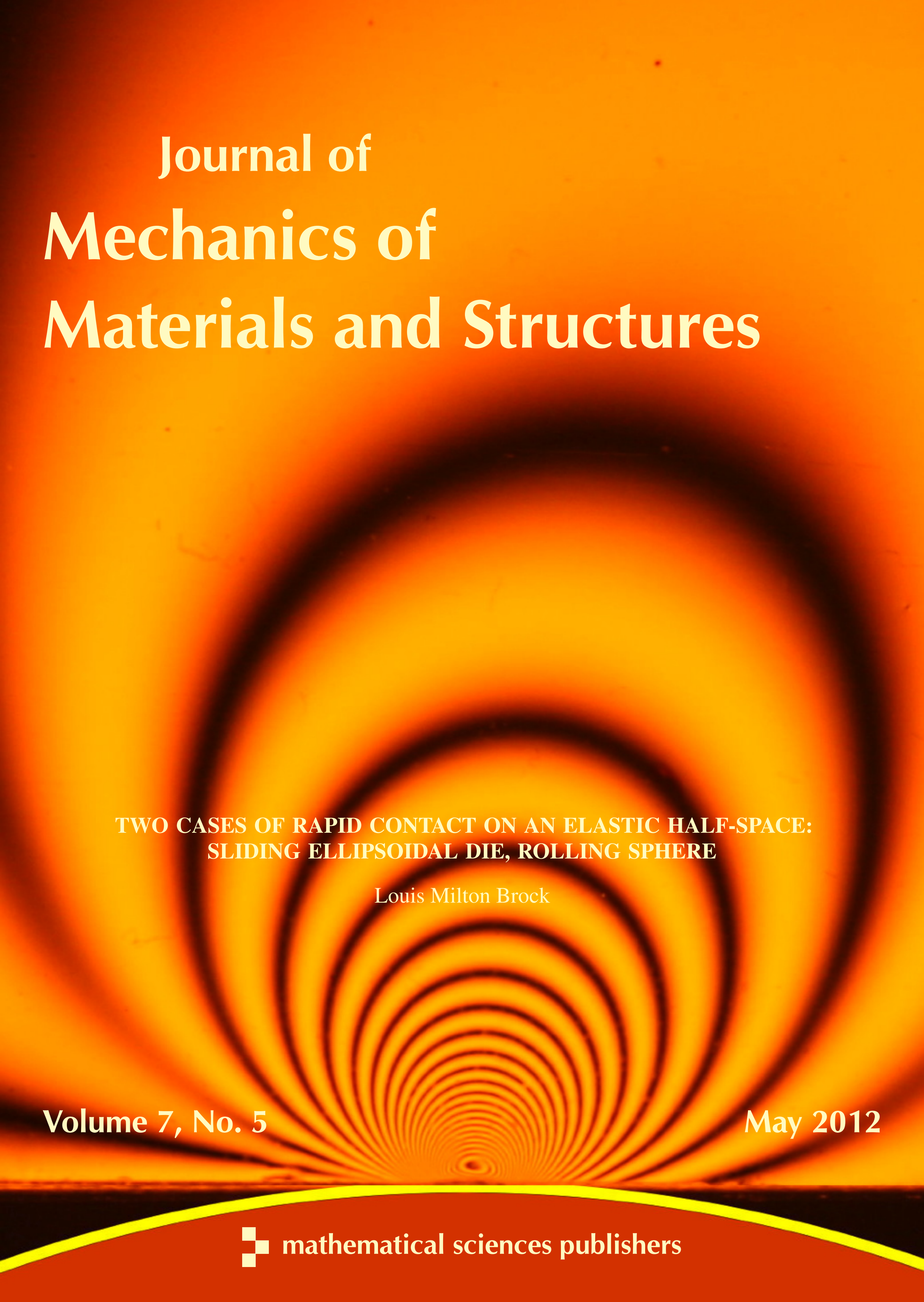
Journal of
Mechanics of
Materials and Structures



TWO CASES OF RAPID CONTACT ON AN ELASTIC HALF-SPACE: SLIDING ELLIPSOIDAL DIE, ROLLING SPHERE

Louis Milton Brock

Volume 7, No. 5 May 2012

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 7, No. 5, 2012
dx.doi.org/10.2140/jomms.2012.7.469

TWO CASES OF RAPID CONTACT ON AN ELASTIC HALF-SPACE: SLIDING ELLIPSOIDAL DIE, ROLLING SPHERE

LOUIS MILTON BROCK

In one case a rigid ellipsoidal die translates over the surface of a half-space. Because of friction, both compression and shear force are required. In the other, a rigid sphere rolls on the surface under a compressive force. Both motions occur along a straight path at constant subcritical speed. A dynamic steady state is treated, that is, the contact zone and its traction remain constant in the frame of the die or sphere. Exact solutions for contact zone traction are derived in analytic form, as well as formulas for the contact zone shape. Axial symmetry is not required in the solution process. Cartesian coordinates are used, but a system of quasipolar coordinates is introduced that allows problem reduction to singular integral equations similar in form to those found in 2D contact.

1. Introduction

Sliding and rolling contact arises in machining, mechanism operation, and vehicle travel [Barwell 1979; Bayer 1994; Blau 1996]. The literature on the mechanics of contact is vast, for example, [Ahmadi et al. 1983; Barber 1983; Johnson 1985; Kalker 1990; Hills et al. 1993]. An important category — e.g., [Craggs and Roberts 1967; Churilov 1978; Rahman 1996] — treats indentation of an elastic surface by a rigid die that also translates over the surface. If speed and resultant forces are constant, then a dynamic steady state may be achieved. In that instance, contact zone geometry and surface traction do not vary in the frame of the moving die. For the thermoelastic solid, Brock and Georgiadis [2000] treat sliding contact opposed by friction and Brock [2004] treats rolling contact without slip by a rigid cylinder. Sliding and rolling speeds are constant, and robust asymptotic solutions in analytic form are given for the dynamic steady state in 2D.

The aforementioned studies are adopted here for 3D isothermal problems of sliding by a rigid ellipsoid and rolling by a rigid sphere. Again sliding is resisted by friction, and rolling without slipping is assumed. Sliding and rolling speeds are constant and subcritical, that is, below the Rayleigh wave speed. Ignoring slipping in rolling is an idealization [Johnson 1985], and one that gives rise to rapid oscillations in thin strips along the contact zone edge. It is noted, however, that strip widths in [Brock 2004] prove to be orders of magnitude smaller than the contact zone radius.

The solution process is standard, e.g., [Hills et al. 1993]: a solution to the unmixed boundary value problem of specified surface traction reduces the mixed contact problem to the solution of integral equations. To this end, the governing equations for the elastic half-space, subjected to a translating zone of (somewhat) arbitrary traction over its surface, are given in the next section. Translation speed is constant and subcritical, and zone geometry and traction do not change during translation. Therefore, as in [Brock and Georgiadis 2000; Brock 2004], a dynamic steady state is assumed. Cartesian coordinates are used,

Keywords: sliding contact, ellipsoid, rolling contact sphere, quasipolar.

and an exact transform solution in the half-space surface spatial variables is obtained. In view of the 3D nature of the problem and the lack of axial symmetry for the ellipsoidal die, quasipolar coordinates, both in transform and spatial planes, are employed during the inversion process. These are defined by a polar angle that sweeps through 180° (π radians) and a radial coordinate that has both positive and negative directions. For points in the contact zone, the resulting displacement expressions reduce to double integrals whose limits are independent of the points. The imposed displacement conditions are then satisfied by requiring the integrands to be solutions of Cauchy singular integral equations that are similar in form to those in the 2D studies [Brock and Georgiadis 2000; Brock 2004]. The contact zone traction is then extracted as analytic functions of the quasipolar coordinates. The normal traction is required to vanish continuously on the contact zone boundary, and to render the resultant compression force as a stationary value for a given sliding/rolling speed. These requirements lead to expressions that define the contact zone geometry.

