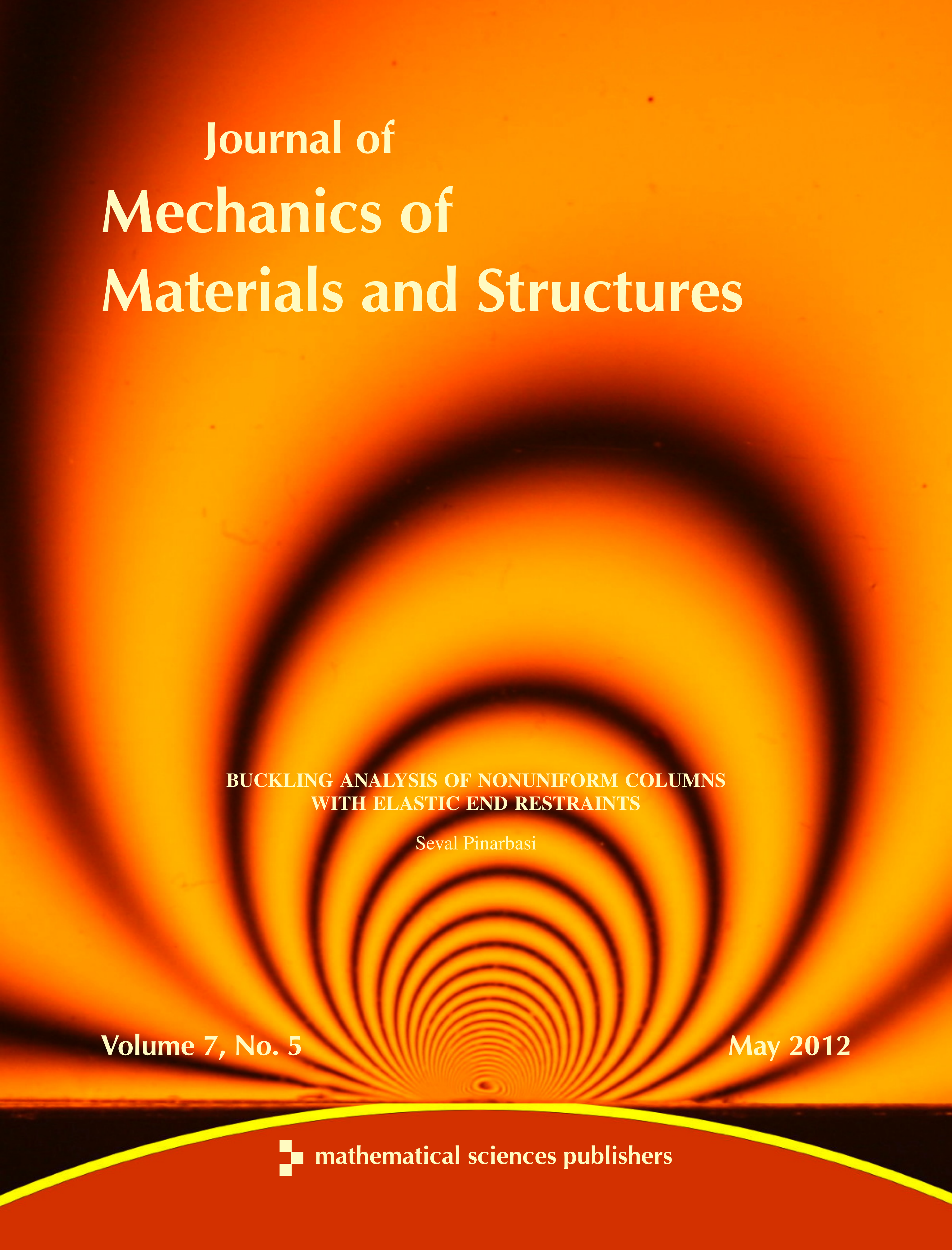

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BUCKLING ANALYSIS OF NONUNIFORM COLUMNS WITH ELASTIC END RESTRAINTS

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Since compression members, such as columns in a multistory building, are mostly the key elements in a structure, even a small decrease in their load carrying capacity can lead to catastrophic failure of the structure. A compression member has to be designed to satisfy not only the strength and serviceability requirements, but also the stability requirements. In fact, the behavior of a slender column is mostly governed by the stability limit states. In an attempt to construct ever-stronger and ever-lighter structures, many engineers currently design slender high strength columns with variable cross sections and various end conditions. Even though buckling behavior of uniform columns with ideal boundary conditions have extensively been studied, there are limited studies in the literature on buckling analysis of nonuniform columns with elastic end restraints since such an analysis requires the solution of more complex differential equations for which it is usually impractical or sometimes even impossible to obtain exact solutions. This paper shows that variational iteration method (VIM) can successfully be used for this purpose. VIM results obtained for columns of constant cross sections, for which exact results are available in the literature, agree with the exact results perfectly, verifying the efficiency of VIM in the analysis of this special type of buckling problem. It is also shown that unlike exact solution procedures, variational iteration algorithms can easily be used even when the variation of column stiffness along its length and/or the end conditions are rather complex.

1. Introduction

Compression members subjected to uniform axial loads are commonly used in many engineering applications. Columns in a multistory building, for example, are the key structural elements which support the heavy weight of the structure. Even a small decrease in their load carrying capacity can lead to catastrophic failure of the structure. Compression members differ from tension members in that the design of the former has to consider not only the strength and serviceability requirements but also the stability requirements. In fact, the behavior of a slender column is mostly governed by the stability limit states. For this reason, many international design specifications include specific provisions on stability of compression members.

Since 1744, when the Swiss mathematician Leonhard Euler published his famous buckling formula, research on stability of slender columns has increasingly continued. This continuous interest on stability problems is based mainly on the desire of constructing "ever-stronger" and "ever-lighter" structures. This "optimum structure" approach has led most engineers to design columns with higher strength and lighter weight. Unfortunately, design engineers are lack of sufficient guidance on design of nonuniform columns since most of the provisions on compression members are developed for uniform columns.

Keywords: variational iteration method, elastic buckling, stability, nonuniform column, elastic end restraints.

Elastic buckling behavior of uniform columns has extensively been investigated by many researchers. For fully developed buckling theory and the related exact solutions, one can refer to one of the classical textbooks on structural stability (e.g., [Timoshenko 1961; Chajes 1974; Wang et al. 2005; Simitses and Hodges 2006]). On the other hand, there are very few studies in the literature on columns with variable flexural stiffness since such an analysis requires the solution of more complex differential equations. In many cases, it is impractical and sometimes even impossible to obtain closed-form solutions to these problems.

When the buckling studies in the literature are examined, it is also seen that most of the studies on column buckling assume ideal end conditions. Such ideal boundary conditions can realistically model the real end conditions in some special structures, such as columns in one-story buildings, vertical and diagonal elements in truss structures and bracing elements in braced frames. However, in a general multistory building, the ends of the columns are neither hinged nor fully fixed or free. Instead, they are commonly connected to beams and the restraining effect of the beams on the column ends strongly depends on the type of the beam-to-column connection. In addition, the behavior of a column in a frame is significantly influenced from the existence and amount of the bracing members in the frame. For this reason, the buckling solutions obtained for columns of ideal end conditions cannot always be safely used for columns with elastic end restraints.

However, as in the case of buckling analysis of nonuniform columns, buckling analysis of columns with elastic end restraints is difficult to handle due to the complex boundary conditions and studies in the literature on this subject are also very limited (e.g., [Eisenberger and Clastornik 1987; Li 2000; 2001; 2003; Ozturk and Sabuncu 2005; Atanackovic and Novakovic 2006; Tan and Yuan 2008; Singh and Li 2009; Atanackovic et al. 2010]). For this reason, most design specifications offer engineers design charts, instead of design formulas, for the design of such framed columns. These "alignment" charts are drawn from the buckling (characteristic) equation derived for uniform columns with elastic end springs, which needs special techniques to solve due to its high nonlinearity, by making some assumptions on the stiffnesses of the restraints (e.g., the assumption of identical slopes at the ends of the beam). Thus, even these charts do not provide exact values. Moreover, they are applicable only to uniform columns. However, as mentioned previously, due to economical and esthetic issues, nowadays, many columns are designed with variable stiffness.

Consequently, there is a need for a practical tool to solve buckling problems of nonuniform columns with elastic end restraints. In recent years, many analytical approaches; such as, variational iteration method (VIM), homotopy perturbation method (HPM), differential quadrature method (DQM) are proposed for the solution of nonlinear equations and many researchers (e.g., [Arbabi and Li 1991; Du et al. 1996; Rosa and Franciosi 1996; Cailo and Elishakoff 2004; Civalek 2004; Aydogdu 2008; Malekzadeh and Karami 2008; Atay 2009; Coşkun 2009; 2010; Huang and Luo 2011; Ozturk and Coşkun 2011; Serna et al. 2011; Yuan and Wang 2011]) have shown that complex engineering problems, such as buckling and vibration problems, can easily be solved using these techniques. A kind of nonlinear analytical technique which was proposed by He [1999], variational iteration method (VIM) has many successful applications to various kinds of nonlinear engineering problems [Abulwafa et al. 2007; Batiha et al. 2007; Coşkun and Atay 2007; Ganji and Sadighi 2007; Ganji et al. 2007; 2008; Sweilam and Khader 2007; Coşkun and Atay 2008; Miansari et al. 2008; Shou and He 2008; Ozturk 2009; Liu and Gurram 2009; Atay 2010; Coşkun et al. 2011; Geng 2011; Yang and Chen 2011]. As shown in [Coşkun and Atay 2009;

Atay and Coşkun 2009; Okay et al. 2010; Pinarbasi 2011], VIM is an effective and powerful technique that can successfully be used in the analysis of elastic stability of compression and flexural members with variable cross sections under different loading and boundary conditions. In this paper, this powerful technique is used to determine the buckling loads of slender columns with elastic end restraints. To the best knowledge of authors, exact solutions to this problem are available only for some particular cases of uniform columns. For this reason, before analyzing the columns with variable cross sections, the buckling loads of columns with constant cross sections are determined using classical variational iteration algorithm and VIM results are compared with the exact results. After verifying the efficiency of VIM in the analysis of this special type of buckling problem, stability of columns with variable flexural stiffness is studied. In the analyses, columns with two different types of stiffness variations along their lengths; linear and exponential variations, and with various end conditions are considered. Buckling loads obtained for these nonuniform columns are computed using classical variational iteration algorithm and compared with those obtained for uniform columns.

