big(C_j \cos(\alpha_j z) + D_j \sin(\alpha_j z)\big)\big], \tag{3-10}$$

where

$$\alpha_j = \frac{\sqrt{2}}{2}\sqrt{\sqrt{k^4 + \phi_j^4 + 2k^2\phi_j^2(2\omega^2+1)} - k^2 - \phi_j^2} \quad \text{and} \quad \beta_j = \frac{k\,\omega\,\phi_j}{\alpha_j},$$

while the solution of (3-9) is

$$L_j^{(s)}(z) = e^{\phi_j z}(T_j + z\,Q_j) + e^{-\phi_j z}(F_j + z\,G_j). \tag{3-11}$$

4. Explicit solution

Now we substitute the expansions of the Plevako functions $L^{(c)}(r, z)$ and $L^{(s)}(r, z)$ into (3-1), (3-2) and (3-4). Denoting the Heaviside step function by $H(x)$, we write

$$u(r, z) = \sum_{j=1}^{\infty}\big(H(h-z)u_j^{(c)}(z) + H(z-h)u_j^{(s)}(z)\big)J_1(\phi_j r) \tag{4-1}$$

and

$$w(r, z) = \sum_{j=1}^{\infty}\big(H(h-z)w_j^{(c)}(z) + H(z-h)w_j^{(s)}(z)\big)J_0(\phi_j r), \tag{4-2}$$

where

$$u_j^{(c)}(z) = -\frac{(1+\nu)\phi_j e^{-2kz}}{E_0}\big(\nu\phi_j^2 L_j^{(c)}(z) + (1-\nu)L_j^{(c)\prime\prime}(z)\big), \tag{4-3}$$

$$u_j^{(s)}(z) = -\frac{(1+\nu)\phi_j e^{-2kh}}{E_0}\big(\nu\phi_j^2 L_j^{(s)}(z) + (1-\nu)L_j^{(s)\prime\prime}(z)\big), \tag{4-4}$$

$$w_j^{(c)}(z) = \frac{e^{-2kz}}{E_0}\big((\nu-1)(L_j^{(c)\prime\prime\prime}(z) - 2kL_j^{(c)\prime\prime}(z)) + (2-\nu)\phi_j^2 L_j^{(c)\prime}(z) + 2k\nu\phi_j^2 L_j^{(c)}(z)\big), \tag{4-5}$$

$$w_j^{(s)}(z) = \frac{(1+\nu)e^{-2kh}}{E_0}\big((\nu-1)L_j^{(s)\prime\prime\prime}(z) + (2-\nu)\phi_j^2 L_j^{(s)\prime\prime}(z)\big). \tag{4-6}$$

The stress field is given by

$$\sigma_z(r, z) = \sum_{j=1}^{\infty}\phi_j^4\big(H(h-z)L_j^{(c)}(z) + H(z-h)L_j^{(s)}(z)\big)J_0(\phi_j r),$$

$$\tau_{rz}(r, z) = -\sum_{j=1}^{\infty}\phi_j^3\big(H(h-z)L_j^{(c)\prime}(z) + H(z-h)L_j^{(s)\prime}(z)\big)J_1(\phi_j r),$$

$$\sigma_r(r, z) = -\sum_{j=1}^{\infty}\phi_j^2\big(H(h-z)L_j^{(c)\prime\prime}(z) + H(z-h)L_j^{(s)\prime\prime}(z)\big)J_0(\phi_j r)$$
$$+\frac{1-\nu}{r}\sum_{j=1}^{\infty}\phi_j\big[H(h-z)\big(L_j^{(c)\prime\prime}(z) + \omega^2\phi_j^2 L_j^{(c)}(z)\big) + H(z-h)\big(L_j^{(s)\prime\prime}(z) + \omega^2\phi_j^2 L_j^{(s)}(z)\big)\big]J_1(\phi_j r),$$