2. General equations

In terms of Cartesian coordinates $\mathbf{x}(x_k)$, an undisturbed, linear isotropic, homogeneous half-space occupies region $x_3 > 0$. A traction distribution is then applied to a finite, simply connected area C of the surface $x_3 = 0$. Boundary contour $\Im(x_1, x_2) = 0$ defines a continuous closed curve, with continuously varying tangent and normal directions and radius of curvature. The lattermost is always directed to the interior of C and the x_1-direction is an axis of symmetry. Area C is then translated in the positive x_1-direction at constant subcritical speed v. It does not change, and the traction distribution remains invariant with respect to it. A dynamic steady state ensues for which half-space response is invariant in the frame of translating C. It is therefore convenient to translate the Cartesian system with C, so that displacement $\mathbf{u}(u_k)$ and traction $\mathbf{T}(\sigma_{ik})$ vary only with $\mathbf{x}(x_k)$, where time differentiation becomes $-v\partial_1$ and ∂_k signifies x_k-differentiation. For $x_3 > 0$ the governing equations can be extracted from [Achenbach 1973] and modified for the dynamic steady state as

$$\mathbf{u} = \mathbf{u}_D + \mathbf{u}_S, \tag{1}$$

$$(\nabla^2 - c^2\partial_1^2)\mathbf{u}_S = 0, \qquad \nabla \cdot \mathbf{u}_S = 0, \tag{2}$$

$$(c_D^2\nabla^2 - c^2\partial_1^2)\mathbf{u}_D = 0, \qquad \nabla \times \mathbf{u}_D = 0, \tag{3}$$

$$\frac{1}{\mu}\mathbf{T} = (c_D^2 - 2)(\nabla \cdot \mathbf{u}_D)\mathbf{1} + 2(\nabla\mathbf{u} + \mathbf{u}\nabla). \tag{4}$$

Here $(\nabla, \mathbf{1}, \nabla^2)$ are the gradient, identity tensor, and Laplacian. Quantities (c, c_D) are, respectively, the contact area speed and dilatational wave speed (v, v_D) that are made dimensionless with respect to the isothermal shear (rotational) wave speed v_S, where

$$c_D = \sqrt{m+1}, \quad m = \frac{1}{1-2\nu}, \quad v_S = \sqrt{\frac{\mu}{\rho}}. \tag{5}$$

Here (ν, μ, ρ) are the Poisson's ratio, shear modulus, and mass density. The boundary conditions for $x_3 = 0$ are that surface traction vanishes for $(x_1, x_2) \notin C$, but

$$\sigma_{13} = \tau_1, \quad \sigma_{23} = \tau_2, \quad \sigma_{33} = \sigma, \quad (x_1, x_2) \in C. \tag{6}$$

Here (τ_1, τ_2, σ) are bounded and continuous functions of $(x_1, x_2) \in C$ but can be integrably singular on the contour $\Im(x_1, x_2) = 0$. In addition $(\mathbf{u}, \mathbf{T})$ should remain finite for $|\mathbf{x}| \to \infty$, $x_3 > 0$.

3. General transform solution

After [van der Pol and Bremmer 1950; Sneddon 1972] the double bilateral Laplace transform is defined as

$$\hat{F} = \iint F(x_1, x_2) \exp(-p_1 x_1 - p_2 x_2)\, dx_1 dx_2. \tag{7}$$

Integration is along the entire $\mathrm{Re}(x_1)$ and $\mathrm{Re}(x_2)$-axes. Application of (7) to (1)–(6) gives for $x_3 > 0$

$$\hat{\mathbf{u}}_D = (p_1, p_2, -\lambda_\pm) D \exp(-\lambda_\pm x_3), \tag{8a}$$

$$\hat{\mathbf{u}}_S = (S_1, S_2, S_3) \exp(-\lambda_S x_3), \quad \lambda_S S_3 = p_1 S_1 + p_2 S_2. \tag{8b}$$

Here coefficients (S_1, S_2, D) are given by

$$\mu Q_R S_1 = \frac{p_2}{\lambda_S} Q_N (p_2 \hat{\tau}_1 - p_1 \hat{\tau}_2) - \lambda_S (Q_K \hat{\tau}_1 + 2 p_1 \lambda_D \hat{\sigma}), \tag{9a}$$

$$\mu Q_R S_2 = \frac{p_1}{\lambda_S} Q_N (p_1 \hat{\tau}_2 - p_2 \hat{\tau}_1) - \lambda_S (Q_K \hat{\tau}_2 + 2 p_2 \lambda_D \hat{\sigma}), \tag{9b}$$

$$\mu Q_R D = Q_K \hat{\sigma} - 2 \lambda_S (p_1 \hat{\tau}_1 + p_2 \hat{\tau}_2). \tag{9c}$$

In (8) and (9) quantities

$$Q_N = Q_K - 2\lambda_D \lambda_S, \quad Q_R = 4(p_1^2 + p_2^2)\lambda_D \lambda_S + Q_K^2, \quad Q_K = (c^2 - 2) p_1^2 - 2 p_2^2, \tag{10a}$$

$$\lambda_S = \sqrt{(c^2 - 1) p_1^2 - p_2^2}, \quad \lambda_D = \sqrt{(s_D^2 c^2 - 1) p_1^2 - p_2^2}, \quad s_D = \frac{1}{c_D}. \tag{10b}$$

Equation (8) is bounded for $x_3 > 0$ only when $\mathrm{Re}(\lambda_S, \lambda_D) \geqslant 0$, so that branch cuts in the (p_1, p_2)-plane is required.

4. Inversion scheme

In view of (8)–(10) and [van der Pol and Bremmer 1950; Sneddon 1972], transform inversion for, say the contribution of σ to u_3, involves the operation

$$-\frac{1}{2\pi i} \int dp_1 \frac{1}{2\pi i} \int dp_2 \frac{\lambda_D}{\mu \Delta} U \iint d\xi_1\, d\xi_2\, \sigma \exp[p_1(x_1 - \xi_1) + p_2(x_2 - \xi_2)]. \tag{11}$$

In (11), $\sigma = \sigma(\xi_1, \xi_2)$ and $U = U(p_1, p_2)$, where

$$U(p_1, p_2) = 2(p_1^2 + p_2^2) \exp(-\lambda_S x_3) + Q_K \exp(-\lambda_D x_3). \tag{12}$$

Double integration is over C, and single integration is along the entire $\mathrm{Im}(p_1)$- and $\mathrm{Im}(p_2)$-axes. This suggests definitions and transformations

$$p_1 = p\cos\psi, \qquad p_2 = p\sin\psi, \tag{13a}$$

$$x = x_1\cos\psi + x_2\sin\psi, \qquad y = x_2\cos\psi - x_1\sin\psi, \tag{13b}$$

$$\xi = \xi_1\cos\psi + \xi_2\sin\psi, \qquad \eta = \xi_2\cos\psi - \xi_1\sin\psi. \tag{13c}$$

