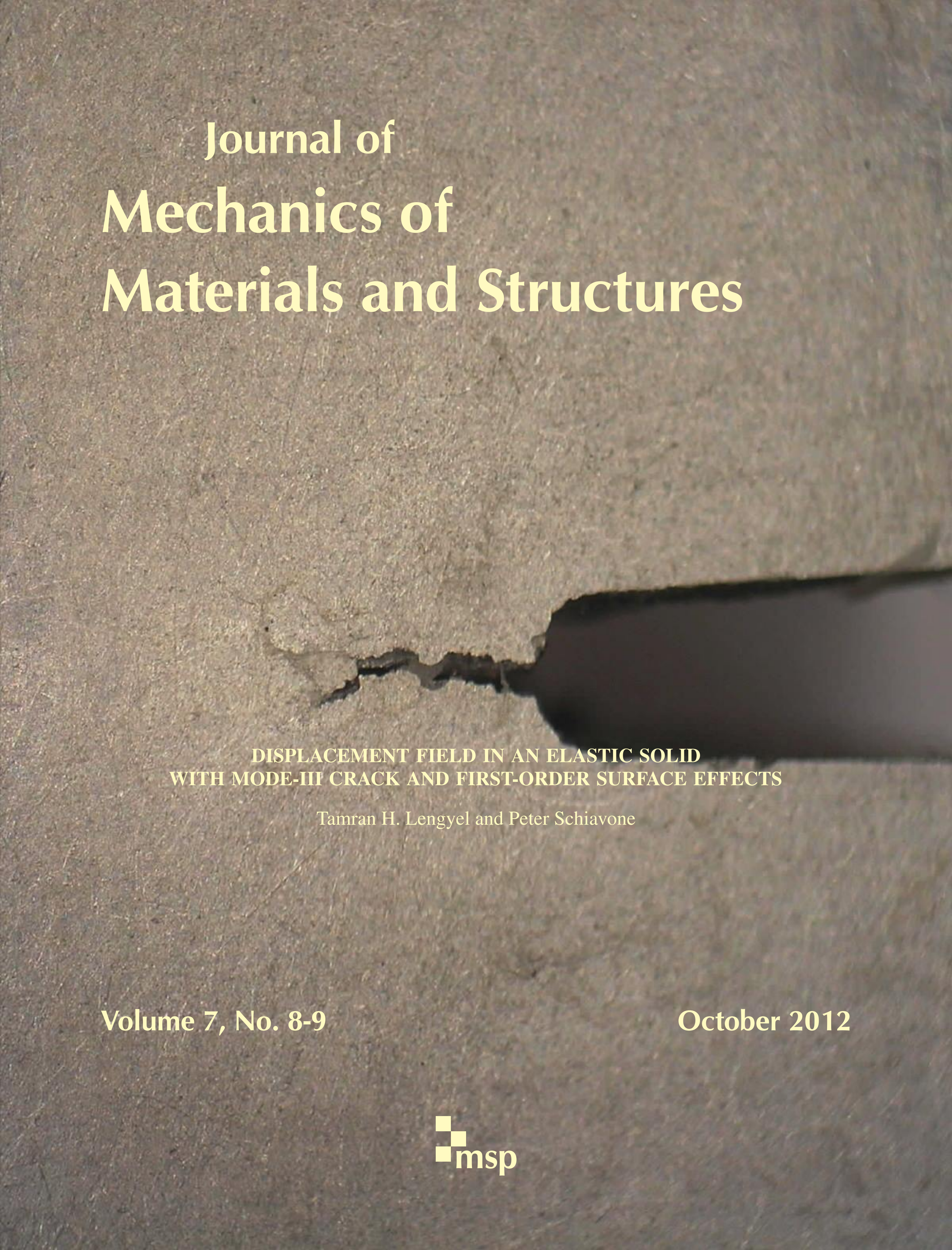

Journal of

Mechanics of Materials and Structures

DISPLACEMENT FIELD IN AN ELASTIC SOLID WITH MODE-III CRACK AND FIRST-ORDER SURFACE EFFECTS

Tamran H. Lengyel and Peter Schiavone

Volume 7, No. 8-9 October 2012

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 7, No. 8-9, 2012
dx.doi.org/10.2140/jomms.2012.7.783