$$\sigma_\theta(r,z) = -\nu \sum_{j=1}^{\infty} \phi_j^2 \left[H(h-z)\left(L_j^{(c)\prime\prime}(z) - \phi_j^2 L_j^{(c)}(z)\right) + H(z-h)\left(L_j^{(s)\prime\prime}(z) - \phi_j^2 L_j^{(s)}(z)\right)\right] J_0(\phi_j r)$$
$$- \frac{1-\nu}{r} \sum_{j=1}^{\infty} \phi_j \left[H(h-z)\left(L_j^{(c)\prime\prime}(z) + \omega^2\phi_j^2 L_j^{(c)}(z)\right) + H(z-h)\left(L_j^{(s)\prime\prime}(z) + \omega^2\phi_j^2 L_j^{(s)}(z)\right)\right] J_1(\phi_j r).$$

Plevako's functions $L_j^{(c)}(z)$ and $L_j^{(s)}(z)$ are now obtained by taking into account the values of the four series of coefficients A_j, B_j, C_j, D_j and T_j, Q_j, F_j, G_j determined by imposing the interface conditions. In both cases, we write the point load applied on the upper face as a Bessel expansion:

$$p(r) = -P\frac{\delta(r)}{r\pi} = \sum_{j=1}^{\infty} p_j\, J_0(\phi_j\, r), \quad \text{where } p_j = \frac{P}{\pi b^2\, J_1^2(\phi_j\, b)}. \tag{4-7}$$

Then the boundary conditions (2-3) and (2-6) give

$$C_j = -A_j - \frac{p_j}{\phi_j^4}, \quad D_j = \frac{p_j(k-\beta_j) - \phi_j^4(2\beta_j\, A_j + \alpha_j\, B_j)}{\phi_j^4 \alpha_j^4}, \quad T_j = 0, \quad Q_j = 0, \tag{4-8}$$

and the interface conditions allow us to obtain the remaining coefficients (see Appendix).

5. Numerical results

The numerical example presented here highlights the effects of material inhomogeneity using the analytic solution obtained in the previous section. We conduct a comparative study of FGM versus homogeneous coatings to examine the different behaviors with attention to the effects of the interface conditions between the coating and the substrate.

We choose a hardened coating ($k < 0$) where $E_0 = 200\,\text{GPa}$, $E_h = 20\,\text{GPa}$ and $\nu = 0.3$. For the comparative analysis, we consider a homogeneous coating with $E_c = E_0$, deposited on a soft homogenous substrate with $E_s = E_h$. In addition, for all cases we assume $h = 500\,\mu\text{m}$, $P = 2000\,\text{N}$ and $b = 20h$.

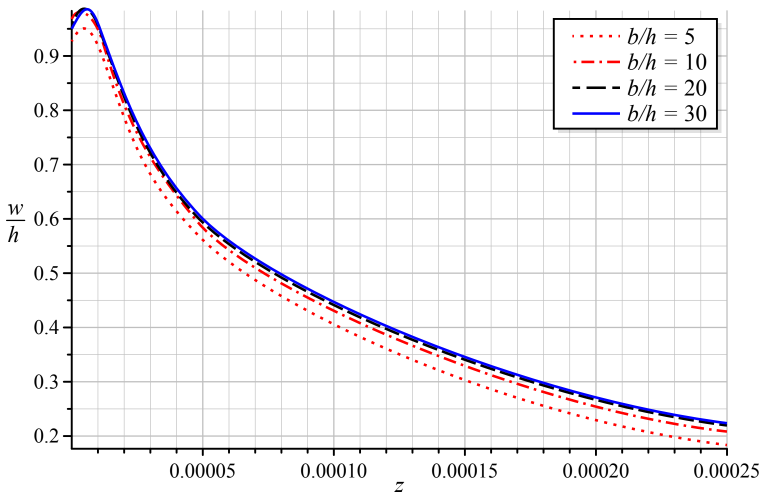

Figure 2. Normalized transversal displacements in the thickness for $b/h = 5, 10, 20, 30$.