In (13), $\mathrm{Re}(p) = 0+$, $-\infty < [\mathrm{Im}(p), x, y, \xi, \eta, \xi_1, \xi_2] < \infty$ and $|\psi| < \pi/2$. Parameters (p, ψ), $(x, \psi; y = 0)$, and $(\xi, \psi; \eta = 0)$ constitute quasipolar coordinate systems, that is,

$$dx_1\,dx_2 = |x|\,dx\,d\psi, \quad d\xi_1\,d\xi_2 = |\xi|\,d\xi d\psi, \quad dp_1\,dp_2 = |p|\,dp\,d\psi. \tag{14}$$

Thus (11) can be written as

$$\frac{1}{i\pi}\int_\Psi d\psi \frac{1}{2\pi i}\int |p|\,dp \frac{\omega_D}{\mu R}\frac{\sqrt{-p}}{p\sqrt{p}} U(p,\psi)\int_{\mathrm{N}} d\eta \int_\Xi d\xi\, \sigma(\xi,\eta)\exp p(x-\xi), \tag{15a}$$

$$U(p,\psi) = 2\exp(-\omega_S x_3\sqrt{-p}\sqrt{p}) + K\exp(-\omega_D x_3\sqrt{-p}\sqrt{p}). \tag{15b}$$

Integration with respect to p is along the positive side of the entire imaginary axis. Subscripts (Ψ, N, Ξ) signify integration over, respectively, the ranges $-\pi/2 < \psi < \pi/2$, $\eta_-(\psi) < \eta < \eta_+(\psi)$, and $x_-(\eta,\psi) < \xi < x_+(\eta,\psi)$. Limits $\eta_\pm(\psi)$ are points on the contour $\Im[\xi_1(\xi,\eta), \xi_2(\xi,\eta)] = 0$ where $d\eta/d\xi = 0$, and limits $x_\pm(\eta,\psi)$ locate the ends of a line parallel to the ξ-axis that spans C for a given η. The restrictions on $(C, \Im)$ imply that $(x_\pm, \eta_\pm)$ exist and are continuous in ψ. In (15) we also have

$$\omega_S = \sqrt{1 - c^2\cos^2\psi}, \quad \omega_D = \sqrt{1 - s_D^2 c^2\cos^2\psi}, \tag{16a}$$

$$N = K + 2\omega_S\omega_D, \quad R = 4\omega_S\omega_D - K^2, \quad K = c^2\cos^2\psi - 2. \tag{16b}$$

The exponential terms in (15b) are made bounded for $x_3 > 0$ by requiring that $\mathrm{Re}(\sqrt{\pm p}) \geqslant 0$ in the p-plane with, respectively, branch cuts $\mathrm{Im}(p) = 0$, $\mathrm{Re}(p) < 0$ and $\mathrm{Im}(p) = 0$, $\mathrm{Re}(p) > 0$. The p-integration is (15a) and can be obtained from Appendix A. The result is that (15a) and counterparts for (τ_1, τ_2) give u_3 for $x_3 > 0$:

$$u_3 = -\frac{1}{\pi^2}\int_\Psi d\psi \frac{\omega_D}{\mu R}\int_{\mathrm{N}} d\eta \int_\Xi d\xi\,\sigma(\xi,\eta)\frac{K(x-\xi)}{(x-\xi)^2 + \omega_D^2 x_3^2} + \frac{2\omega_S(x-\xi)}{(x-\xi)^2+\omega_S^2 x_3^2} \tag{17}$$
$$-\frac{1}{\pi^2}\int_\Psi d\psi \frac{\omega_S}{\mu R}\int_{\mathrm{N}} d\eta \int_\Xi d\xi\,[\tau_1(\xi,\eta)\cos\psi + \tau_2(\xi,\eta)\sin\psi]\frac{2\omega_D^2 x_3}{(x-\xi)^2+\omega_D^2 x_3^2} + \frac{K x_3}{(x-\xi)^2+\omega_S^2 x_3^2}.$$

Here $x = x_1\cos\psi + x_2\sin\psi$, and for $x_3 = 0$, $(x_1, x_2) \in C$, (17) gives (see Appendix A):

$$u_3 = \frac{1}{\pi}\int_\Psi \frac{d\psi}{\mu R}\int_{\mathrm{N}} d\eta\left[c^2\omega_D\cos^2\psi \frac{(vp)}{\pi}\int_\Xi \sigma(\xi,\eta)\frac{d\xi}{\xi - x} - N\mathrm{T}(x,\eta)\right], \tag{18a}$$

$$\mathrm{T}(x,\eta) = \tau_1(x,\eta)\cos\psi + \tau_2(x,\eta)\sin\psi. \tag{18b}$$

Here (vp) signifies principal value integration. Similar results can be obtained for (u_1, u_2). When $\psi = 0$, the term R in (16b) is the 2D Rayleigh function. It can be shown that $R \geqslant 0$ $(0 \leqslant c^2 < c_R^2)$ and $R \leqslant 0$ $(c_R^2 < c^2 < 1)$, where root c_R^2 defines the Rayleigh wave speed $v = c_R v_S$. Vanishing R can be associated with critical behavior [Georgiadis and Barber 1993]. Here of course $R = 0$ when $v = v_R$ only for $\psi = 0$.

5. Sliding contact with friction

Consider that (σ, τ_1, τ_2) result from the sliding of a rigid die at subcritical speed v in the positive x_1-direction. The die is an ellipsoid that, when it touches but does not indent the surface $x_3 = 0$, can be described in the translating **x**-coordinate by

$$C_1 x_1^2 + C_2 x_2^2 + C_3\left(x_3 + \frac{1}{\sqrt{C_3}}\right)^2 = 1. \tag{19}$$

Here (C_1, C_2, C_3) are positive constants, their inverses have dimensions of length squared, and (19) is consistent with the symmetry assumed for C, which of course is now a contact zone. If a rigid body motion U_3 into the surface accompanies translation, indentation occurs. This requires compressive force F_3 in the x_3-direction. For small deformations indentation is defined by $u_3 = u_3^C$, $(x_1, x_2) \in C$, where

$$u_3^C = U_3 - \frac{1}{2\sqrt{C_3}}(C_1 x_1^2 + C_2 x_2^2). \tag{20}$$

It is noted that $|\mathbf{x}|$ is now the distance from the surface point in the contact zone that undergoes the largest normal displacement. That is, the validity of the asymptotic expressions increases with this distance. If sliding is resisted by friction with kinetic coefficient γ, die translation also requires a shear force $F_1 = \gamma F_3$ in the positive x_1-direction. It is assumed that die translation and die/surface slip essentially coincide, that is, $(\tau_2 \approx 0, \tau_1 = \gamma\sigma)$. In view of (18) the contact problem must then satisfy for $(x_1, x_2) \in C$ the equation

$$\frac{1}{\pi}\int_\Psi d\eta \int_{\mathrm{N}} d\eta \left[\frac{c^2\omega_D \cos^2\psi}{\pi\mu R}(vp)\int_\Xi \sigma(\xi,\eta)\frac{d\xi}{\xi - x} - \frac{N}{\mu R}\Gamma\sigma(x,\eta)\right] = u_3^C, \quad \Gamma = \gamma\cos\psi. \tag{21}$$