msp

DISPLACEMENT FIELD IN AN ELASTIC SOLID WITH MODE-III CRACK AND FIRST-ORDER SURFACE EFFECTS

TAMRAN H. LENGYEL AND PETER SCHIAVONE

We analyze a nonstandard boundary-value problem for Laplace's equation characterizing the displacement field arising from the antiplane deformations of an infinite elastic solid containing a sharp finite crack when first-order surface effects are included on the crack faces. The surface effects are incorporated using the continuum-based surface/interface model of Gurtin and Murdoch. We establish a uniqueness result for the displacement field and use complex variable methods to reduce the problem to a series of integral equations which are solved numerically using an efficient, stable, yet convenient finite element discretization method. Our results demonstrate the effect of the surface on the displacement field and its implications for the corresponding stress distributions in the vicinity of the crack.

1. Introduction

The incorporation of surface mechanics into mathematical models describing deformation of various elastic structures has drawn an increasing amount of attention in the literature recently (see, for example, [Gibbs 1906; Orowan 1970; Gurtin and Murdoch 1975; 1978; Chan and Larché 1982; Benveniste and Aboudi 1984; Thomson and Chuang 1986; Cammarata 1994; Steigmann and Ogden 1997a; 1997b; Wang et al. 2007; 2010a; 2010b; Zhu 2008; Altenbach et al. 2009; 2010a; 2010b; 2011] and the references contained therein). The concept is particularly significant when considering continuum models of deformation at the nanoscale where the high surface-area-to-volume ratio means that the separate contributions of the surface can no longer be ignored.

In the case of antiplane deformations of a linearly elastic and homogeneous isotropic solid containing a sharp crack, Kim et al. [2010a; 2010b] have incorporated surface effects on the faces of the crack using the Gurtin–Murdoch surface elasticity model [Gurtin and Murdoch 1975; 1978; Ru 2010]. The resulting mathematical model gave rise to a nonstandard boundary-value problem for the (harmonic) antiplane displacement. Using complex variable methods, Kim et al. presented various conclusions regarding the *stress* distributions in the vicinity of the crack and how they are affected by the surface mechanics. A rigorous analysis of the actual displacement boundary-value problem in terms of uniqueness and determination of the displacement solution was never conducted. The determination of the displacement field is important in identifying the deformation of the body; in particular, the change of shape of the crack faces in response to the presence of surface effects. To this end, it is extremely important to ensure that the mathematical model being used to describe the displacement field is "correct" or well-posed in the sense that any solution found by numerical methods is necessarily unique. In fact, as we discuss in Section 4, the uniqueness result, in particular, has significant implications not only for the determination

Keywords: displacement field, mode-III crack, antiplane deformations, surface elasticity, integral equations.

of the displacement field but also for the corresponding stress distributions in the vicinity of the crack tips.

In this paper we undertake a rigorous formulation and analysis of the displacement boundary-value problem first mentioned in [Kim et al. 2010a; 2010b]. We establish a uniqueness result for the displacement field and use complex variable methods to reduce the problem to a series of integral equations which are solved numerically using an efficient and stable, yet straightforward numerical method. Our results demonstrate the contribution of the mechanics of the crack surfaces on the displacement field and the implications for the corresponding stress distributions at the crack tips.

2. Formulation

We consider antiplane deformations of a linearly elastic and homogeneous isotropic solid occupying a cylindrical region in $\mathbb{R}^3$ with generators parallel to the x_3-axis of a rectangular Cartesian coordinate system (x_1, x_2, x_3). Suppose that the cylinder contains a single internal finite crack running the length of the cylinder and occupying the region $L = \{-a \le x_1 \le a, x_2 = 0^{\pm}\}$ of a typical cross section S. The crack length $2a$ is assumed to be much smaller than any characteristic length in S. The faces of the crack are assumed to have separate elasticities which is described by an adaptation of the Gurtin–Murdoch surface elasticity model [Ru 2010]. The theory of antiplane elasticity leads to a description of the deformation of the solid characterized by the unknown antiplane displacement function $w(x_1, x_2)$ defined in S.