2. Elastic buckling of columns with elastic end restraints

General buckling equation and related boundary conditions. Consider an axially loaded column of variable flexural rigidity EI along its length L with elastic end restraints as shown in Figure 1, left. Assume that the lateral displacement and rotation of the top end of the column are restrained, respectively, by an extensional spring with elastic spring constant α_0 and a rotational spring with elastic spring constant β_0. Further assume that similar springs with spring constants α_L and β_L restrain the bottom end of the column.

Figure 1, middle, shows the buckled shape of such a column under a uniaxial load of P. In the figure, M_A, M_B and V show support reactions. As can be seen from that figure, the origin of x-y coordinate system is located at the top end of the column. The equilibrium equation at an arbitrary section of the

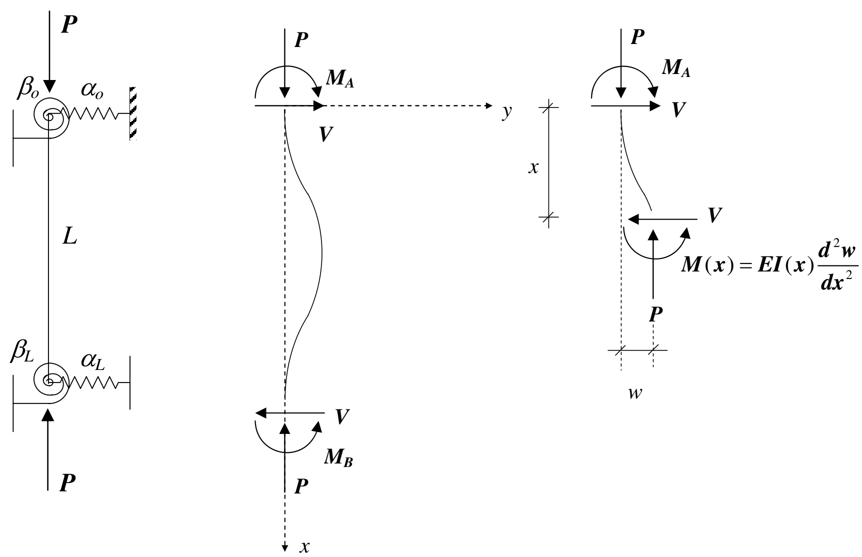


Figure 1. An axially loaded column with elastic end restraints. Left: undeformed shape. Middle: deformed (buckled) shape. Right: free body diagram for internal forces.

column can be written from the free body diagram shown in Figure 1, right as

$$M(x) + Pw(x) - Vx - M_A = 0, \tag{1}$$

where $w(x)$, or simply w, is the displacement component in y direction. Using the well-known moment-curvature relation

$$M(x) = EI(x)\frac{d^2w}{dx^2}, \tag{2}$$

Equation (1) can be rewritten as

$$EI(x)\frac{d^2w}{dx^2} + Pw = Vx + M_A. \tag{3}$$

Differentiation of (3) with respect to x gives shear force in the column at any section:

$$V = EI(x)\frac{d^3w}{dx^3} + \frac{d[EI(x)]}{dx}\frac{d^2w}{dx^2} + P\frac{dw}{dx}. \tag{4}$$

Further differentiation of (4) with respect to x yields the governing equation of the buckling problem:

$$\frac{d^4w}{dx^4} + \frac{2}{EI(x)}\frac{d[EI(x)]}{dx}\frac{d^3w}{dx^3} + \frac{1}{EI(x)}\left(P + \frac{d^2[EI(x)]}{dx^2}\right)\frac{d^2w}{dx^2} = 0. \tag{5}$$

It is to be noted that the governing equation (5) is applicable to all columns regardless of their end conditions.

Using (2) and (3), the boundary conditions at the top and bottom end of the column can be written as

$$\text{at } x = 0; \quad \beta_0\frac{dw}{dx} = EI(x)\frac{d^2w}{dx^2} \quad \text{and} \quad \alpha_0 w = -\left(EI(x)\frac{d^3w}{dx^3} + \frac{d[EI(x)]}{dx}\frac{d^2w}{dx^2} + P\frac{dw}{dx}\right) \tag{6}$$

and

$$\text{at } x = L; \quad \beta_L\frac{dw}{dx} = -EI(x)\frac{d^2w}{dx^2} \quad \text{and} \quad \alpha_L w = EI(x)\frac{d^3w}{dx^3} + \frac{d[EI(x)]}{dx}\frac{d^2w}{dx^2} + P\frac{dw}{dx}. \tag{7}$$

Columns with constant stiffness. When flexural stiffness of the column does *not* change along its length, in other words, when $EI(x) = EI$, the governing equation (5) and the related boundary conditions (6) and (7) reduce to the simpler forms

$$\frac{d^4w}{dx^4} + \frac{P}{EI}\frac{d^2w}{dx^2} = 0 \tag{8}$$

with

$$\frac{d^2w}{dx^2} - \frac{\beta_0}{EI}\frac{dw}{dx} = 0 \quad \text{and} \quad \frac{d^3w}{dx^3} + \frac{P}{EI}\frac{dw}{dx} + \frac{\alpha_0}{EI}w = 0 \qquad \text{at } x = 0, \tag{9}$$

and

$$\frac{d^2w}{dx^2} + \frac{\beta_L}{EI}\frac{dw}{dx} = 0 \quad \text{and} \quad \frac{d^3w}{dx^3} + \frac{P}{EI}\frac{dw}{dx} - \frac{\alpha_L}{EI}w = 0 \qquad \text{at } x = L. \tag{10}$$

For easier computations, these equations can be written in nondimensional form as

$$(\bar{w})'''' + \lambda(\bar{w})'' = 0 \tag{11}$$

with

$$(\bar{w})'' - \bar{\beta}_0(\bar{w})' = 0 \quad \text{and} \quad (\bar{w})''' + \lambda(\bar{w})' + \bar{\alpha}_0\bar{w} = 0 \qquad \text{at } \bar{x} = 0, \tag{12}$$

$$(\bar{w})'' + \bar{\beta}_L(\bar{w})' = 0 \quad \text{and} \quad (\bar{w})''' + \lambda(\bar{w})' - \bar{\alpha}_L\bar{w} = 0 \qquad \text{at } \bar{x} = 1, \tag{13}$$

where $\bar{w} = w/L$ and $\bar{x} = x/L$, primes denote differentiation with respect to $\bar{x}$, the normalized spring stiffnesses are

$$\bar{\beta}_0 = \frac{\beta_0 L}{EI}, \quad \bar{\beta}_L = \frac{\beta_L L}{EI}, \quad \bar{\alpha}_0 = \frac{\alpha_0 L^3}{EI} \qquad \text{and} \quad \bar{\alpha}_L = \frac{\alpha_L L^3}{EI} \tag{14}$$

and the normalized critical load is

$$\lambda = \frac{PL^2}{EI}. \tag{15}$$

Since exact solutions are available in the literature for uniform columns and since these solutions correspond to limiting conditions for variable stiffness cases, before studying the buckling problems of nonuniform columns, the buckling loads of uniform columns are to be determined and compared with the exact solutions available in the literature.

Columns with variable stiffness.

Columns with linearly varying stiffness. When flexural stiffness of the column decrease along its length linearly, i.e., when

$$EI(x) = EI(1 - b\tfrac{x}{L}), \tag{16}$$

where b is a constant determining the "sharpness" of the stiffness change along the column length, the governing equation becomes

$$\frac{d^4w}{dx^4} - \frac{2b/L}{(1-bx/L)}\frac{d^3w}{dx^3} + \frac{P}{EI(1-bx/L)}\frac{d^2w}{dx^2} = 0, \tag{17}$$

which can be written in nondimensionalized form as follows:

$$(\bar{w})'''' - \frac{2b}{(1-b\bar{x})}(\bar{w})''' + \frac{\lambda}{(1-b\bar{x})}(\bar{w})'' = 0. \tag{18}$$

Similarly, the related boundary conditions can be expressed in nondimensional form:

$$\text{at } \bar{x} = 0; \qquad (\bar{w})'' - \bar{\beta}_0(\bar{w})' = 0, \qquad (\bar{w})''' - b(\bar{w})'' + \lambda(\bar{w})' + \bar{\alpha}_0\bar{w} = 0, \tag{19}$$

$$\text{at } \bar{x} = 1; \qquad (\bar{w})'' + \frac{\bar{\beta}_L}{(1-b)}(\bar{w})' = 0, \qquad (\bar{w})''' - \frac{b}{(1-b)}(\bar{w})'' + \frac{\lambda}{(1-b)}(\bar{w})' - \frac{\bar{\alpha}_L}{(1-b)}\bar{w} = 0. \tag{20}$$

Columns with exponentially varying stiffness. If the bending stiffness of the column changes exponentially along its length, i.e., if

$$EI(x) = EIe^{-a(x/L)}, \tag{21}$$

where a is a positive constant determining the "sharpness" of the stiffness change, the governing equation becomes

$$\frac{d^4w}{dx^4} - \frac{2a}{L}\frac{d^3w}{dx^3} + \left(\frac{P}{EIe^{-a(x/L)}} + \frac{a^2}{L^2}\right)\frac{d^2w}{dx^2} = 0, \tag{22}$$

which, when written in nondimensionalized form, becomes

$$(\bar{w})'''' - 2a(\bar{w})''' + (\lambda e^{a\bar{x}} + a^2)(\bar{w})'' = 0. \tag{23}$$