We adopted this value for b because, according to a numerical sensitivity analysis, greater values of b do not affect the final results in these cases. To this end, in Figure 2 we plot the normalized transversal displacement for the case of a perfectly bonded interface condition, for the ratios $b/h = 5, 10, 20, 30$. Similar results can be obtained for radial displacement and stresses. We stress that the chosen value does not represent a specific material, but is used to better show the effect of the graded properties.

We now present some plots of normalized displacements and stresses throughout the thickness for two systems: a functionally graded coating on a homogeneous substrate (FGC) and a homogeneous coating on a homogeneous substrate (HC). For both systems, we consider the perfectly bonded (PB) and frictionless contact (FC) interface conditions.

In Figure 3, the normalized transversal displacement w/h at $r = 1/10h$ reveals, on the free surface, an increase of the order of 15% in the graded coating case compared to the homogeneous coating, The variation through the thickness of the normalized radial displacement, near the interface for $r = 1/10h$, is shown in Figure 4. It reveals very different behaviors for the two interface conditions. We observe that

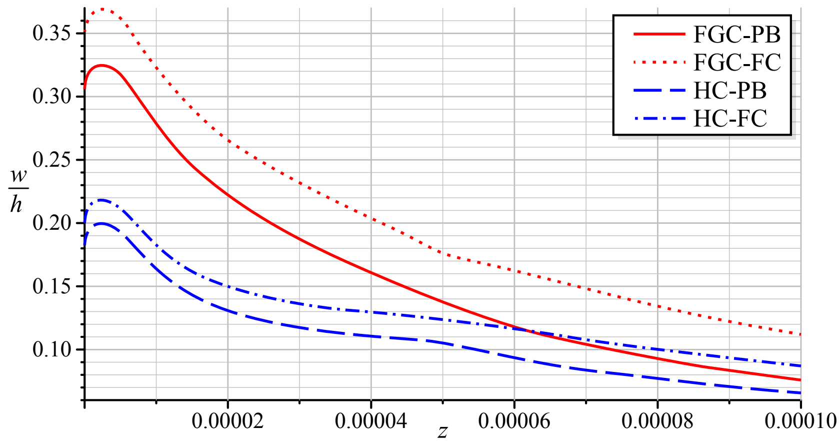

Figure 3. Normalized transversal displacements in the thickness for $r = 1/10\,h$.

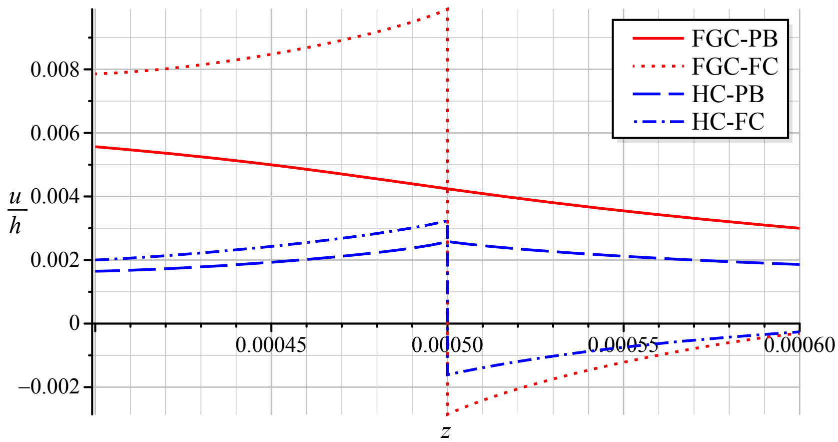

Figure 4. Normalized radial displacements near the interface for $r = 1/10\,h$.