In order to formulate the boundary-value problem for w, we first note that the nature of antiplane deformations are such that [Milne-Thompson 1962]

$$w^+ = -w^- \quad \text{everywhere in } S. \tag{2.1}$$

Here w^+ and w^- represent the antiplane displacements in the upper and lower half-planes $x_2 > 0$ and $x_2 < 0$, respectively. Consequently, since we assume that the material remains connected (no tearing), $w = 0$ at the crack tips $x_1 = \pm a$ and $x_2 = 0$. Finally, we assume that the crack faces are subjected to a loading $\sigma_{23} = P(x_1)$, where $P(x_1)$ is a Hölder-continuous function, and that the remote boundaries of the cross section are free of loading, that is, $\sigma_{23} = 0$ and $R^2 = x_1^2 + x_2^2 \to \infty$.

We obtain the following two-dimensional discontinuous boundary-value problem for the Laplace operator Δ:

$$\Delta w(x_1, x_2) = 0, \qquad (x_1, x_2) \in S \setminus L, \tag{2.2}$$

$$\mp\beta \frac{\partial^2 w}{\partial x_1^2}(x_1, 0^{\pm}) + P(x_1) = \mu \frac{\partial w}{\partial x_2}(x_1, 0^{\pm}), \qquad -a < x_1 < a. \tag{2.3}$$

For $R = \sqrt{x_1^2 + x_2^2} \to \infty$, as in the statement of the standard exterior Dirichlet problem for Laplace's equation [Kress 1999], it is required that

$$w(x_1, x_2) = w_\infty + O\left(\frac{1}{R}\right), \tag{2.4}$$

uniformly for all directions. Here β is a parameter representing the elasticity of the surface layer and μ is the shear modulus in the bulk material [Ru 2010]. We note that a variation of the boundary-value problem (2.2)–(2.4) has been solved numerically in an infinite strip using Fourier transforms and reduction to a Riemann–Hilbert problem [Antipov and Schiavone 2011].

It is convenient to introduce the dimensionless transformations

$$x \equiv \frac{x_1}{a}, \quad y \equiv \frac{x_2}{a}, \quad \tilde{P}(x) \equiv \frac{P(ax)}{P_0}, \quad \gamma \equiv \frac{\beta}{a\mu}, \quad u(x, y) \equiv \frac{\mu}{aP_0} w(ax, ay), \tag{2.5}$$

where P_0 is the root-mean-squared value of the input stress along the crack face defined by

$$P_0 = \sqrt{\frac{1}{2a} \int_{-a}^{a} [P(t)]^2 \, dt},$$

γ is defined as the dimensionless surface parameter, and u is the normalized antiplane displacement field. The boundary-value problem (2.2)–(2.4) then becomes

$$\Delta u(x, y) = 0, \qquad (x, y) \in S \setminus L, \tag{2.6}$$

$$\mp\gamma \frac{\partial^2 u}{\partial x^2}(x, 0^{\pm}) + \tilde{P}(x) = \frac{\partial u}{\partial y}(x, 0^{\pm}), \qquad -1 < x < 1, \tag{2.7}$$

$$u = u_\infty + O\big(1/\tilde{R}\big), \quad \tilde{R} = \sqrt{x^2 + y^2} \to \infty, \tag{2.8}$$

where u_∞ is a constant. Finally, the corresponding dimensionless shear stresses τ_x and τ_y are defined using the above transformations as

$$\sigma_{13} = \mu \frac{\partial w}{\partial x_1} = P_0 \frac{\partial u}{\partial x} \equiv P_0 \tau_x, \quad \sigma_{23} = \mu \frac{\partial w}{\partial x_2} = P_0 \frac{\partial u}{\partial y} \equiv P_0 \tau_y,$$

with the total normalized stress magnitude τ given by

$$\tau^2 \equiv (\tau_x)^2 + (\tau_y)^2.$$

Note that (2.1) along with the assumption that the material remains connected (no tearing) at the crack tips implies that

$$\tau_x^+ = -\tau_x^-, \quad \tau_y^+ = \tau_y^-.$$

3. Uniqueness of solution

We shall prove that the boundary-value problem (2.6)–(2.8) can have at most one solution. Consider the difference $v(x, y)$ of any two solutions of the boundary-value problem (2.6)–(2.8). We prove that $v(x, y) \equiv 0$ in $S \setminus L$ by adapting the methods used in [Knowles and Pucik 1973]. Clearly v satisfies the corresponding homogeneous boundary-value problem. Referring to Figure 1, we surround the crack by a smooth curve $\partial\Omega$ of "radius" ϵ, contained entirely within the area $B \subset S$. The region exterior to $\partial\Omega$ is denoted by Ω and the outer boundary of B by ∂B. We decompose $\partial\Omega$ as