Similarly, the related boundary conditions can be expressed in nondimensional form as

$$(\bar{w})'' - \bar{\beta}_0(\bar{w})' = 0 \quad \text{and} \quad (\bar{w})''' - a(\bar{w})'' + \lambda(\bar{w})' + \bar{\alpha}_0\bar{w} = 0, \qquad \text{at } \bar{x} = 0, \tag{24}$$

$$(\bar{w})'' + \bar{\beta}_L e^a(\bar{w})' = 0 \quad \text{and} \quad (\bar{w})''' - a(\bar{w})'' + \lambda e^a(\bar{w})' - \bar{\alpha}_L e^a\bar{w} = 0 \qquad \text{at } \bar{x} = 1. \tag{25}$$

3. VIM formulations for the studied buckling problems

According to the variational iteration method (VIM) [He 1999], a general homogeneous nonlinear differential equation can be written in the form

$$Lw(x) + Nw(x) = 0, \tag{26}$$

where L is a linear operator and N is a nonlinear operator, and the "correction functional" is

$$w_{n+1}(x) = w_n(x) + \int_0^x \lambda(\xi)\big(Lw_n(\xi) + N\tilde{w}_n(\xi)\big)\,d\xi. \tag{27}$$

In (27), $\lambda(\xi)$ is a general Lagrange multiplier that can be identified optimally via variational theory, w_n is the n-th approximate solution and $\tilde{w}_n$ denotes a restricted variation, i.e., $\delta\tilde{w}_n = 0$. As summarized in [He et al. 2010] for a fourth order differential equation such as the equations of the problem considered in this paper, $\lambda(\xi)$ equals to

$$\lambda(\xi) = \frac{(\xi - x)^3}{6}. \tag{28}$$

The original variational iteration algorithm proposed in [He 1999] has the iteration formula

$$w_{n+1}(x) = w_n(x) + \int_0^x \lambda(\xi)\big(Lw_n(\xi) + Nw_n(\xi)\big)\,d\xi. \tag{29}$$

In a recent paper, He et al. [2010] proposed two additional variational iteration algorithms for solving various types of differential equations. These algorithms can be expressed as follows:

$$w_{n+1}(x) = w_0(x) + \int_0^x \lambda(\xi)\big(Nw_n(\xi)\big)\,d\xi, \tag{30}$$

$$w_{n+2}(x) = w_{n+1}(x) + \int_0^x \lambda(\xi)\big(Nw_{n+1}(\xi) - Nw_n(\xi)\big)\,d\xi. \tag{31}$$

Thus, the three VIM iteration algorithms for (18), as an example, can be written as

$$\begin{aligned}
\bar{w}_{n+1}(x) = \bar{w}_n(x) \quad &+ \int_0^x \frac{(\xi - x)^3}{6}\left(\bar{w}_n''''(\xi) - \frac{2b}{1-b\xi}\bar{w}_n'''(\xi) + \frac{\lambda}{1-b\xi}\bar{w}_n''(\xi)\right)d\xi,\\
\bar{w}_{n+1}(x) = \bar{w}_0(x) \quad &+ \int_0^x \frac{(\xi - x)^3}{6}\left(-\frac{2b}{1-b\xi}\bar{w}_n'''(\xi) + \frac{\lambda}{1-b\xi}\bar{w}_n''(\xi)\right)d\xi,\\
\bar{w}_{n+2}(x) = \bar{w}_{n+1}(x) \quad &+ \int_0^x \frac{(\xi - x)^3}{6}\left(-\frac{2b}{1-b\xi}(\bar{w}_{n+1}'''(\xi) - \bar{w}_n'''(\xi)) + \frac{\lambda}{1-b\xi}(\bar{w}_{n+1}''(\xi) - \bar{w}_n''(\xi))\right)d\xi.
\end{aligned}$$

Similar algorithms can easily be written for (11) and (23). In order to determine the most effective VIM algorithm to be used in the current study, one single case of a buckling equation (linearly varying stiffness case with $b = 0.3$) is solved using all three algorithms. Parallel to the findings of Pinarbasi [2011], all iteration algorithms yield exactly the same results. For this reason, the classical VIM algorithm is decided to be used throughout the study.

4. Buckling loads for columns with elastic restraints

The general buckling problems formulated in Section 2 are specialized to three different end conditions shown in Figure 2. In Case I (left), the bottom end of the column which is free to rotate ($\beta_L \to 0$) is laterally restrained with an extensional spring (with α_L) while the top end of the column is fixed ($\alpha_0 \to \infty$, $\beta_0 \to \infty$). Such a column can exist in a single story frame where the beam-to-column connections are simple shear connections. Case II (Figure 2, middle) investigates an interior column in a multistory building whose lateral stiffness is provided by laterally stiff elements such as lateral bracings or reinforced concrete walls. In such a "sway-prevented structure", the relative lateral displacement of one end of the column with respect to the other end is so small that it is neglected. For this reason, in Case II, the stiffnesses of linear springs are assumed to approach infinity ($\alpha_0 \to \infty$, $\alpha_L \to \infty$) while rotational spring stiffnesses (β_0 and β_L) are let have any value. In Case III (Figure 2, right), the relative lateral displacement of one end of the column with respect to the other end is *not* small so it cannot be neglected. Such columns can be seen in a "sway-permitted" structure whose lateral stiffness is provided only by flexural stiffnesses of frame members. For simplicity, the lateral stiffness of the extensional spring at the top end of the column is taken zero, while rotational spring stiffnesses (β_0 and β_L) can have any value.

Columns with constant stiffness. The exact solution to the differential equation (11) has the form

$$\bar{w} = C_1 \sin\sqrt{\lambda}\bar{x} + C_2 \cos\sqrt{\lambda}\bar{x} + C_3\bar{x} + C_4, \tag{32}$$

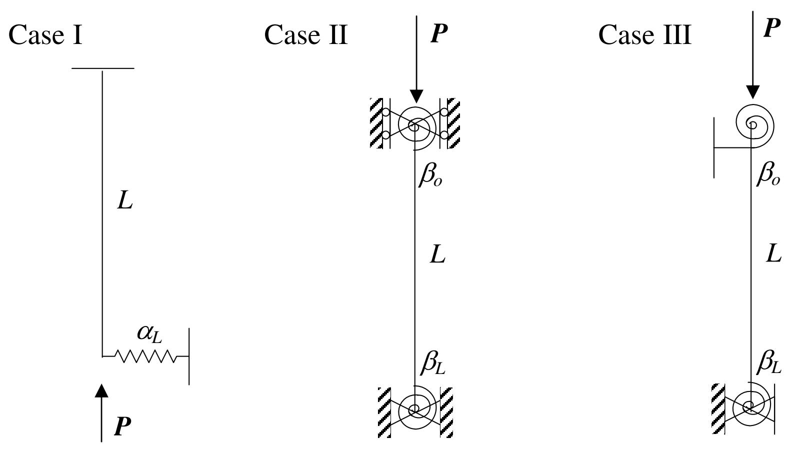
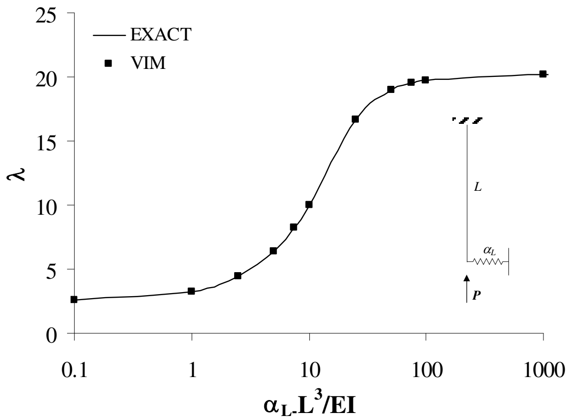


Figure 2. The three cases (boundary conditions) studied in the paper. Case I: $\alpha_0 \to \infty$, $\beta_0 \to \infty$, $\beta_L \to 0$. Case II: $\alpha_0 \to \infty$, $\alpha_L \to \infty$. Case III: $\alpha_0 \to 0$, $\alpha_L \to \infty$.



	λ	
$\alpha_L L^3/EI$	EXACT	VIM
0.1	2.54841	2.54841
1	3.27349	3.27349
2.5	4.46442	4.46442
5	6.39207	6.39207
7.5	8.23092	8.23092
10	9.95634	9.95634
25	16.6435	16.6435
50	18.9922	18.9922
75	19.4958	19.4958
100	19.7035	19.7035
1000	20.1496	20.1496

Figure 3. Case I — columns with constant stiffness — variation of normalized buckling load with normalized linear spring stiffness.

where C_i (i=1,2,3,4) are evaluated from the related boundary conditions. In Case I, the boundary conditions are

$$[(\bar{w})']_{\bar{x}=0} = 0, \quad [\bar{w}]_{\bar{x}=0} = 0, \quad [(\bar{w})'']_{\bar{x}=1} = 0 \quad \text{and} \quad [(\bar{w})''' + \lambda(\bar{w})' - \bar{\alpha}_L \bar{w}]_{\bar{x}=1} = 0. \tag{33}$$

By substituting (32) into these boundary conditions, four homogeneous equations are obtained. These equations can be put into matrix form:

$$[M(\lambda)]\{C\} = \{0\}, \tag{34}$$

where $\{C\} = \{C_1\ C_2\ C_3\ C_4\}^T$. Thus, the problem reduces to an eigenvalue problem. For a nontrivial solution, the determinant of the coefficient matrix has to be zero. The smallest possible real root of the characteristic equation, which is obtained by equating the determinant of the coefficient matrix to zero, gives the nondimensional buckling load in the first buckling mode. For some particular values of α_L, the exact values are calculated and plotted in Figure 3, in a semilogarithmic scale.