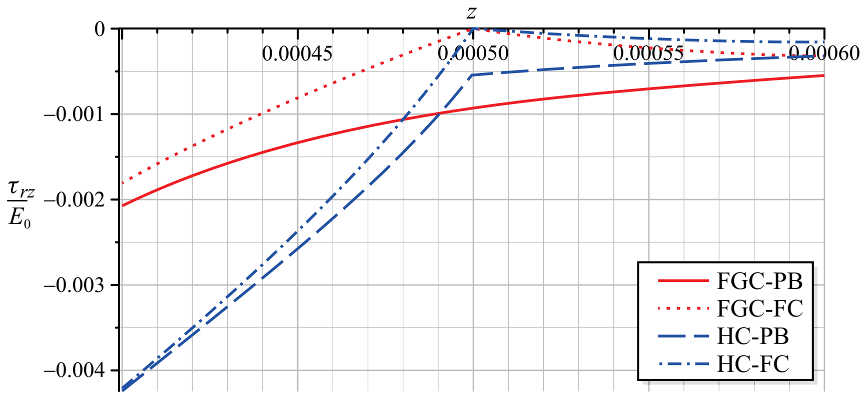

Figure 5. Normalized shear stresses near the interface for $r = 1/10\,h$.

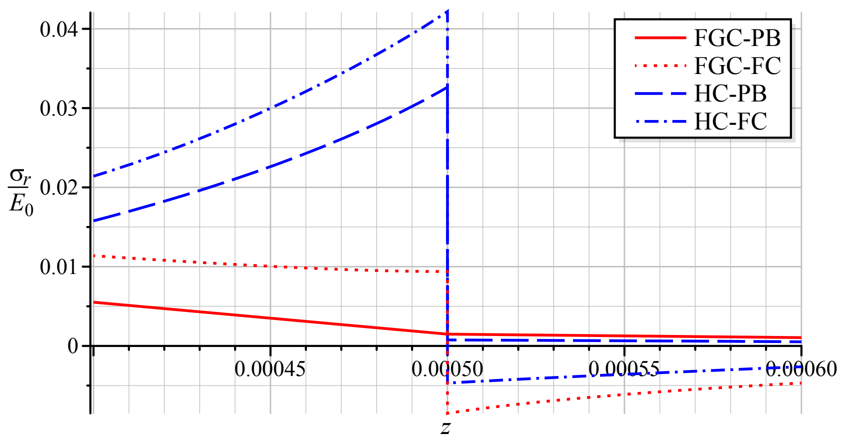

Figure 6. Normalized radial stresses near the interface for $r = 1/10\,h$.

in the graded coating, the radial displacement (FGC-PB) is about twice the corresponding value of the corresponding displacement for the homogeneous coating (HC-PB). In the frictionless contact case, the value is about three times as much. In Figure 5, we show the variations of the normalized shear stress. For FGM-PB, it has a smooth behavior at the interface, unlike the other cases. Furthermore, the FGC-PB stress is about twice the corresponding value for HC-PB. Finally, in Figure 6, we present the normalized radial stress near the interface. We observe that only the FGC-PB case is continuous, and its interface value is about $1/20$ times the corresponding value of the HC-PB in the coating.

An analysis of the radial stress solution, calculated in $r = 0$ and $z = h$ and for the HC-PB case, shows that

$$\sigma_r^{(\text{HC-PB})}(0, h^-) = \frac{f(E_0/E_h)}{h^2}, \quad \sigma_r^{(\text{FGC-PB})}(0, h) = \frac{g(E_0/E_h)}{h^2}$$

for suitable functions f and g. To better understand this we recall Boussinesq's half-space solution for