In light of (7), (20), and Appendix A, u_3^C can be written as

$$u_3^C = -\frac{1}{\pi}\int_\Psi d\eta \int_{\mathrm{N}} d\eta \int_\Xi d\xi \frac{d}{dx}\delta(x-\xi)u_3^C(\xi,\eta), \tag{22a}$$

$$u_3^C(\xi,\eta) = U_3 - \frac{C_1}{2\sqrt{C_3}}(\xi\cos\psi - \eta\sin\psi)^2 - \frac{C_2}{2\sqrt{C_3}}(\eta\cos\psi + \xi\sin\psi)^2. \tag{22b}$$

Here δ is the Dirac function. Equation (21) thereby reduces to matching the integrands of double integration in (ψ, η). Parameter ξ in $\sigma(\xi, \eta)$ is an integration variable representing parameter x that itself depends on coordinate (x_1, x_2) and integration variable ψ. However, as noted in light of (13) for $y = 0$, (x_1, x_2) can be replaced by quasipolar coordinates (x, ψ). Thus traction σ itself can be found by dropping η, and (21) and (22) are reduced to

$$-c^2\omega_D\cos^2\psi\frac{(vp)}{\pi}\int_\Xi \sigma(\xi,\psi)\frac{d\xi}{\xi - x} + \Gamma N\sigma(x,\psi) = \mu R G(x,\psi), \tag{23a}$$

$$G(x,\psi) = \frac{-Ax}{\sqrt{C_3}}, \qquad A = C_1\cos^2\psi + C_2\sin^2\psi. \tag{23b}$$

In view of Appendix B we introduce unknowns $g(x)$ and Ω in the representation

$$\frac{1}{\mu}\sigma(x) = g(x)\cos\pi\Omega + \mathrm{I}(g;x)\sin\pi\Omega \quad (x_- < x < x_+). \tag{24}$$

Experience [Brock and Georgiadis 2000; Brock 2004] with 2D analysis indicates that $\Omega \neq \Omega(x)$. Dependence of $(\sigma, g, \Omega, x_\pm)$ on ψ is implicit in (25). Use of (25), (B3a), (B3b), and (B4) in (24) gives a classical [Erdogan 1985] linear relation in (g, I):

$$[\Gamma N \sin \pi\Omega - c^2\omega_D \cos^2\psi \cos\pi\Omega]\mathrm{I}(g; x) + [\Gamma N \cos\pi\Omega + c^2\omega_D\cos^2\psi \sin\pi\Omega]g(x) = RG(x, \psi). \quad (25)$$

Eigenvalue Ω is chosen to make the coefficient of $\mathrm{I}(g; x)$ vanish, and $g(x)$ follows:

$$\Omega = -\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{-\gamma N}{c^2\omega_D \cos\psi}\right), \quad g(x) = \frac{R}{\Delta}\frac{Ax}{\sqrt{C_3}}, \quad (26a)$$

$$\Delta = \sqrt{\Gamma_\Psi^2 N^2 + (c^2\omega_D\cos^2\psi)^2} = \cos\psi\sqrt{\gamma^2N^2 + (c^2\omega_D\cos\psi)^2}, \quad (26b)$$

$$\sin\pi\Omega = -\frac{c^2\omega_D}{\Delta}\cos^2\psi, \quad \cos\pi\Omega = -\frac{\gamma N}{\Delta}\cos^2\psi. \quad (26c)$$

In (26a), $-\frac{1}{2} < \Omega < 0$ for $0 < v < v_S$. The polynomial form of $g(x)$ allows the use of (B6) (second equation) to evaluate (24):

$$\sigma(x, \psi) = \frac{\mu R}{\Delta}\frac{A}{\sqrt{C_3}}\left(\frac{x_+ - x}{x - x_-}\right)^\Omega (x + \Omega L), \quad L = x_+ - x_-(x_- < x < x_+). \quad (27)$$

Dependence on ψ is now more explicit. The negative Ω gives σ an integrable singularity as $x \to x_+$. Signorini conditions for contact [Georgiadis and Barber 1993] prohibit singular gradients at the contact zone boundary. Therefore (27) leads to

$$\sigma(x, \psi) = -\frac{\mu A}{\sqrt{C_3}}\frac{R}{\Delta}(x_+ - x)^{1+\Omega}(x - x_-)^{-\Omega}, \quad (28a)$$

$$x_+ = -\Omega L, \quad x_- = -(1+\Omega)L. \quad (28b)$$

For $0 \leqslant v < v_R$, $R \geqslant 0$ for $|\psi| \leqslant \pi/2$, and (28a) satisfies the Signorini condition that nonwelded contact cannot not involve tensile stress. It is also noted that the (R, Δ)-ratio in (28a) is finite for $|\psi| \leqslant \pi/2$.

6. Contour of *C*

Equation (28b) defines in part contour $\Im$ and, because $\Omega(-\psi) = \Omega(\psi)$ and $\Omega(\pm\pi/2) = -\frac{1}{2}$, does not violate the symmetry of C. The unknown contact zone span L depends on c and is an even function of ψ. It is determined by requiring that (τ_1, τ_2, σ) be consistent with the resultant force system acting on the die. Here $(x_\pm, \sigma)$ and therefore τ_1 are even functions of ψ, and $\tau_2 \approx 0$. Thus the condition that there is no resultant force in the x_2-direction and no resultant torque about the x_3-axis is automatically satisfied. The condition that resultant force in the (x_1, x_3)-directions is $F_1 = \gamma F_3$ and $-F_3$, respectively, is met if

$$\int_\Psi d\psi \int_\Xi \sigma(\xi, \psi)|\xi|\, d\xi = -F_3. \quad (29)$$

Equation (29) is an integral equation for $L(c, \psi)$. One solution approach is based on the observation that, for a given value $x_3 > 0$, projection of die (19) onto the x_1x_2-plane is an elliptical area bounded by contour $C_1x_1^2 + C_2x_2^2 =$ constant. In terms of (x, ψ), if the span of C along the x_1-axis ($\psi = 0$) is L_1, then the span L for a given $|\psi| < \pi/2$ is

$$L = \sqrt{\frac{C_1}{A_1}} L_1. \tag{30}$$

A simple assumption is that (30) also holds in (28b) for $(C, \Im)$, where L_1 is an unknown function of c. Here, however, it is argued that, for a given speed (c), F_3 should be stationary with respect to (28a). That is,

$$\int_\Psi d\psi \int_\Xi \delta\sigma(\xi, \psi)|\xi|\, d\xi = 0. \tag{31}$$

This requirement is satisfied when at every $x_- < x < x_+$, $|\psi| < \pi/2$,

$$\delta\sigma = \frac{\partial\sigma}{\partial x}\delta x + \frac{\partial\sigma}{\partial\psi}\delta\psi = 0. \tag{32}$$

Here ψ and x are held constant in the first and second coefficients, respectively, and $(\delta x, \delta\psi)$ are arbitrary. Differentiation of (28a) shows that

$$x = -(1+2\Omega)L: \quad \frac{\partial\sigma}{\partial x} = 0, \quad \frac{\partial^2\sigma}{\partial x^2} > 0. \tag{33a}$$

The second term then vanishes for $x = -(1+2\Omega)L$ if

$$-\frac{\partial}{\partial\psi}\left(\frac{RA}{\Delta\sqrt{C_3}} QL\right) = 0, \quad Q = (1+\Omega)^{1+\Omega}(-\Omega)^{-\Omega}. \tag{33b}$$