Even though the differential equation to be solved in this case is relatively simple, when the exact solution is tried to be obtained, finding the smallest root of the resulting characteristic equation which contains trigonometric functions can be somewhat difficult. It is observed that the result is very sensitive to the initial guess. So, one should be aware of that a couple of trials may be required to find the correct root of the characteristic equation.

The same problem is also studied using VIM. The initial approximation is selected as a third degree polynomial with four unknown coefficients A_i (i=1,2,3,4):

$$\bar{w}_0 = A_1(\bar{x})^3 + A_2(\bar{x})^2 + A_3\bar{x} + A_4. \tag{35}$$

Using the first iteration algorithm and conducting nine iterations, $\bar{w}_9$ is obtained. Through substitution in the boundary conditions (33), four homogeneous equations are obtained. Similar to the exact solution procedure, by making the determinant of the coefficient matrix of these equations equal to zero, the characteristic equation for the related bucking problem is obtained. The roots of the characteristic equation give the normalized buckling loads. Since the characteristic equation is a polynomial, one can easily

	λ								
$\beta_0 L/EI$	$\beta_L=\beta_0$			$\beta_L=0$			$\beta_L\to\infty$		
	Exact	VIM (9 iter)	VIM (17 iter)	Exact	VIM (9 iter)	VIM (17 iter)	Exact	VIM (9 iter)	VIM (17 iter)
0	9.870	9.8696	9.8696	9.870	9.8696	9.8696	20.191	20.1907	20.1907
0.5	11.772	11.7719	11.7719	10.798	10.7978	10.7978	21.659	21.6594	21.6594
1	13.492	13.4924	13.4924	11.598	11.5982	11.5982	22.969	22.9688	22.9688
2	16.463	16.4634	16.4634	12.894	12.8944	12.8944	25.182	25.1822	25.1822
4	20.957	20.9568	20.9568	14.660	14.6602	14.6602	28.397	28.3971	28.3969
10	28.168	28.1683	28.1677	17.076	17.0763	17.0763	33.153	33.1546	33.1532
20	**30.355**	**32.7846**	**32.7819**	18.417	18.4173	18.4173	35.902	35.9059	35.9019
∞	39.478	39.4916	39.4784	20.191	20.1908	20.1907	39.478	39.4916	39.4784

Table 1. Case II — columns with constant stiffness — comparison of VIM solutions with exact solutions [Wang et al. 2005].

	λ								
$\beta_0 L/EI$	$\beta_L=\beta_0$			$\beta_L=0$			$\beta_L\to\infty$		
	Exact	VIM (9 iter)	VIM (17 iter)	Exact	VIM (9 iter)	VIM (17 iter)	Exact	VIM (9 iter)	VIM (17 iter)
0	0.000	0.0000	0.0000	0.000	0.0000	0.0000	2.4674	2.46740	2.46740
0.5	0.922	0.9220	0.9220	0.4268	0.42676	0.42676	3.3731	3.37309	3.37309
1	1.7071	1.7071	1.7071	0.7402	0.74017	0.74017	4.1159	4.11586	4.11586
2	2.9607	2.9607	2.9607	1.1597	1.15966	1.15966	5.2392	5.23920	5.23920
4	4.6386	4.6386	4.6386	1.5992	1.59919	1.59919	6.6071	6.60712	6.60712
10	6.9047	6.9047	6.9047	**2.0517**	**2.04167**	**2.04167**	8.1955	8.19547	8.19547
20	8.1667	8.1667	8.1667	2.2384	2.23840	2.23840	8.9583	8.95831	8.95831
∞	9.8696	9.8696	9.8696	2.4674	2.46740	2.46740	9.8696	9.86960	9.86960

Table 2. Case III — columns with constant stiffness — comparison of VIM solutions with exact solutions [Wang et al. 2005].

compute its all roots. Selecting the smallest root is no more tedious. For comparison, VIM results are also plotted in Figure 3, which shows perfect agreement with the exact results.

For Case II and Case III, the characteristic equations of the buckling problems were derived by Wang et al. [2005]. They also tabulated exact results for some particular values of spring stiffnesses. In order to evaluate the efficiency of VIM, approximate solutions are obtained for the same values of spring stiffnesses using classical iteration algorithm and VIM results are compared with the exact results given in [Wang et al. 2005] in Tables 1 and 2. The same initial approximation chosen in Case I, namely, Equation (35), is used also in these two cases. Normalized buckling loads are computed for two different number of iterations; nine and seventeen.

From (12) and (13), for uniform columns, the boundary conditions for Case II become

$$[(\bar{w})'' - \bar{\beta}_0(\bar{w})']_{\bar{x}=0} = 0, \quad [\bar{w}]_{\bar{x}=0} = 0, \quad [(\bar{w})'' + \bar{\beta}_L(\bar{w})']_{\bar{x}=1} = 0 \quad \text{and} \quad [\bar{w}]_{\bar{x}=1} = 0 \tag{36}$$

and the boundary conditions for Case III become

$$[(\bar{w})'' - \bar{\beta}_0(\bar{w})']_{\bar{x}=0} = 0, \quad [(\bar{w})''' + \lambda(\bar{w})']_{\bar{x}=0} = 0, \quad [(\bar{w})'' + \bar{\beta}_L(\bar{w})']_{\bar{x}=1} = 0, \quad [\bar{w}]_{\bar{x}=1} = 0. \tag{37}$$

From Tables 1 and 2, it can be seen that even the VIM results obtained with nine iterations are sufficiently close to the exact results. Still, by increasing the number of iterations, the exact results can be obtained even when spring stiffnesses converge infinity. One can see that only one result in Table 1, shown in bold, does not match. This corresponds to the case when β_0= β_L= 20. Considering that all other results match perfectly, this discrepancy may be due to a misprint in the reference. A similar, but smaller, mismatch occurs in Table 2, when β_0= 10 and β_L= 0.

Figure 3 and Tables 1 and 2 clearly show that VIM is a powerful technique in predicting buckling loads of uniform columns with elastic restraints. The excellent match of VIM solutions with exact results also encourages the use of this practical technique in buckling problems of nonuniform columns, whose exact solutions are impractical or sometimes even impossible to derive.

Columns with variable stiffness. Although it is somewhat easy to derive closed form solutions for buckling problems of uniform columns, which has a fourth order homogenous differential equation with constant coefficients, it may be relatively difficult to obtain exact results for buckling of nonuniform columns. To the best knowledge of author, there are no such solutions available in the literature. For this reason, in this section of the paper, only the VIM results obtained using the classical VIM iteration algorithm will be presented.

Similar to the constant stiffness cases studied in the previous section, the iterations in variable stiffness cases are initiated with the simple approximation given in (35). To simplify the integration processes, the variable coefficients in the iteration integrals are expanded in series using nine terms and the normalized buckling loads are obtained from ninth approximate solution.

For each case illustrated in Figure 2, the normalized buckling loads of columns with variable (linearly/exponentially varying) stiffness are computed using classical VIM iteration algorithm for various values of normalized spring stiffness(es) (i.e., for various values of α_L for Case I and of β_0 and β_L for Case II and Case III) and for various degrees of stiffness changes (i.e., for various values of b or a). The numerical results are presented in Tables 3 and 4 for Case I, Tables 5–10 for Case II, and Tables 11–16 for Case III. The tabulated results can be used directly by structural engineers designing columns with linearly or exponentially varying stiffness along their lengths restrained with nonclassical elastic end supports.

It can be valuable to investigate the effect of the degree of stiffness nonlinearity on buckling loads of nonuniform columns by plotting some representative graphs from the above tabulated results. In the following plots, four particular cases of linear (b={0, 0.3, 0.5, 0.7}) and exponential (a={0, 0.5, 1.0, 2.0}) stiffness changes are studied for each end conditions illustrated in Figure 2. As can be inferred from Figure 4, whose two parts plot the variation of bending stiffness of a column with the selected stiffness changes through its length, the cases for b=0 and a=0 actually correspond to the uniform stiffness cases.

	$\alpha_L L^3/EI$								
b	0	0.1	0.25	0.5	1	2.5	5	10	100
0.0	2.4674	2.5484	2.6698	2.8716	3.2735	4.4644	6.3921	9.9563	19.7035
0.1	2.3928	2.4734	2.5940	2.7946	3.1940	4.3761	6.2843	9.7821	18.7228
0.2	2.3155	2.3956	2.5154	2.7147	3.1112	4.2835	6.1696	9.5904	17.7134
0.3	2.2351	2.3145	2.4335	2.6313	3.0246	4.1857	6.0464	9.3767	16.6704
0.4	2.1511	2.2299	2.3479	2.5440	2.9337	4.0819	5.9128	9.1353	15.5871
0.5	2.0643	2.1424	2.2593	2.4534	2.8389	3.9723	5.7681	8.8606	14.4553
0.6	1.9801	2.0574	2.1730	2.3650	2.7460	3.8630	5.6184	8.5544	13.2674
0.7	1.9170	1.9936	2.1083	2.2985	2.6757	3.7777	5.4922	8.2475	12.0251
0.8	1.8623	1.9384	2.0522	2.2410	2.6147	3.7020	5.3692	7.8866	10.6673

Table 3. Case I — columns with linearly varying stiffness.

	$\alpha_L L^3/EI$								
a	0	0.1	0.25	0.5	1	2.5	5	10	100
0.00	2.4674	2.5484	2.6698	2.8716	3.2735	4.4644	6.3921	9.9563	19.7035
0.25	2.2868	2.3667	2.4863	2.6851	3.0807	4.2499	6.1288	9.5241	17.4010
0.50	2.1121	2.1121	2.3085	2.5041	2.8929	4.0380	5.8616	9.0572	15.3231
0.75	1.9438	2.0211	2.1369	2.3290	2.7104	3.8290	5.5895	8.5514	13.4555
1.00	1.7821	1.8581	1.9717	2.1601	2.5335	3.6230	5.3114	8.0046	11.7834
1.50	1.4803	1.5532	1.6622	1.8424	2.1980	3.2199	4.7329	6.8056	8.9663
2.00	1.2105	1.2800	1.3837	1.5546	1.8894	2.8285	4.1188	5.5513	6.7559
2.50	0.9780	1.0435	1.1409	1.3005	1.6097	2.4465	3.4737	4.3719	5.0448
3.00	0.7850	0.8451	0.9340	1.0789	1.3559	2.0716	2.8276	3.3552	3.7360

Table 4. Case I — columns with exponentially varying stiffness.