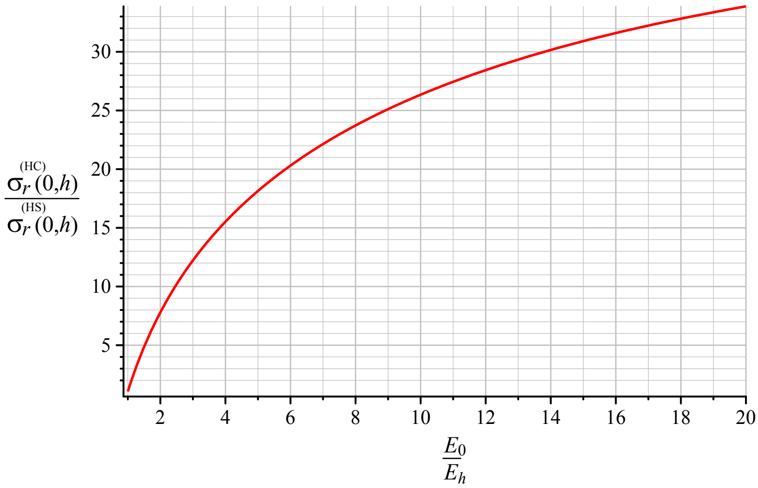

Figure 7. Radial stress ratios at the interface of the homogeneous coating.

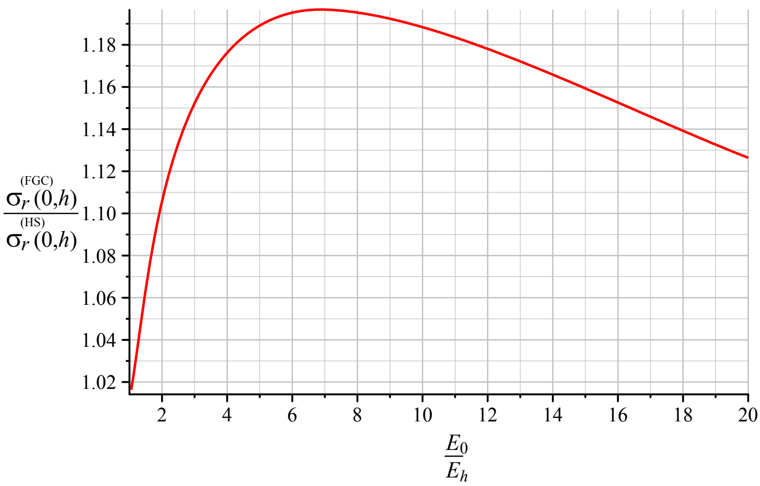

Figure 8. Radial stress ratios at the interface of the graded coating.

$r = 0$ and $z = h$, given by

$$\sigma_r^{(\mathrm{HS})}(0, h) = \frac{P(1-2\nu)}{4\pi h^2},$$

and plot in Figures 7 and 8 the ratios

$$\frac{\sigma_r^{(\mathrm{HC\text{-}PB})}(0, h^-)}{\sigma_r^{(\mathrm{HS})}(0, h)}, \qquad \frac{\sigma_r^{(\mathrm{FGC\text{-}PB})}(0, h)}{\sigma_r^{(\mathrm{HS})}(0, h)}$$

as functions of E_0/E_h. These plots show that the value at the interface of the radial stress for FGC is near the corresponding value of Boussinesq's solution. In contrast, this value for HC is quickly increasing with the ratio between the elastic properties.

6. Concluding remarks

We have determined an exact three-dimensional axisymmetric elastic solution for a functionally graded coating subjected to a point-force load. We assumed inhomogeneity governed by exponential gradation along the coating thickness, and considered two different interface conditions (perfectly bonded and frictionless contact). We compared the solution to the case of a homogeneous coating on a homogeneous substrate with different elastic properties. The analytical solution, obtained in terms of Bessel expansions, highlighted the effects of inhomogeneity on the behavior of the displacement and stress fields at the interface.