Separation of variables and integration leads to

$$L = \frac{C_1}{A}\frac{R_1\Delta}{R\Delta_1}\frac{Q_1}{Q}L_1, \qquad Q_1 = (1+\Omega_1)^{1+\Omega_1}(-\Omega_1)^{-\Omega_1}, \tag{34a}$$

$$R_1 = 4\omega_{1D}\omega_{1S} - K_1^2, \qquad \Delta_1 = \sqrt{\gamma^2 N_1^2 + (c^2\omega_{1D})^2}, \tag{34b}$$

$$N_1 = 2\omega_{1D}\omega_{1S} + K_1, \qquad \Omega_1 = -\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{-\gamma N_1}{c^2\omega_{1D}}\right), \tag{34c}$$

$$\omega_{1S} = \sqrt{1-c^2}, \qquad \omega_{1D} = \sqrt{1 - \frac{c^2}{c_D^2}}, \quad K_1 = c^2 - 2. \tag{34d}$$

For $L = L_2$, that is, $|\psi| = \pi/2$, (34a) and its static $(c = 0)$ and smooth sliding $(c \neq 0, \gamma = 0)$ limit cases give, respectively,

$$L_2 = \frac{C_1}{C_2}\frac{\sqrt{c_D^4 + \gamma^2}}{c_D^2 - 1}\frac{R_1 Q_1}{\Delta_1}L_1, \quad L_2 = \frac{C_1}{C_2}L_1, \quad L_2 = \frac{C_1}{C_2}\frac{c_D^2}{2(c_D^2 - 1)}\frac{R_1}{c^2\omega_{1D}}L_1. \tag{35}$$

Equations (30), (34a) and (35) in light of (28b) allow several observations:

(I) Except for a circular projection profile and smooth contact $(C_1 = C_2, \gamma = 0)$, the ratio of spans along the axes of symmetry is not maintained in C.

γ	$c=0.1$	$c=0.2$	$c=0.3$	$c=0.4$	$c=0.5$	$c=0.6$
0.1	0.9938	0.9712	0.9330	0.8781	0.8034	0.7049
0.2	0.9948	0.9721	0.9339	0.8789	0.8042	0.7055
0.3	0.9963	0.9736	0.9354	0.8803	0.8045	0.7066

Table 1. Values of the dimensionless ratio C_2L_2/C_1L_1 for pairs of (γ, c).

(II) In the static case with an elliptic projection profile, the difference between spans is enhanced in C.

(III) Die translation speed ($c \neq 0$) accentuates this effect.

(IV) Friction leaves C with symmetry only with respect to the x_1-axis, whether die translation occurs ($c \neq 0$) or not ($c = 0$).

Results in [Rahman 1996] also exhibit a sliding speed effect on span ratio. Moreover, stationary principles are features of static smooth contact [Barber 1992]. The effect noted in (III) is illustrated in Table 1 with values of ratio C_2L_2/C_1L_1 for values of (γ, c).

Substituting (28) and (34a) in (29), and applying integration results from Appendix B along with an integration variable change, gives, finally, an equation for L_1 as a function of c:

$$F_3 = \left(\frac{R_1Q_1}{\Delta_1}\right)^3 (C_1L_1)^3 \frac{\mu}{\sqrt{C_3}} \int_\Psi \frac{d\psi}{Q^3}\left(\frac{\Delta}{AR}\right)^2 \int_{-(1+\Omega)}^{-\Omega} (-\Omega - t)^{1+\Omega}(t+1+\Omega)^{-\Omega}|t|\,dt. \tag{36}$$

In view of (28b) and the monotonic variation of Ω with ψ and its range $\left(-\frac{1}{2} < \Omega < 0\right)$, span L does not cross $\Im$.

7. Rolling without slipping

Consider that rolling without slipping by a rigid sphere of radius r_0 is what produces the translating C. Rolling at constant speed requires no force in the x_1-direction, but indentation needs a compressive force, which we call F_3, to be imposed. Thus (19) and (20) are replaced by

$$x_1^2 + x_2^2 + (x_3 + r_0)^2 = r_0^2, \quad u_3^C = U_3 - \frac{1}{2r_0}(x_1^2 + x_2^2). \tag{37}$$

In view of (18) and its counterparts for (u_1, u_2), the contact problem in this case gives coupled equations:

$$-c^2\omega_D \cos^2\psi \frac{(vp)}{\pi} \int_\Xi d\xi \frac{\sigma(\xi,\psi)}{\xi - x} + N\mathrm{T}(x,\psi) = \mu R G_3(x,\psi), \tag{38a}$$

$$-N\sigma(x,\psi)\cos\psi + \frac{(vp)}{\pi\omega_S} \int_\Xi \frac{d\xi}{\xi - x}[N_1\tau_1(\xi,\psi) + N_{12}\tau_2(\xi,\psi)] = \mu R G_1(x,\psi), \tag{38b}$$

$$-N\sigma(x,\psi)\sin\psi + \frac{(vp)}{\pi\omega_S} \int_\Xi \frac{d\xi}{\xi - x}[N_{12}\tau_1(\xi,\psi) + N_2\tau_2(\xi,\psi)] = \mu R G_2(x,\psi). \tag{38c}$$

Coefficients (N_1, N_2, N_{12}) are

$$(N_1, N_2) = M(\cos^2\psi, \sin^2\psi) - M, \quad N_{12} = M\sin\psi\cos\psi, \tag{39a}$$

$$M = K + 4\omega_S\omega_D - 2\omega_S^2 = 2N + c^2\cos^2\psi, \quad G_k(x,\psi) = \frac{\partial}{\partial x}u_k^C(x,0). \tag{39b}$$

Linear analysis [Johnson 1985] of contact surface kinematics and experience with the 2D rolling cylinder [Brock 2004] suggests in view of (37) that (u_1^C, u_2^C) are such that

$$[G_1(x,\psi), G_2(x,\psi)] = G(x,\psi)(\cos\psi, \sin\psi), \quad G_3(x,\psi) = -\frac{x}{r_0}, \quad G(x,\psi) = -V_0 + \frac{x^2}{2r_0^2}. \tag{40}$$

In light of Appendix B, we write

$$\frac{1}{\mu}(\sigma, \tau_k) = (g, g_k)\cos\pi\Omega + [\mathrm{I}(g;x), \mathrm{I}(g_k;x)]\sin\pi\Omega. \tag{41}$$

Here $k = (1,2)$ and, as in (26), dependence of $(\sigma, \tau_1, \tau_2, \Omega)$ and (g, g_1, g_2) on ψ is implicit. Use of Appendix B then gives the set of equations

$$\mathbf{K}\begin{bmatrix}\mathrm{I}(g;x)\\ \mathrm{I}(g_1;x)\\ \mathrm{I}(g_2;x)\end{bmatrix} + \mathbf{M}\begin{bmatrix}g(x)\\ g_1(x)\\ g_2(x)\end{bmatrix} = R\begin{bmatrix}G_3\\ G\cos\psi\\ G\sin\psi\end{bmatrix}, \tag{42a}$$

$$\mathbf{K} = \begin{bmatrix}-c^2\omega_D\cos^2\psi\cos\pi\Omega & N\cos\psi\sin\pi\Omega & N\sin\psi\sin\pi\Omega\\ -N\cos\psi\sin\pi\Omega & \dfrac{N_1}{\omega_S}\cos\pi\Omega & \dfrac{N_{12}}{\omega_S}\cos\pi\Omega\\ -N\sin\psi\sin\pi\Omega & \dfrac{N_{12}}{\omega_S}\cos\pi\Omega & \dfrac{N_2}{\omega_S}\cos\pi\Omega\end{bmatrix}, \tag{42b}$$

$$\mathbf{M} = \begin{bmatrix}c^2\omega_D\cos^2\psi\sin\pi\Omega & N\cos\psi\cos\pi\Omega & N\sin\psi\cos\pi\Omega\\ -N\cos\psi\cos\pi\Omega & -\dfrac{N_1}{\omega_S}\sin\pi\Omega & -\dfrac{N_{12}}{\omega_S}\sin\pi\Omega\\ -N\sin\psi\cos\pi\Omega & -\dfrac{N_{12}}{\omega_S}\sin\pi\Omega & -\dfrac{N_2}{\omega_S}\sin\pi\Omega\end{bmatrix}. \tag{42c}$$