	$\beta_0 L/EI$								
b	0	0.1	0.25	0.5	1	2	4	10	100
0.0	9.8696	10.0666	10.3511	10.7978	11.5982	12.8944	14.6602	17.0763	19.7970
0.1	9.3716	9.5634	9.8402	10.2741	11.0493	12.2985	13.9866	16.2690	18.8042
0.2	8.8635	9.0498	9.3183	9.7384	10.4868	11.6860	13.2922	15.4364	17.7834
0.3	8.3434	8.5237	8.7832	9.1885	9.9079	11.0537	12.5733	14.5737	16.7298
0.4	7.8087	7.9824	8.2321	8.6213	9.3093	10.3974	11.8247	13.6751	15.6365
0.5	7.2560	7.4224	7.6614	8.0327	8.6863	9.7116	11.0399	12.7326	14.4948
0.6	6.6812	6.8396	7.0665	7.4180	8.0334	8.9897	10.2107	11.7371	13.2950
0.7	6.0825	6.2318	6.4451	6.7745	7.3475	8.2278	9.3329	10.6842	12.0333
0.8	5.4696	5.6090	5.8077	6.1131	6.6402	7.4393	8.4228	9.5952	10.7371

Table 5. Case II — columns with linearly varying stiffness, β_L=0.

	$\beta_0 L/EI$								
b	0	0.1	0.25	0.5	1	2	4	10	100
0.0	9.8696	10.2656	10.8447	11.7719	13.4924	16.4634	20.9568	28.1683	37.9572
0.1	9.3716	9.7676	10.3458	11.2696	12.9768	15.9024	20.2681	27.1131	36.0973
0.2	8.8635	9.2599	9.8377	10.7582	12.4511	15.3254	19.5477	25.9988	34.1762
0.3	8.3434	8.7407	9.3187	10.2362	11.9131	14.7283	20.2726	24.8132	32.1791
0.4	7.8087	8.2078	8.7867	9.7015	11.3601	14.1053	17.9768	23.5401	30.0877
0.5	7.2560	7.6579	8.2386	9.1507	10.7869	13.4462	17.0968	22.1551	27.8773
0.6	6.6812	7.0870	7.6700	8.5778	10.1829	12.7305	16.1153	20.6196	25.5111
0.7	6.0825	6.4914	7.0740	7.9702	9.5239	11.9159	14.9736	18.8703	22.9324
0.8	5.4696	5.8740	6.4438	7.3060	8.7630	10.9259	13.5782	16.8166	20.0636

Table 6. Case II — columns with linearly varying stiffness, $\beta_L = \beta_0$.

	$\beta_0 L/EI$								
b	0	0.1	0.25	0.5	1	2	4	10	100
0.0	20.1907	20.4982	20.9462	21.6594	22.9688	25.1822	28.3971	33.1546	38.7118
0.1	19.1685	19.4679	19.9039	20.5971	21.8669	24.0044	27.0859	31.5885	36.7606
0.2	18.1179	18.4087	18.8318	19.5035	20.7310	22.7876	25.7281	29.9663	34.7519
0.3	17.0330	17.3144	17.7236	18.3722	19.5541	21.5237	24.3143	28.2765	32.6709
0.4	15.9057	16.1770	16.5709	17.1942	18.3265	20.2020	22.8317	26.5041	30.4993
0.5	14.7245	14.9845	15.3615	15.9569	17.0343	18.8066	21.2619	24.6272	28.2130
0.6	13.4714	13.7186	14.0766	14.6405	15.6564	17.3134	19.5769	22.6134	25.7757
0.7	12.1185	12.3509	12.6868	13.2143	14.1593	15.6846	17.7327	20.4122	23.1324
0.8	10.6238	10.8384	11.1478	11.6318	12.4924	13.8631	15.6644	17.9511	20.2078

Table 7. Case II — columns with linearly varying stiffness, $\beta_L \to \infty$.

	$\beta_0 L/EI$								
a	0	0.1	0.25	0.5	1	2	4	10	100
0.00	9.8696	10.0666	10.3511	10.7978	11.5982	12.8944	14.6602	17.0763	19.7970
0.25	8.6951	8.8800	9.1463	9.5628	10.3039	11.4894	13.0723	15.1763	17.4678
0.50	7.6345	7.8078	8.0570	8.4449	9.1301	10.2115	11.6253	13.4490	15.3706
0.75	6.6807	6.8432	7.0761	7.4371	8.0696	9.0535	10.3113	11.8848	13.4891
1.00	5.8266	5.9789	6.1965	6.5322	7.1152	8.0080	9.1226	10.4735	11.8071
1.50	4.3885	4.5224	4.7123	5.0019	5.4948	6.2237	7.0879	8.0690	8.9779
2.00	3.2634	3.3813	3.5470	3.7962	4.2104	4.7983	5.4560	6.1537	6.7615
2.50	2.3955	2.4998	2.6448	2.8592	3.2054	3.6734	4.1640	4.6491	5.0474
3.00	1.7329	1.8261	1.9540	2.1391	2.4273	2.7948	3.1528	3.4818	3.7373

Table 8. Case II — columns with exponentially varying stiffness, β_L=0.

	$\beta_0 L/EI$								
a	0	0.1	0.25	0.5	1	2	4	10	100
0.00	9.8696	10.2656	10.8447	11.7719	13.4924	16.4634	20.9568	28.1683	37.9572
0.25	8.6951	9.0912	9.6684	10.5871	12.2742	15.1312	19.3093	25.6463	33.6012
0.50	7.6345	8.0318	8.6080	9.5184	11.1681	13.8959	17.7352	23.2317	29.6633
0.75	6.6807	7.0803	7.6563	8.5575	10.1638	12.7453	16.2284	20.9375	26.1113
1.00	5.8266	6.2294	6.8054	7.6958	9.2507	11.6682	14.7864	18.7760	22.9202
1.50	4.3885	4.8003	5.3753	6.2343	7.6558	9.6994	12.1029	14.8873	17.5178
2.00	3.2634	3.6860	4.2546	5.0617	6.3049	7.9434	9.7142	11.6021	13.2518
2.50	2.3955	2.8297	3.3810	4.1094	5.1398	6.3912	7.6528	8.9047	9.9273
3.00	1.7329	2.1777	2.6960	3.3199	4.1300	5.0524	5.9299	6.7435	7.3689

Table 9. Case II — columns with exponentially varying stiffness, $\beta_L = \beta_0$.

	$\beta_0 L/EI$								
a	0	0.1	0.25	0.5	1	2	4	10	100
0.00	20.1907	20.4982	20.9462	21.6594	22.9688	25.1822	28.3971	33.1546	38.7118
0.25	17.7938	18.0823	18.5020	19.1681	20.3842	22.4186	25.3196	29.4819	34.1545
0.50	15.6379	15.9085	16.3014	16.9228	18.0507	19.9163	22.5250	26.1504	30.0674
0.75	13.7046	13.9583	14.3258	14.9051	15.9497	17.6565	19.9937	23.1361	26.4052
1.00	11.9763	12.2141	12.5577	13.0972	14.0633	15.6210	17.7068	20.4171	23.1330
1.50	9.0679	9.2767	9.5767	10.0436	10.8665	12.1541	13.7950	15.7797	17.6281
2.00	6.7879	6.9712	7.2329	7.6360	8.3327	9.3851	10.6528	12.0744	13.3081
2.50	5.0249	5.1861	5.4144	5.7615	6.3477	7.1965	8.1554	9.1490	9.9556
3.00	3.6800	3.8224	4.0220	4.3206	4.8104	5.4843	6.1918	6.8675	7.3829

Table 10. Case II — columns with exponentially varying stiffness, $\beta_L \to \infty$.

	$\beta_0 L/EI$								
b	0	0.1	0.25	0.5	1	2	4	10	100
0.0	0.0000	0.0968	0.2305	0.4268	0.7402	1.1597	1.5992	2.0417	2.4188
0.1	0.0000	0.0967	0.2300	0.4250	0.7347	1.1453	1.5703	1.9922	2.3473
0.2	0.0000	0.0966	0.2295	0.4232	0.7288	1.1300	1.5395	1.9403	2.2732
0.3	0.0000	0.0965	0.2289	0.4211	0.7224	1.1133	1.5067	1.8856	2.1959
0.4	0.0000	0.0964	0.2283	0.4189	0.7153	1.0953	1.4714	1.8276	2.1148
0.5	0.0000	0.0963	0.2276	0.4164	0.7075	1.0754	1.4331	1.7655	2.0293
0.6	0.0000	0.0961	0.2268	0.4136	0.6987	1.0534	1.3912	1.6987	1.9384
0.7	0.0000	0.0960	0.2260	0.4106	0.6891	1.0291	1.3455	1.6269	1.8420
0.8	0.0000	0.0961	0.2255	0.4079	0.6795	1.0041	1.2981	1.5528	1.7433

Table 11. Case III — columns with linearly varying stiffness, β_L=0.