$$\partial\Omega = \bigcup_{i=1}^{4} \partial\Omega_i,$$

where $\partial\Omega_1$ and $\partial\Omega_2$ are the upper ($y = +\epsilon$) and lower ($y = -\epsilon$) portions of the straight portion of $\partial\Omega$, respectively ($\partial\Omega_{1,2} : x \in [-1, 1], y = \pm\epsilon$); and $\partial\Omega_3$ ($\pi/2 \le \varphi \le 3\pi/2$) and $\partial\Omega_4$ ($-\pi/2 \le \varphi \le \pi/2$) are the left and right curved portions of $\partial\Omega$, respectively (in cylindrical polar coordinates).

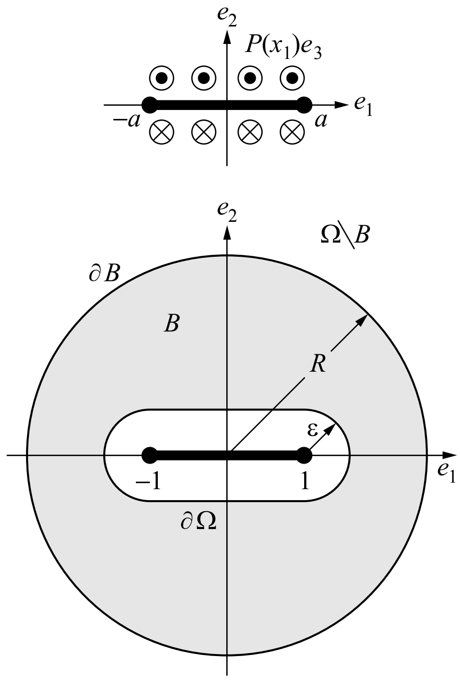

Figure 1. Crack surrounded by a smooth curve $\partial\Omega$ of "radius" ϵ, contained entirely within the area $B \subset S$.

We define the values of v on the boundary of $\partial\Omega$ to be

$$\begin{aligned} v_\epsilon^\pm &\equiv v(x, \pm\epsilon) && \text{on } \partial\Omega_{1,2},\\ v_r &\equiv v(\mp 1 + \epsilon\cos(\varphi), \epsilon\sin(\varphi)) && \text{on } \partial\Omega_{3,4}. \end{aligned}$$

Green's first identity in the bounded domain B yields

$$\int_B |\nabla v|^2 \, dA = \oint_{\partial B} v \frac{\partial v}{\partial \boldsymbol{n}} \, ds - \oint_{\partial\Omega} v \frac{\partial v}{\partial \boldsymbol{n}} \, ds. \tag{3.1}$$

As $R \to \infty$ the region $B \to \Omega$ and further as $\epsilon \to 0$, the region $\Omega \to S \setminus L$. As $R \to \infty$, the asymptotic behavior of (2.8) yields for (3.1)

$$\begin{aligned} \int_\Omega |\nabla v|^2 \, dA &= -\oint_{\partial\Omega} v \frac{\partial v}{\partial \boldsymbol{n}} \, ds = -\sum_{i=1}^{4} \int_{\partial\Omega_i} v \frac{\partial v}{\partial \boldsymbol{n}} \, ds \\ &= -\int_{-1}^{1} v_\epsilon^+ (\nabla v_\epsilon^+ \cdot (-\boldsymbol{e}_2))(-dx) - \int_{-1}^{1} v_\epsilon^- (\nabla v_\epsilon^- \cdot \boldsymbol{e}_2) \, dx - 2\epsilon \int_{-\pi/2}^{\pi/2} v_r (\nabla v_r \cdot (-\boldsymbol{e}_r)) \, d\theta \\ &= -\int_{-1}^{1} v_\epsilon^+ \frac{\partial v_\epsilon}{\partial y}^+ dx - \int_{-1}^{1} v_\epsilon^- \frac{\partial v_\epsilon}{\partial y}^- dx + 2\epsilon \int_{-\pi/2}^{\pi/2} v_r \frac{\partial v_r}{\partial r} \, d\theta = -I_1 - I_2 + 2\epsilon I_3. \end{aligned} \tag{3.2}$$