In (42b) $|\mathbf{K}|$ vanishes for eigenvalues $\Omega = (\Omega_\pm, \Omega_0)$ given by

$$\Omega_\pm = \Omega_0 \mp i\mathrm{P}, \quad \Omega_0 = -\tfrac{1}{2}, \tag{43a}$$

$$\mathrm{P} = \frac{1}{\pi}\ln\sqrt{\frac{1+q}{1-q}}, \quad q = \frac{-N}{\sqrt{\omega_S\omega_D}c^2\cos^2\psi}. \tag{43b}$$

Quantity $q > 0$ for $|\psi| \leqslant \pi/2$ when $0 < v < v_R$, and $(\Omega_0, \Omega_\pm)$ is associated with eigenfunction sets $(g_0, g_\pm)$ and $(g_k^0, g_k^\pm)$, $k = (1,2)$. Given that $|\mathbf{K}| = 0$, the first term in (42a) disappears by choosing

$$g_1^\pm(x) = \mathrm{H}_\pm(x)\cos\psi, \quad g_2^\pm(x) = \mathrm{H}_\pm(x)\sin\psi, \tag{44a}$$

$$N\sin\pi\Omega_\pm\mathrm{H}_\pm(x) = c^2\omega_F\cos^2\psi\cos\pi\Omega_\pm g_\pm(x), \tag{44b}$$

$$g_1^0(x) = g_0(x)\sin\psi, \quad g_2^0(x) = -g_0(x)\cos\psi. \tag{44c}$$

Substitution into the remaining term in (42a) gives three equations for $(g_0, g_\pm)$. From (43) it follows that

$$\sin\pi\Omega_\pm = -\cosh\pi\mathrm{P} = -\frac{\sqrt{\omega_S\omega_D}c^2\cos^2\psi}{\sqrt{(1-\omega_S\omega_D)R}}, \quad \cos\pi\Omega_\pm = \mp i\sinh\pi\mathrm{P} = \frac{\pm iN}{\sqrt{(1-\omega_S\omega_D)R}}. \tag{45a}$$

Their use in the three equations then leads to the expressions

$$g_0(x) = 0, \quad g_\pm(x) = \frac{\sqrt{R}}{2\sqrt{1-\omega_S\omega_D}}\left[\sqrt{\frac{\omega_S}{\omega_D}}G_3(x,\psi) \pm iG(x,\psi)\right]. \tag{46}$$

Use of (41) and polynomial integration results from Appendix B lead to closed-form expressions for (σ, τ_1, τ_2). In light of (43a) these are singular as $x \to x_+$, and enforcement of the Signorini conditions leads to requirements

$$\left[\mathrm{P}\frac{L}{r_0} + \sqrt{\frac{\omega_S}{\omega_D}}\right]\left(x_+ - \frac{L}{2}\right) = 0, \quad L = x_+ - x_-, \tag{47a}$$

$$V_0 + \frac{L}{r_0}\sqrt{\frac{\omega_S}{\omega_D}} - \frac{1}{2r_0^2}\left[\left(x_+ - \frac{L}{2}\right)^2 + \frac{L^2}{8}(1-4\mathrm{P}^2)\right] = 0. \tag{47b}$$

Expressions for (V_0, x_+) follow as

$$V_0 = \frac{L}{r_0}\left[\frac{L}{16r_0}(1-4\mathrm{P}^2) - \mathrm{P}\sqrt{\frac{\omega_S}{\omega_D}}\right], \quad x_\pm = \pm\frac{L}{2}. \tag{48}$$

Use of (48) in the closed-form expressions for (σ, τ_1, τ_2) gives

$$\sigma(x,\psi) = \frac{-\mu\mathrm{B}Q}{2r_0\cos\Phi}\sqrt{L^2-4x^2}\cos\left[\mathrm{P}\ln\frac{L-2x}{L+2x} - \Phi\right], \tag{49a}$$

$$\begin{bmatrix}\tau_1(x,\psi)\\ \tau_2(x,\psi)\end{bmatrix} = \frac{-\mu\mathrm{B}Q}{2r_0\cos\Phi}\sqrt{\frac{\omega_D}{\omega_S}}\begin{bmatrix}\cos\psi\\ \sin\psi\end{bmatrix}\sqrt{L^2-4x^2}\sin\left[\mathrm{P}\ln\frac{L-2x}{L+2x} - \Phi\right], \tag{49b}$$

$$Q = \frac{\sqrt{R}}{\sqrt{1-\omega_S\omega_D}}, \quad \Phi = \tan^{-1}\frac{x\mathrm{P}}{2\mathrm{B}r_0}, \quad \mathrm{B} = \frac{\mathrm{P}L}{2r_0} + \sqrt{\frac{\omega_S}{\omega_D}}. \tag{49c}$$

Equation (49) is integrable in x but the rapid oscillatory behavior due to the logarithmic term implies that compression is guaranteed only in a region $|x| < r_C < L/2$ of C. Under the reasonable assumption that $(L, r_C) \ll r_0$ it can be shown that

$$r_C = \frac{L}{2}\frac{1-\exp(-\pi/2\mathrm{P})}{1+\exp(-\pi/2\mathrm{P})}. \tag{50}$$

For a steel solid ($\nu = 1/3$, $\mu = 75$ GPa) calculations of (43) for various combinations of $|\psi| < \pi/2$ and $0 < c < c_R$ give $\exp(-\pi/2\mathrm{P}) \approx O(10^{-9})$. As in [Brock 2004], therefore, the boundary strip is orders of magnitude smaller than the contact zone span L. In (49) (σ, τ_1) and τ_2 are, respectively, even and odd functions of ψ. It follows that shear traction on C produces neither resultant force in the (x_1, x_2)-direction nor resultant moment about the x_3-axis. Compressive force F_3 is the resultant of σ so that (29) again provides an integral equation for span L in (49). Application of the stationary principle in (33) for this

$c=0.1$	$c=0.2$	$c=0.3$	$c=0.4$	$c=0.5$	$c=0.6$
0.994	0.9733	0.9384	0.8882	0.8204	0.7318

Table 2. Values of the dimensionless ratio L_2/L_1 for values of c.

case gives for $|\psi| < \pi/2$

$$x = 0: \quad \frac{\partial \sigma}{\partial x} = 0, \quad \frac{\partial^2 \sigma}{\partial x^2} > 0. \tag{51}$$

It can then be shown that L is given by

$$L = \frac{\mathrm{B}_1 Q_1}{\mathrm{B} Q} L_1, \quad Q_1 = \frac{\sqrt{R_1}}{\sqrt{1-(\omega_S \omega_D)_1}}, \quad \mathrm{B}_1 = \sqrt{\frac{\omega_{1S}}{\omega_{1D}}} + \frac{\mathrm{P}_1 L_1}{2r_0}. \tag{52}$$