	$\beta_0 L/EI$								
b	0	0.1	0.25	0.5	1	2	4	10	100
0.0	0.0000	0.1967	0.4798	0.9220	1.7071	2.9607	4.6386	6.9047	9.4865
0.1	0.0000	0.1965	0.4788	0.9180	1.6933	2.9182	4.5311	6.6604	9.0232
0.2	0.0000	0.1963	0.4775	0.9134	1.6773	2.8698	4.4121	6.3987	8.5430
0.3	0.0000	0.1961	0.4760	0.9078	1.6585	2.8142	4.2791	6.1165	8.0429
0.4	0.0000	0.1957	0.4741	0.9010	1.6358	2.7492	4.1288	5.8099	7.5185
0.5	0.0000	0.1952	0.4715	0.8921	1.6075	2.6713	3.9561	5.4726	6.9637
0.6	0.0000	0.1942	0.4673	0.8791	1.5695	2.5738	3.7520	5.0946	6.3687
0.7	0.0000	0.1916	0.4587	0.8569	1.5131	2.4437	3.5003	4.6591	5.7178
0.8	0.0000	0.1845	0.4392	0.8142	1.4207	2.2575	3.1740	4.1383	4.9853

Table 12. Case III — columns with linearly varying stiffness, $\beta_L = \beta_0$.

	$\beta_0 L/EI$								
b	0	0.1	0.25	0.5	1	2	4	10	100
0.0	2.4674	2.6634	2.9430	3.3731	4.1159	5.2392	6.6071	8.1955	9.6752
0.1	2.2928	2.4857	2.7604	3.1821	3.9076	4.9969	6.3089	7.8103	9.1890
0.2	2.1154	2.3048	2.5743	2.9869	3.6937	4.7466	5.9993	7.4106	8.6869
0.3	1.9346	2.1203	2.3839	2.7866	3.4729	4.4864	5.6760	6.9935	8.1658
0.4	1.7495	1.9310	2.1883	2.5800	3.2437	4.2140	5.3359	6.5553	7.6214
0.5	1.5589	1.7357	1.9857	2.3650	3.0036	3.9262	4.9746	6.0903	7.0476
0.6	1.3608	1.5323	1.7740	2.1390	2.7488	3.6176	4.5850	5.5902	6.4348
0.7	1.1522	1.3172	1.5490	1.8972	2.4732	3.2797	4.1559	5.0413	5.7679
0.8	0.9276	1.0846	1.3042	1.6316	2.1661	2.8978	3.6682	4.4207	5.0217

Table 13. Case III — columns with linearly varying stiffness, $\beta_L \to \infty$.

	$\beta_0 L/EI$								
a	0	0.1	0.25	0.5	1	2	4	10	100
0.00	0.0000	0.0968	0.2305	0.4268	0.7402	1.1597	1.5992	2.0417	2.4188
0.25	0.0000	0.0965	0.2293	0.4224	0.7265	1.1240	1.5278	1.9208	2.2456
0.50	0.0000	0.0963	0.2279	0.4177	0.7117	1.0862	1.4542	1.8001	2.0774
0.75	0.0000	0.0961	0.2265	0.4125	0.6957	1.0462	1.3787	1.6803	1.9148
1.00	0.0000	0.0958	0.2248	0.4068	0.6783	1.0041	1.3015	1.5617	1.7582
1.50	0.0000	0.0951	0.2210	0.3936	0.6392	0.9132	1.1436	1.3307	1.4644
2.00	0.0000	0.0943	0.2162	0.3775	0.5934	0.8141	0.9835	1.1115	1.1986
2.50	0.0000	0.0931	0.2098	0.3569	0.5390	0.7067	0.8237	0.9064	0.9603
3.00	0.0000	0.0908	0.1999	0.3285	0.4722	0.5896	0.6643	0.7142	0.7457

Table 14. Case III — columns with exponentially varying stiffness, β_L=0.

					$\beta_0 L/EI$				
a	0	0.1	0.25	0.5	1	2	4	10	100
0.00	0.0000	0.1967	0.4798	0.9220	1.7071	2.9607	4.6386	6.9047	9.4865
0.25	0.0000	0.1963	0.4772	0.9120	1.6727	2.8559	4.3782	6.3250	8.4094
0.50	0.0000	0.1957	0.4740	0.9004	1.6335	2.7421	4.1111	5.7711	7.4488
0.75	0.0000	0.1951	0.4701	0.8867	1.5895	2.6202	3.8410	5.2466	6.5926
1.00	0.0000	0.1943	0.4656	0.8709	1.5404	2.4917	3.5716	4.7537	5.8301
1.50	0.0000	0.1921	0.4538	0.8323	1.4289	2.2214	3.0468	3.8658	4.5463
2.00	0.0000	0.1892	0.4382	0.7845	1.3033	1.9450	2.5555	3.1061	3.5282
2.50	0.0000	0.1852	0.4184	0.7288	1.1695	1.6731	2.1077	2.4651	2.7204
3.00	0.0000	0.1799	0.3942	0.6669	1.0314	1.4111	1.7059	1.9282	2.0780

Table 15. Case III — columns with exponentially varying stiffness, $\beta_L = \beta_0$.

					$\beta_0 L/EI$				
a	0	0.1	0.25	0.5	1	2	4	10	100
0.00	2.4674	2.6634	2.9430	3.3731	4.1159	5.2392	6.6071	8.1955	9.6752
0.25	2.0666	2.2553	2.5237	2.9344	3.6366	4.6801	5.9165	7.3018	8.5478
0.50	1.7254	1.9076	2.1656	2.5580	3.2220	4.1898	5.3033	6.5057	7.5498
0.75	1.4364	1.6124	1.8608	2.2361	2.8638	3.7595	4.7581	5.7959	6.6662
1.00	1.1924	1.3628	1.6022	1.9614	2.5545	3.3812	4.2723	5.1623	5.8834
1.50	0.8153	0.9759	1.1990	1.5284	2.0559	2.7527	3.4489	4.0890	4.5742
2.00	0.5525	0.7047	0.9134	1.2152	1.6803	2.2555	2.7820	3.2263	3.5426
2.50	0.3722	0.5171	0.7127	0.9880	1.3919	1.8521	2.2336	2.5290	2.7278
3.00	0.2507	0.3891	0.5720	0.8203	1.1621	1.5137	1.7751	1.9618	2.0818

Table 16. Case III — columns with exponentially varying stiffness, $\beta_L \to \infty$.

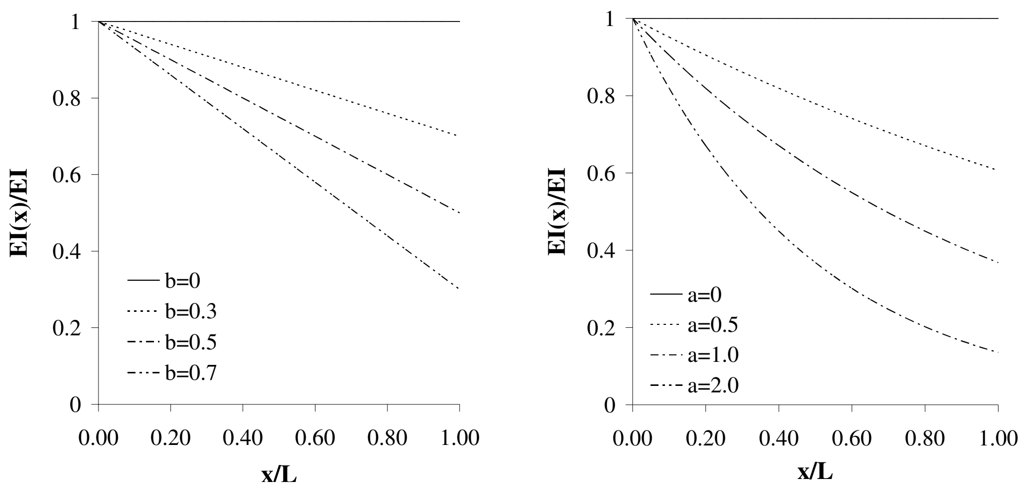


Figure 4. Stiffness variations studied in the paper in more detail. Left: linear variation in stiffness. Right: exponential variation in stiffness.

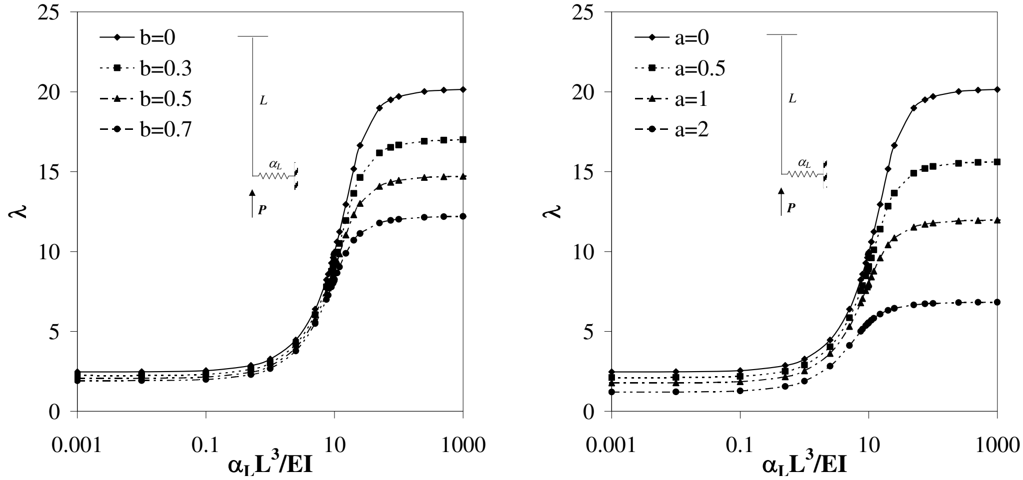


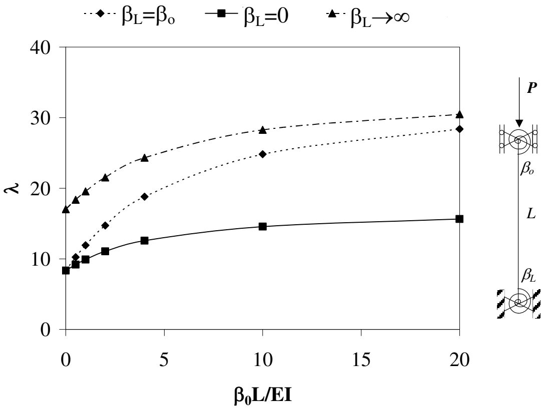
Figure 5. Case I — columns with variable stiffness: variation of normalized buckling load with normalized linear spring stiffness. Left: linear variation in stiffness. Right: exponential variation in stiffness.