The integrals I_1 and I_2 are determined using the boundary condition (2.7) and the smoothness of the solution v [Kress 1999]. Noting that $v^+ = -v^-$ for antiplane displacement, we have, as $\epsilon \to 0$,

$$\lim_{\epsilon\to 0} I_1 = \lim_{\epsilon\to 0}\int_{-1}^{1} v_\epsilon^+ \frac{\partial v_\epsilon}{\partial y}^+ dx = \int_{-1}^{1} v^+ \frac{\partial v}{\partial y}^+ dx = -\gamma\int_{-1}^{1} v^+ \frac{\partial^2 v}{\partial x^2}^+ dx \equiv \nu,$$
$$\lim_{\epsilon\to 0} I_2 = \lim_{\epsilon\to 0}\int_{-1}^{1} v_\epsilon^- \frac{\partial v_\epsilon}{\partial y}^- dx = \int_{-1}^{1} v^- \frac{\partial v}{\partial y}^- dx = \gamma\int_{-1}^{1} v^- \frac{\partial^2 v}{\partial x^2}^- dx = -\nu.$$

Also,

$$\lim_{\epsilon\to 0}|I_3| = \lim_{\epsilon\to 0}\left|\int_{-\pi/2}^{\pi/2} v_r \frac{\partial v_r}{\partial r}\, d\theta\right| = \left|\int_{-\pi/2}^{\pi/2} v(\pm 1, 0)\frac{\partial v}{\partial \boldsymbol{n}}(\pm 1, 0)\, d\theta\right|$$
$$\leq \int_{-\pi/2}^{\pi/2} |v(\pm 1, 0)\nabla v(\pm 1, 0)|\, d\theta = \int_{-\pi/2}^{\pi/2} V_0^2\, d\theta = \pi V_0^2.$$

Taking the limit as $\epsilon \to 0$ on the right-hand side of (3.2) then yields:

$$\int_{S\backslash L} |\nabla v|^2\, dA = 0.$$

Note that in the case when $\partial v/\partial \boldsymbol{n}$ has a logarithmic singularity at the crack tips, the integral I_3 remains uniformly bounded as $\epsilon \to 0^+$ since the integrand remains weakly singular [Kress 1999].

Hence, $v = \text{const.}$ in $S\backslash L$. However, given that $v^+ = -v^-$, $-1 \leq x \leq 1$, it follows that, at the crack tips, $v(-1, 0) = v(1, 0) = 0$, from which it follows that $v \equiv 0$ (or, $u_2 = u_1$) in $S \setminus L$. Consequently, as a result of the linearity of the problem there can be at most one solution of (2.6)–(2.8).

4. Integral equation

Define the complex antiplane displacement potential $\phi(z)$, $z \equiv x + iy$, as

$$\phi(z) \equiv u(x, y) + iv(x, y).$$

Here $u(x, y)$ is the displacement field from (2.6)–(2.8) with $u_\infty = 0$ and $v(x, y)$ is the conjugate function. Define the function $\theta(x)$ by $\theta(x) \equiv \lim_{y\to 0^+} u(x, y)$. Then θ is the normalized antiplane displacement on the upper crack face, where

$$u(x, 0^\pm) = \pm\theta(x), \quad -1 \leq x \leq 1.$$

Then, the displacement jump across the crack is given by

$$2\theta(t) = [\phi(t)]^+ - [\phi(t)]^-$$

and $\phi(z)$ can be represented in the form of a Cauchy integral [Muskhelishvili 1963]:

$$\phi(z) = \frac{1}{\pi i}\int_L \frac{\theta(t)}{t - z}\, dt,$$

so that, as a consequence of (2.1),

$$\theta(-1) = \theta(1) = 0. \tag{4.1}$$

Using the Plemelj formula [Muskhelishvili 1963], after some rearrangement, we obtain the following integrodifferential equation for the displacement on the crack:

$$\frac{1}{\pi}\frac{d}{dx}\left[\int_L \frac{\theta(t)}{t-x}\,dt\right]+\gamma\theta''(x)=\tilde{P}(x), \quad -1\le x\le 1.$$

Using integration by parts, we can write this equation as a Fredholm equation of the second kind:

$$\gamma\theta(x)-\frac{1}{\pi}\int_L \theta(t)\ln|t-x|\,dt=\zeta_2(x)-C_1x-C_2, \tag{4.2}$$

where the function $\zeta_2(x)$ is defined by

$$\zeta_2(x)=\int_0^x\int_0^t \tilde{P}(s)\,ds\,dt$$

and C_1 and C_2 are constants of integration.