Equation (52) is transcendental in (L, L_1), but becomes a linear relation under the assumption that $(L, L_1) \ll r_0$:

$$L = \sqrt{\frac{\omega_D \omega_{1S}}{\omega_S \omega_{1D}}} \frac{Q_1}{Q} L_1, \tag{53a}$$

$$L_2 = L_1 \quad (c=0), \qquad L_2 = \frac{1}{2}\sqrt{\frac{c_D^2+1}{c_D^2-1}} \sqrt{\frac{c_D \omega_{1S}}{\omega_{1D}}} Q_1 L_1 \quad (c \neq 0). \tag{53b}$$

In light of the same assumption (29) reduces to the relation for unknown span L_1 along the x_1-axis of C:

$$\frac{1}{\mu r_0^2} F_3 = 2\left[Q_1 \sqrt{\frac{\omega_{1S}}{\omega_{1D}}} \frac{L_1}{2r_0}\right]^3 \int_\Psi \frac{\omega_D}{\omega_S Q^2} d\psi \int_0^1 t\, dt \sqrt{1-t^2} \cos \mathrm{P} \ln \frac{1-t}{1+t}. \tag{54}$$

Study of (53) shows that in this case, contact zone C does not preserve the circular profile of the projected die area except in the static limit ($c=0$). Values of the span ratio L_2/L_1 are given in Table 2 for $c \neq 0$, and show that increasing speed tends to "squeeze" C onto the x_1-axis. Equation (53a) does not define an ellipse. However, an elliptical approximation for C can use (53b) to define the ratio of semimajor and seminor dimensions.

8. Comment on supercritical/subseismic behavior

As noted above sliding contact solutions for speeds $v_R < v < v_S$ violate Signorini conditions [Georgiadis and Barber 1993]. In particular, in the sliding die problem here (28a) for normal stress in C still holds for this speed range, and is still both bounded and continuous on zone boundary $\Im$. However, $R = 0$ along spans defined by $|\psi| = \Psi_R$, and $R < 0$ for $|\psi| < \Psi_R$, where

$$\Psi_R = \tan^{-1} \sqrt{\frac{c^2}{c_R^2} - 1}. \tag{55}$$

That is, contact zone traction changes in a continuous fashion from compression in two fan-shaped regions $\Psi_R < |\psi| < \pi/2$ to tension in two fan-shaped regions $|\psi| < \Psi_R$. The situation for the rolling sphere case is more complicated: For $|\psi| < \Psi_R$, $R < 0$ and eigenvalues are now

$$\Omega_0 = -\frac{1}{2}, \quad \Omega_\pm = \mp\frac{i}{\pi}\ln\sqrt{\frac{q+1}{q-1}}. \tag{56}$$

Equation (56) and behavior of R show that traction is continuous as $|\psi| \to \Psi_R$ but, because $\Omega_\pm$ have no real part, traction on $\Im$ in fan-shaped regions $|\psi| < \Psi_R$ is both oscillatory in nature and not continuous. These two results suggest that contact actually does not occur for $|\psi| < \Psi_R$. Such a consideration is beyond the scope of this paper. Careful study of die/sphere-contact zone separation for $|\psi| < \Psi_R$ is required. Moreover, $\Psi_R \to 21.25°$ as $v \to v_S$ for a typical [Achenbach 1973] value $c_R = 0.932$. That is, separation would involve a substantial portion of the projection area in the subseismic limit.

9. Summary comments

Combining quasipolar coordinates with an analysis defined in terms of Cartesian coordinates produces solutions that can be seen as awkward hybrids. However, analytical expressions for contact zone traction in the quasipolar system are readily extracted. In any event, the approach was adopted in order to address problems without axial symmetry. Indeed no degree of symmetry is required. That imposed on contact zone C served to guarantee that a simple resultant force system could produce die sliding and rolling in the specified direction. Rolling contact by a sphere was considered because its 2D counterpart, the cylinder, is a standard rolling problem.

The assumption that key geometric features of the area projected by the rigid die onto the contact surface are preserved in the contact zone shape, or that the zone is essentially elliptical, is often accurate [Johnson 1985; Hills et al. 1993; Bayer 1994; Blau 1996]. It also avoids an iteration process based on, for example, maintenance of compression everywhere in the zone. Here, in addition to the Signorini requirement of bounded traction on the zone boundary, a requirement that resultant compressive force be stationary with respect to the traction was imposed. This gave expressions that defined the contact zone geometry. These indicated that the contact zone is often a distortion of the projected area. In particular, sliding/rolling speed tends to "flatten" the contact zone onto the line of travel, see [Rahman 1996]. Friction in sliding destroys any projection area symmetry, save that with respect to the line of travel. It is hoped that these results afford some insight into problems of 3D dynamic contact.

Appendix A

Consider integrals involving real parameters (X, Y) over the entire $\mathrm{Im}(p)$-axis P:

$$\frac{1}{2\pi i}\int_{\mathrm{P}} |p|\left(1, \frac{\sqrt{-p}}{\sqrt{p}}\right)\exp[pX - Y\sqrt{-p}\sqrt{p}]\frac{dp}{p} \quad (Y \geqslant 0). \tag{A1}$$

$\mathrm{Re}(\sqrt{\pm p}) \geqslant 0$ in the p-plane with, respectively, branch cuts $\mathrm{Im}(p) = 0$, $\mathrm{Re}(p) < 0$ and $\mathrm{Im}(p) = 0$, $\mathrm{Re}(p) > 0$. Specifically, for $\mathrm{Re}(p) = 0+$ and, respectively, $\mathrm{Im}(p) = q > 0$ and $\mathrm{Im}(p) = q < 0$:

$$\sqrt{-p} = \left|\frac{q}{2}\right|^{1/2}(1 \mp i), \quad \sqrt{p} = \left|\frac{q}{2}\right|^{1/2}(1 \pm i). \tag{A2}$$

Use of (A2) reduces (A1) to

$$\frac{1}{i\pi}\int_0^\infty (\cos qX, \sin qX)\exp(-Yq)\,dq. \tag{A3}$$

From standard [Peirce and Foster 1956] tables (A3) is evaluated as

$$\frac{1}{i\pi}\left(\frac{Y}{X^2+Y^2}, \frac{X}{X^2+Y^2}\right). \tag{A4}$$

It is noted that the X-derivative of (A4) gives the evaluation for integrals

$$\frac{1}{2\pi i}\int_{\mathrm{P}} |p|\left(1, \frac{\sqrt{-p}}{\sqrt{p}}\right)\exp(-pX - Y\sqrt{-p}\sqrt{p})\,dp \quad (Y > 0). \tag{A5}$$

It is also noted [Stakgold 1967] that

$$\frac{1}{\pi}\frac{Y}{X^2+Y^2} \to \delta(X) \quad (Y \to 0). \tag{A6}$$

Here δ is the Dirac function.