Figure 5 shows the variation of normalized buckling load with normalized linear spring stiffness for columns of variable stiffness with the end conditions considered in Case I. Recalling that the cases for b=0 and a=0 correspond to uniform columns, it can be seen from these graphs that as the sharpness of the stiffness variation (a or b) increases, the buckling load of the column decreases considerably especially if the spring stiffness is large. Figure 5 also shows that there is no need to increase the spring stiffness beyond a critical value because further increases will result in no change in buckling load. For a particular case, this "critical" value of the spring stiffness can easily be determined using VIM.

Figures 6 and 7 show the variation of normalized buckling load with normalized rotational spring stiffnesses for columns of, respectively, linearly and exponentially variable flexural stiffness with the boundary conditions considered in Case II. Similarly, Figures 8 and 9 show the effect of rotational spring stiffnesses on normalized buckling load for columns of, respectively, linearly and exponentially variable flexural stiffness with the boundary conditions considered in Case III. Comparison of the graphs presented in Figures 6 and 7 with those given in Figures 8 and 9 clearly shows the importance of the lateral bracing of the columns. Case II columns with lateral bracing have much larger elastic buckling loads compared to Case III columns which are free to displace in lateral direction.

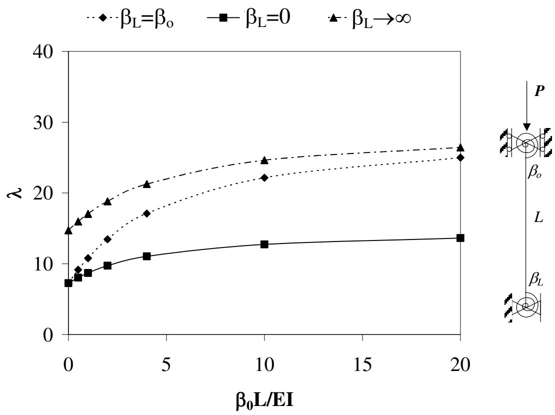
5. Conclusions

In an attempt to construct ever-stronger and ever-lighter structures, many engineers currently design slender high strength columns with variable cross sections and various end conditions. Even though buckling behavior of uniform columns with ideal boundary conditions are extensively studied, there are limited studies in the literature on buckling analysis of nonuniform columns with elastic end restraints. This is due to the fact that such an analysis requires the solution of more complex differential equations for which it is usually impractical or sometimes even impossible to obtain exact solutions.



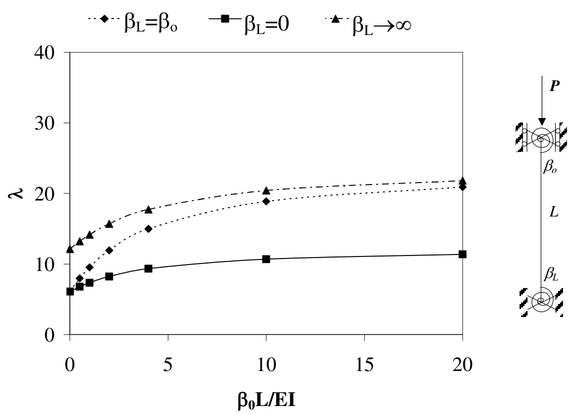
$\beta_0 L/EI$	λ		
	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	8.34341	8.34341	17.033
0.5	10.2362	9.18849	18.3722
1	11.9131	9.90792	19.5541
2	14.7283	11.0537	21.5237
4	18.7877	12.5733	24.3143
10	24.8132	14.5737	28.2765
20	28.3848	15.6475	30.4787

a. b=0.3



$\beta_0 L/EI$	λ		
	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	7.25597	7.25597	14.7245
0.5	9.1507	8.03271	15.9569
1	10.7869	8.68631	17.0343
2	13.4462	9.71162	18.8066
4	17.0968	11.0399	21.2619
10	22.1551	12.7326	24.6272
20	24.9815	13.6171	26.4395

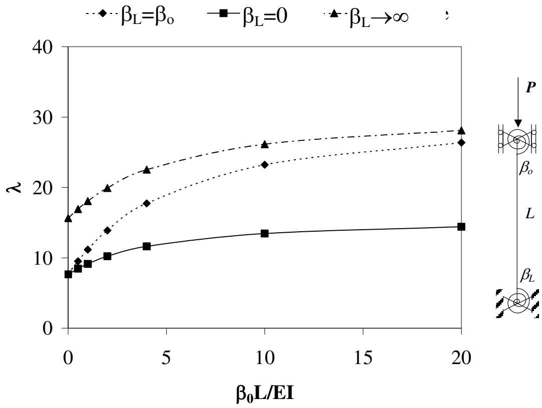
b. b=0.5



$\beta_0 L/EI$	λ		
	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	6.08246	6.08246	12.1185
0.5	7.97022	6.77453	13.2143
1	9.52393	7.3475	14.1593
2	11.9159	8.2278	15.6846
4	14.9736	9.33292	17.7327
10	18.8703	10.6842	20.4122
20	20.9134	11.3675	21.8001

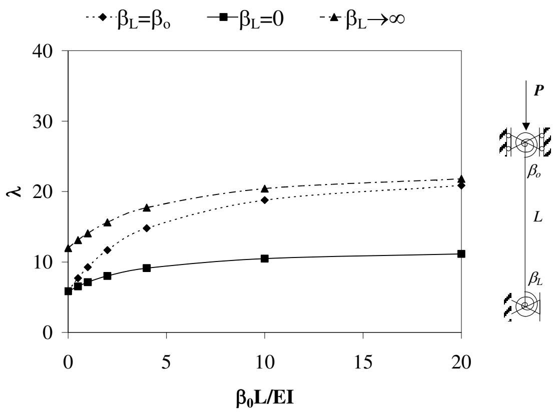
c. b=0.7

Figure 6. Case II — variation of normalized buckling load with normalized rotational spring stiffnesses for columns with linearly varying stiffness.



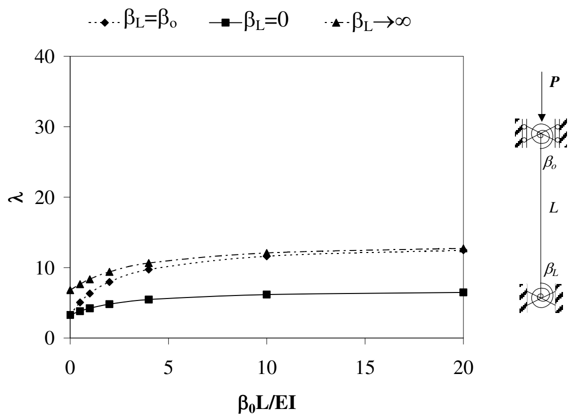
	λ		
β_0L/EI	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	7.63449	7.63449	15.6379
0.5	9.51836	8.44491	16.9228
1	11.1681	9.13009	18.0507
2	13.8959	10.2115	19.9163
4	17.7352	11.6253	22.525
10	23.2317	13.449	26.1504
20	26.3844	14.411	28.125

a. a=0.5



	λ		
β_0L/EI	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	5.82655	5.82655	11.9763
0.5	7.69583	6.53223	13.0972
1	9.25074	7.11518	14.0633
2	11.6682	8.00799	15.621
4	14.7864	9.12257	17.7068
10	18.776	10.4735	20.4171
20	20.864	11.1509	21.8082

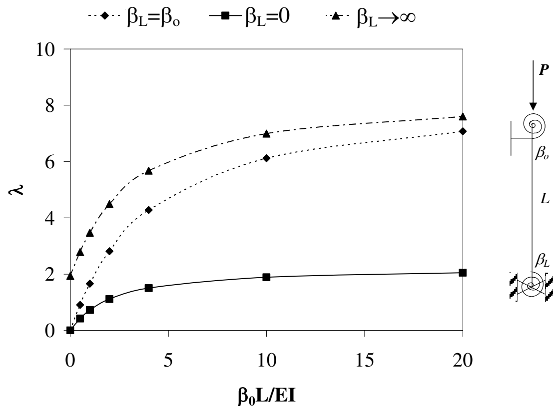
b. a=1



	λ		
β_0L/EI	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	3.26337	3.26337	6.78791
0.5	5.06173	3.79622	7.63600
1	6.30488	4.2104	8.33274
2	7.94339	4.79829	9.38508
4	9.71424	5.45597	10.6528
10	11.6021	6.15367	12.0744
20	12.4645	6.47049	12.7239

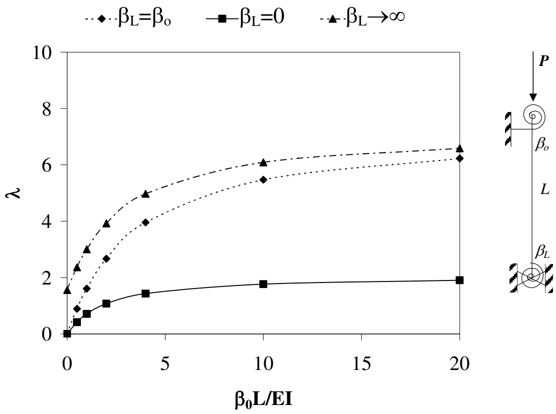
c. a=2

Figure 7. Case II — variation of normalized buckling load with normalized rotational spring stiffnesses for columns with exponentially varying stiffness.