Appendix

We explicitly write the coefficients A_j, B_j, F_j, G_j of Section 4 in the case of perfectly bonded interface conditions. We first introduce the quantity

$$\Lambda = \phi_j^4\, k\big(d_1 \cos(\alpha_j h)\sin(\alpha_j h) + d_2 \cos^2(\alpha_j h) + d_3 + d_4\, e^{2\,h\beta_j} + d_5\, e^{-2\,h\beta_j}\big),$$

where

$$\begin{aligned}
d_1 &= -8\,\alpha_j\phi_j k\nu + 4\,(\beta_j^2 - 2\,k^2\nu)\alpha_j,\\
d_2 &= 8\,\phi_j^2 k\nu + 4\,(2\,k^2\nu - \beta_j^2)\phi_j + 8\,\nu\,k(k-\beta_j)(k+\beta_j),\\
d_3 &= -2\,\phi_j^3 - 4\,\phi_j^2 k\nu + (6\,\beta_j^2 - 2\,(1+2\,\nu)k^2)\phi_j + 4\,k\beta_j^2\nu - 4\,k^3\nu,\\
d_4 &= \phi_j^3 + \beta_j\phi_j^2 - (\beta_j^2 + 2\,k\nu\,\beta_j - (1-2\,\nu)k^2)\phi_j - \beta_j^3 - 2\,k\beta_j^2\nu + (1-2\,\nu)k^2\beta_j,\\
d_5 &= \phi_j^3 - \beta_j\phi_j^2 - (\beta_j^2 - 2\,k\nu\,\beta_j - (1-2\,\nu)k^2)\phi_j + \beta_j^3 - 2\,k\beta_j^2\nu + (2\,\nu-1)k^2\beta_j.
\end{aligned}$$

The constants of integration are

$$\begin{aligned}
A_j &= (p_j/\Lambda)\,\big(a_1 \cos(\alpha_j h)\sin(\alpha_j h) + a_2 \cos^2(\alpha_j h) + a_3\, e^{-2\,h\beta_j} + a_4\big),\\
B_j &= (p_j/\Lambda)\,\big(b_1 \cos(\alpha_j h)\sin(\alpha_j h) + b_2 \cos^2(\alpha_j h) + b_3\, e^{-2\,h\beta_j} + b_4\big),
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= 4\,\alpha_j k^3\nu + (4\,\phi_j\nu - 2\,\beta_j)\alpha_j k^2 - 2\,\beta_j(\beta_j+\phi_j)\alpha_j k - 4\,\beta_j(\beta_j^2-\phi_j^2)(\nu-1)\alpha_j,\\
a_2 &= -4\,k^4\nu + (2\,\beta_j - 4\,\phi_j\nu)k^3 + (4\,\beta_j^2\nu - 4\,\phi_j^2\nu + 2\,\phi_j\beta_j)k^2 - 2\,\beta_j(\beta_j^2-\phi_j^2-\phi_j\beta_j)k\\
&\qquad + 4\,\phi_j\beta_j(\nu-1)(\beta_j^2-\phi_j^2),\\
a_3 &= (\beta_j-\phi_j)\big((1-2\,\nu)k^3 + 2\,\beta_j\nu\,k^2 - (\beta_j^2-\phi_j^2)k\big),\\
a_4 &= -2\,\nu\,k^2(\beta_j^2-\phi_j^2) + 2\,k^4\nu + ((1+2\,\nu)\phi_j - \beta_j)k^3 - 2\,\phi_j\beta_j(\beta_j^2-\phi_j^2)(\nu-1)\\
&\qquad + (\phi_j^3 + \beta_j^3 - 3\,\phi_j\beta_j^2 - \beta_j\phi_j^2)k
\end{aligned}$$