Appendix B

Consider region Ξ of the $\mathrm{Re}(t)$-axis defined as $x_- < t < x_+$ and function

$$W(x) = \int_\Xi \frac{\Omega(t)\,dt}{t-x}, \qquad |\mathrm{Re}\,\Omega(t)| < 1. \tag{B1}$$

For $x \in \Xi$, $W(x \pm i0) = w(x) \pm i\pi\Omega(x)$, where $w(x)$ is the principal value

$$w(x) = (vp)\int_\Xi \frac{\Omega(t)\,dt}{t-x}. \tag{B2}$$

For a function $g(t)$ that is bounded and piecewise continuous in Ξ the following relations hold for $x \in \Xi$:

$$G(x) = g(x)\cos\pi\Omega(x) + \mathrm{I}(g; x)\sin\pi\Omega(x), \tag{B3a}$$

$$\frac{1}{\pi}(vp)\int_\Xi \frac{G(t)}{t-x}\,dt = -g(x)\sin\pi\Omega(x) + \mathrm{I}(g; x)\cos\pi\Omega(x), \tag{B3b}$$

$$\int_\Xi G(t)\,dt = \int_\Xi g(t)\exp[-w(t)]\,dt. \tag{B3c}$$

In (B3a) and (B3b) the functional

$$\mathrm{I}(g; x) = \frac{1}{\pi}\exp w(x)(vp)\int_\Xi \frac{g(t)}{t-x}\exp[-w(t)]\,dt. \tag{B4}$$

For $x \notin \Xi$ (B3b) is replaced with

$$\frac{1}{\pi}\int_\Xi \frac{G(t)}{t-x}\,dt = \frac{1}{\pi}\exp W(x)\int_\Xi \frac{g(t)}{t-x}\exp[-w(t)]\,dt. \tag{B5}$$

For $\Omega(t) = \Omega < 0$ (constant), polynomial forms of $g(t)$ give explicit results, for example, for $g(t) = (t^0, t, t^2)$ the right-hand sides of (B3a) are

$$\left(\frac{x_+ - x}{x - x_-}\right)^{\Omega}, \quad (x + \Omega L)\left(\frac{x_+ - x}{x - x_-}\right)^{\Omega}, \quad \left[x^2 + \Omega(x + x_+) - \frac{\Omega}{2}(1 - \Omega)L^2\right]\left(\frac{x_+ - x}{x - x_-}\right)^{\Omega}. \tag{B6}$$

Similarly, the right-hand sides of (B5) are, respectively,

$$\frac{1}{\sin \pi\Omega}\left[\left(\frac{x - x_+}{x - x_-}\right)^{\Omega} - 1\right], \tag{B7a}$$

$$\frac{1}{\sin \pi\Omega}\left[x\left(\frac{x - x_+}{x - x_-}\right)^{\Omega} - x - \Omega L\right], \tag{B7b}$$

$$\frac{1}{\sin \pi\Omega}\left[x^2\left(\frac{x - x_+}{x - x_-}\right)^{\Omega} - x^2 - \Omega(x + x_+) + \frac{\Omega}{2}(1 - \Omega)L^2\right]. \tag{B7c}$$

Here $L = x_+ - x_-$ is the width of Ξ. The results in (B1)–(B7) are standard, and in this case, taken from [Brock and Georgiadis 2000; Brock 2004].

References

[Achenbach 1973] J. D. Achenbach, *Wave propagation in elastic solids*, North-Holland Series in Applied Mathematics and Mechanics **16**, North-Holland, Amsterdam, 1973.

[Ahmadi et al. 1983] N. Ahmadi, L. M. Keer, and T. Mura, "Non-Hertzian contact stress analysis for an elastic half-space: normal and sliding contact", *Int. J. Solids Struct.* **19**:4 (1983), 357–373.

[Barber 1983] J. R. Barber, "The solution of elasticity problems for the half-space by the method of Green and Collins", *Appl. Sci. Res.* **40**:2 (1983), 135–157.

[Barber 1992] J. R. Barber, *Elasticity*, Solid Mechanics and its Applications **12**, Kluwer, Dordrecht, 1992.

[Barwell 1979] F. T. Barwell, *Bearing systems: principles and practice*, Oxford University Press, Oxford, 1979.

[Bayer 1994] R. G. Bayer, *Mechanical wear prediction and prevention*, Mechanical Engineering **91**, Marcel Dekker, New York, 1994.

[Blau 1996] P. J. Blau, *Friction science and technology*, Mechanical Engineering **100**, Marcel Dekker, New York, 1996.

[Brock 2004] L. M. Brock, "Coupled thermoelastic analysis of perfect rolling contact: solution behavior near yield initiation", *J. Therm. Stresses* **27**:5 (2004), 383–403.

[Brock and Georgiadis 2000] L. M. Brock and H. G. Georgiadis, "Siding contact with friction of a thermoelastic solid at subsonic, transonic and supersonic speeds", *J. Therm. Stresses* **23**:7 (2000), 629–656.

[Churilov 1978] V. A. Churilov, "Action of an elliptical stamp moving at a constant speed on an elastic half-space", *Prikl. Mat. Mekh.* **42**:6 (1978), 1074–1079. In Russian; translated in *J. Appl. Math. Mech.* **42**:6 (1978), 1176–1182.

[Craggs and Roberts 1967] J. W. Craggs and A. M. Roberts, "On the motion of a heavy cylinder over the surface of an elastic half-space", *J. Appl. Mech.* (*ASME*) **34**:1 (1967), 207–209.

[Erdogan 1985] F. Erdogan, "Mixed boundary value problems in mechanics", pp. 1–86 in *Mechanics today*, vol. 4, edited by S. Nemat-Nasser, Pergamon, New York, 1985.

[Georgiadis and Barber 1993] H. G. Georgiadis and J. R. Barber, "On the super-Rayleigh/subseismic elastodynamic indentation problem", *J. Elasticity* **31**:3 (1993), 141–161.

[Hills et al. 1993] D. A. Hills, D. Nowell, and A. Sackfield, *Mechanics of elastic contacts*, Butterworth-Heinemann, Oxford, 1993.

[Johnson 1985] K. L. Johnson, *Contact mechanics*, Cambridge University Press, Cambridge, 1985.

[Kalker 1990] J. J. Kalker, *Three-dimensional bodies in elastic contact*, Solid Mechanics and its Applications **2**, Kluwer, Dordrecht, 1990.

[Peirce and Foster 1956] B. O. Peirce and R. M. Foster, *A short table of integrals*, 4th ed., Ginn, New York, 1956.

[van der Pol and Bremmer 1950] B. van der Pol and H. Bremmer, *Operational calculus: based on the two-sided Laplace integral*, Cambridge University Press, Cambridge, 1950.

[Rahman 1996] M. Rahman, "Hertz problem for a rigid punch moving across the surface of a semi-infinite elastic solid", *Z. Math. Phys.* **47**:4 (1996), 601–615.

[Sneddon 1972] I. N. Sneddon, *The use of integral transforms*, McGraw-Hill, New York, 1972.

[Stakgold 1967] I. S. Stakgold, *Boundary value problems in mathematical physics*, vol. 1, MacMillan, New York, 1967. Reprinted in Classics in Applied Mathematics **29**, SIAM, Philadelphia, 2000.

Received 6 Mar 2012. Revised 27 Mar 2012. Accepted 30 May 2012.

LOUIS MILTON BROCK: brock@engr.uky.edu
Department of Mechanical Engineering, University of Kentucky, Lexington, KY 40506-0503, United States
http://www.engr.uky.edu/me/faculty_staff/brock.html

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

jomms.net

Founded by Charles R. Steele and Marie-Louise Steele

Cover design: Alex Scorpan — Cover photo: Wikimedia Commons

See http://jomms.net for submission guidelines.

JoMMS (ISSN 1559-3959) is published in 10 issues a year. The subscription price for 2012 is US $555/year for the electronic version, and $735/year (+$60 shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720–3840.

JoMMS peer-review and production is managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
http://msp.org/

A NON-PROFIT CORPORATION

Typeset in LaTeX

Journal of Mechanics of Materials and Structures

Volume 7, No. 5 May 2012