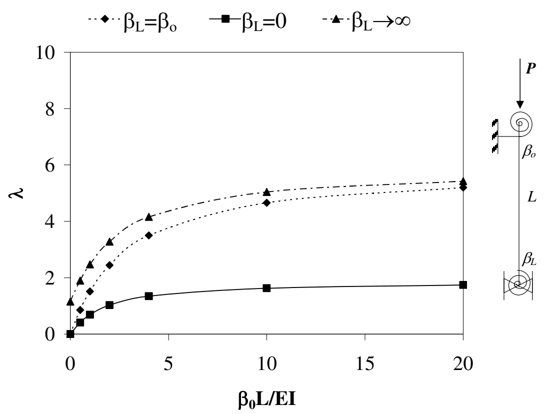
	λ		
β_0L/EI	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	0.00000	0.00000	1.93455
0.5	0.90783	0.42114	2.78663
1	1.65849	0.72239	3.47294
2	2.81421	1.11334	4.48636
4	4.27910	1.50670	5.67600
10	6.11654	1.88562	6.99354
20	7.07751	2.04883	7.60366

a. b=0.3



	λ		
β_0L/EI	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	0.00000	0.00000	1.55885
0.5	0.89207	0.41639	2.36504
1	1.60747	0.70749	3.00361
2	2.67126	1.07538	3.92620
4	3.95605	1.43309	4.97456
10	5.47257	1.76553	6.09031
20	6.22766	1.90520	6.59212

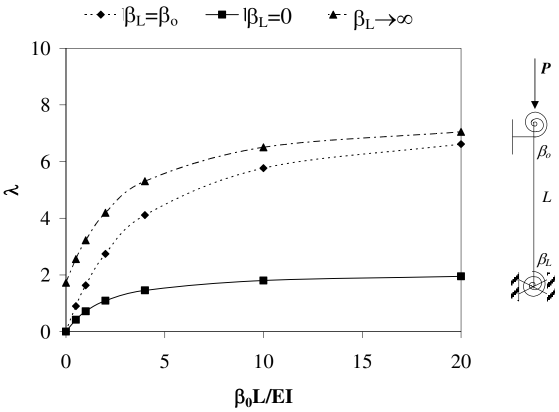
b. b=0.5



	λ		
β_0L/EI	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	0.00000	0.00000	1.15216
0.5	0.85687	0.41056	1.89724
1	1.51307	0.68909	2.47320
2	2.44373	1.02909	3.27971
4	3.50031	1.34553	4.15588
10	4.65906	1.62689	5.04125
20	5.20407	1.74162	5.42537

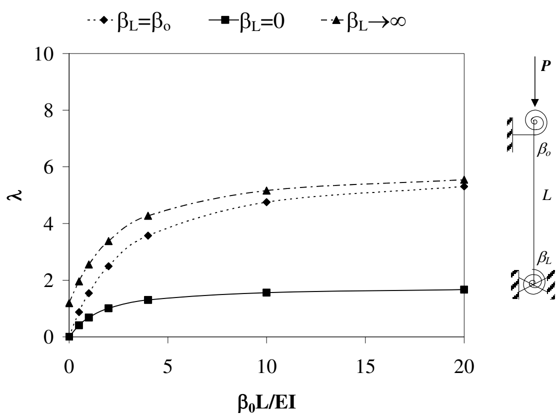
c. b=0.7

Figure 8. Case III — variation of normalized buckling load with normalized rotational spring stiffnesses for columns with linearly varying stiffness.



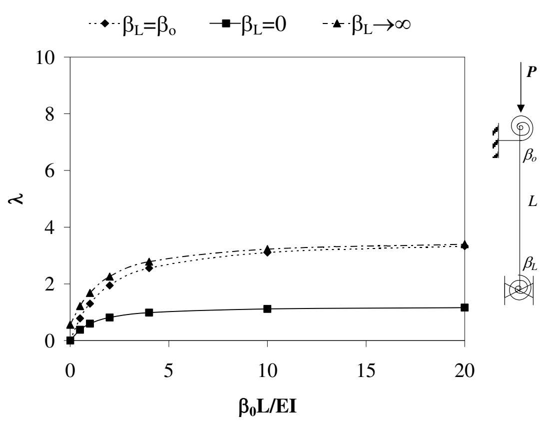
	λ		
$\beta_0 L/EI$	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	0.00000	0.00000	1.72544
0.5	0.90036	0.41772	2.55800
1	1.63354	0.71170	3.22202
2	2.74207	1.08621	4.18976
4	4.11106	1.45422	5.30329
10	5.77112	1.80014	6.50565
20	6.61537	1.94665	7.05180

a. a=0.5



	λ		
$\beta_0 L/EI$	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	0.00000	0.00000	1.19237
0.5	0.87089	0.40682	1.96144
1	1.54044	0.67830	2.55445
2	2.49166	1.00407	3.38124
4	3.57160	1.30152	4.27229
10	4.75369	1.56165	5.16229
20	5.30830	1.66670	5.54461

b. a=1



	λ		
$\beta_0 L/EI$	$\beta_L=\beta_0$	$\beta_L=0$	$\beta_L\to\infty$
0	0.00000	0.00000	0.55250
0.5	0.78447	0.37748	1.21519
1	1.30331	0.59337	1.68031
2	1.94497	0.81411	2.25547
4	2.55547	0.98351	2.78196
10	3.10614	1.11153	3.22630
20	3.33138	1.15897	3.39786

c. a=2

Figure 9. Case III — variation of normalized buckling load with normalized rotational spring stiffnesses for columns with exponentially varying stiffness.

This paper shows that the variational iteration method (VIM) can successfully be used to determine the buckling loads of slender columns with elastic end restraints. To the best knowledge of author, exact solutions to this problem are available only for some particular cases of uniform columns. For this reason, before analyzing the columns with variable cross sections, the buckling loads of columns with constant cross sections are determined using classical variational iteration algorithm and VIM results are compared to the exact results, which show perfect match. After verifying the efficiency of VIM in the analysis of this special type of buckling problem, the columns with variable flexural stiffness are analyzed using this practical technique. It is shown that unlike exact solution procedures, variational iteration algorithms can easily be used even when the column stiffness change along its length exponentially or linearly and/or the end conditions are rather complex.

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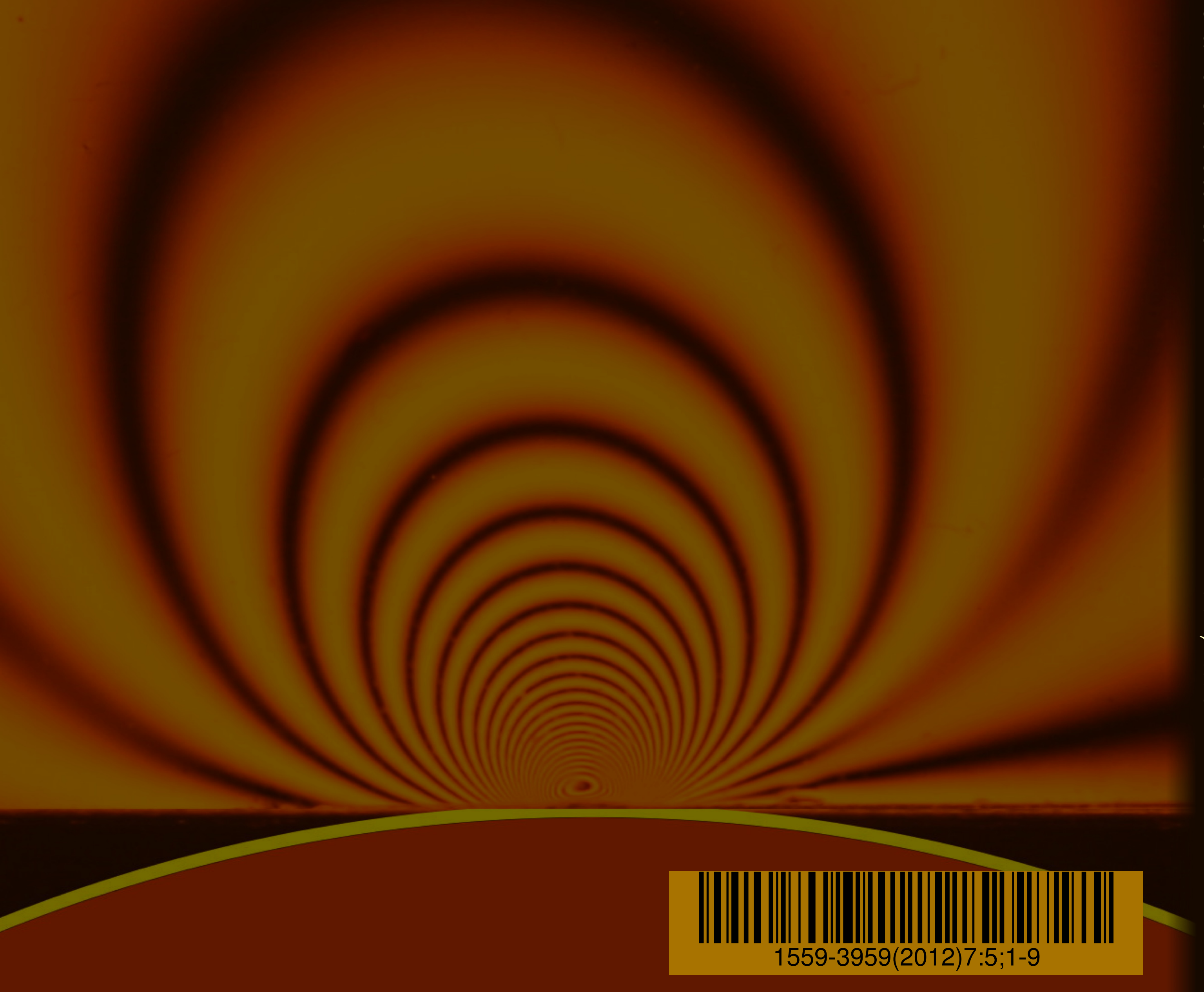

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