and

$$b_1 = -4k^4\nu+(2\beta_j-4\phi_j\nu)k^3+(4\beta_j^2\nu-4\phi_j^2\nu+2\phi_j\beta_j)k^2+2\beta_j(\phi_j^2+\phi_j\beta_j-\beta_j^2)k+4\phi_j\beta_j(\nu-1)(\beta_j^2-\phi_j^2),$$
$$b_2 = -4\alpha_j k^3\nu+(2\beta_j-4\phi_j\nu)\alpha_j k^2+2\beta_j(\beta_j+\phi_j)\alpha_j k+4\beta_j(\beta_j^2-\phi_j^2)(\nu-1)\alpha_j,$$
$$b_3 = \alpha_j(\beta_j-\phi_j)(k-\beta_j)(k-2k\nu+2\beta_j-2\beta_j\nu),$$
$$b_4 = 2\alpha_j k^3\nu-(\beta_j-\phi_j-2\phi_j\nu)\alpha_j k^2-\beta_j(\beta_j+\phi_j)\alpha_j k-2\beta_j(\beta_j^2-\phi_j^2)(\nu-1)\alpha_j.$$

The remaining coefficients are

$$F_j = e^{h(k+\phi_j)}\big[p_j f_1 e^{-h\beta_j}+A_j\big(f_2\sin(\alpha_j h)+f_3\cos(\alpha_j h)\big)+B_j\big(f_4\sin(\alpha_j h)+f_5\cos(\alpha_j h)\big)\big],$$
$$G_j = e^{h(k+\phi_j)}\big[p_j g_1 e^{-h\beta_j}+A_j\big(g_2\sin(\alpha_j h)+g_3\cos(\alpha_j h)\big)+B_j\big(g_4\sin(\alpha_j h)+g_5\cos(\alpha_j h)\big)\big],$$

where

$$f_1 = \big((1-h\phi_j)(k-\beta_j)-(\alpha_j^2+\beta_j^2+k^2-2\,\beta_j k)h\big)\sin(\alpha_j h)/(\alpha_j\phi_j^4)-(1-h\phi_j)\cos(\alpha_j h)/\phi_j^4,$$
$$f_2 = \big(-2\,\beta_j+(2\,\phi_j\beta_j+2\,\beta_j k-2\,\beta_j^2-\alpha_j^2)h\big)e^{-h\beta_j}/\alpha_j+e^{h\beta_j}\alpha_j h,$$
$$f_3 = 2\sinh(h\beta_j)(1-h\beta_j-hk-h\phi_j),$$
$$f_4 = 2\sinh(h\beta_j)(1-hk-h\phi_j)-2\cosh(h\beta_j)\beta_j h,$$
$$f_5 = -2\sinh(h\beta_j)\alpha_j h,$$
$$g_1 = (\alpha_j^2+\beta_j^2+k^2-2\,\beta_j k+\phi_j k-\phi_j\beta_j)\ \sin(\alpha_j h)/(\alpha_j\phi_j^4)-\alpha_j\cos(\alpha_j h)\phi_j,$$
$$g_2 = -e^{h\beta_j}\alpha_j-(2\,\phi_j\beta_j+2\,\beta_j k-2\,\beta_j^2-\alpha_j^2)e^{-h\beta_j}/\alpha_j,$$
$$g_3 = 2\sinh(h\beta_j)(\beta_j+k+\phi_j),$$
$$g_4 = 2\cosh(h\beta_j)\beta_j+2\sinh(h\beta_j)(k+\phi_j),$$
$$g_5 = 2\sinh(h\beta_j)\alpha_j.$$

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Received 31 Jan 2012. Revised 17 Mar 2012. Accepted 5 Apr 2012.

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JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

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Cover design: Alex Scorpan — Cover photo: Mando Gomez, www.mandolux.com

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JoMMS (ISSN 1559-3959) is published in 10 issues a year. The subscription price for 2012 is US $555/year for the electronic version, and $735/year (+$60 shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720–3840.

JoMMS peer-review and production is managed by EditFLOW® from Mathematical Sciences Publishers.

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mathematical sciences publishers
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A NON-PROFIT CORPORATION
Typeset in LaTeX

Journal of Mechanics of Materials and Structures

Volume 7, No. 4 April 2012