The normalized displacement field in the bulk material is then described by

$$u(x,y)=\operatorname{Re}\phi(z)=\operatorname{Im}\left\{\frac{1}{\pi}\int_L\frac{\theta(t)}{t-z}\,dt\right\}.$$

The corresponding stresses on the crack are then given by

$$\begin{aligned}\tau_x(x,0^{\pm})&=\pm\theta'(x)\ (\equiv\pm\theta_x),\\ \tau_y(x,0^{\pm})&=\tilde{P}(x)\mp\gamma\theta''(x)\ (\equiv\theta_y), \quad -1<x<1,\end{aligned}$$

where $\theta'(x)$ is given by

$$\gamma\theta'(x)-\frac{1}{\pi}\int_L\theta'(t)\ln|t-x|\,dt=\zeta_1(x)-C_3, \tag{4.3}$$

C_3 is an arbitrary constant, and

$$\zeta_1(x)=\int_0^x\tilde{P}(t)\,dt.$$

The constants C_1, C_2, C_3 are determined using the crack-tip conditions (4.1) and the requirement that

$$\int_{-1}^{1}\theta'(t)\,dt=0$$

which follows from (4.1) as a result of the antiplane nature, (2.1), of the deformations considered here.

Note that (4.3) leads to

$$\gamma\theta''(x)-\frac{1}{\pi}\int_L\theta''(t)\ln|t-x|\,dt=\tilde{P}(x)-\left[\frac{1}{\pi}\theta'(t)\ln|t-x|\right]_{t=-1}^{t=1}. \tag{4.4}$$

From (4.4), if the stresses τ_x ($\propto\theta'(\pm1)$) at the crack tips are finite and nonzero, then the stresses τ_y at the crack tips ($\tau_y\propto\theta''(\pm1)$) cannot be finite even for a finite externally applied stress function ($\tilde{P}$). This result is in agreement with the recent papers [Kim et al. 2013; Walton 2012] where it was noted that by imposing the further condition $\theta'\equiv0$ at the crack tips (to achieve finite stress [Kim et al. 2010a; 2010b]) we effectively overdetermine the problem so that, contrary to the results reported in these two references, there can be no solutions of (2.2)–(2.4) with finite stress at the crack tips. In fact, this is clear from the uniqueness theorem proved above: once the unique displacement field is determined in the body one cannot then arbitrarily assign values to the derivatives of the displacement field at the crack tips. This fact is further reflected in the next section when we actually compute θ and its derivatives.

5. Numerical solution

The displacement and slope on the crack face can be found by solving (4.2) and (4.3), which are accommodated by the general form

$$\gamma f(x) - \frac{1}{\pi}\int_L f(t)\ln|t-x|\,dt = \zeta(x) \tag{5.1}$$

for corresponding functions f and ζ. Equation (5.1) is a Fredholm integral equation of the second kind, which (provided γ is not an eigenvalue for the corresponding integral operator) is well-known to have a unique continuous solution for each continuous $\zeta(x)$ on L. We have approximated the solution for appropriate values of γ using a discretized (finite-element) residual formulation based on a total of N linear elements leading to a system of $N+1$ linear equations for $N+1$ unknowns [Lengyel 2011].

The numerical method can accommodate different input load profiles ($\tilde{P}(x)$). However, if the crack length is relatively small, it is sufficient to assume a constant input stress ($\therefore\ \tilde{P}=\pm 1$). In this way, the solution to the classical antiplane crack problem with a constant mode-III input load ($\tilde{P}=-1$ for convenience) has been solved in [Muskhelishvili 1963; Sih 1965; Broberg 1998], where it is shown that on the (+) crack face:

$$\theta(x) = u(x,0^+) = \sqrt{1-x^2}, \qquad -1 \le x \le 1, \tag{5.2}$$

$$\theta_x(x) = \tau_x(x,0^+) = -\frac{x}{\sqrt{1-x^2}}, \qquad -1 < x < 1, \tag{5.3}$$

$$\theta_y(x) = \tau_y(x,0^+) = -1, \qquad -1 < x < 1. \tag{5.4}$$

Note the singularities in both normalized stress components at the crack tips (when $x=\pm 1$). Plots of (5.2) and (5.3) are compared with the numerical model used here (in the case $\gamma=0$) in Figure 2. The method shows fast and stable convergence to the exact solution in this case. In Figure 3 we plot the

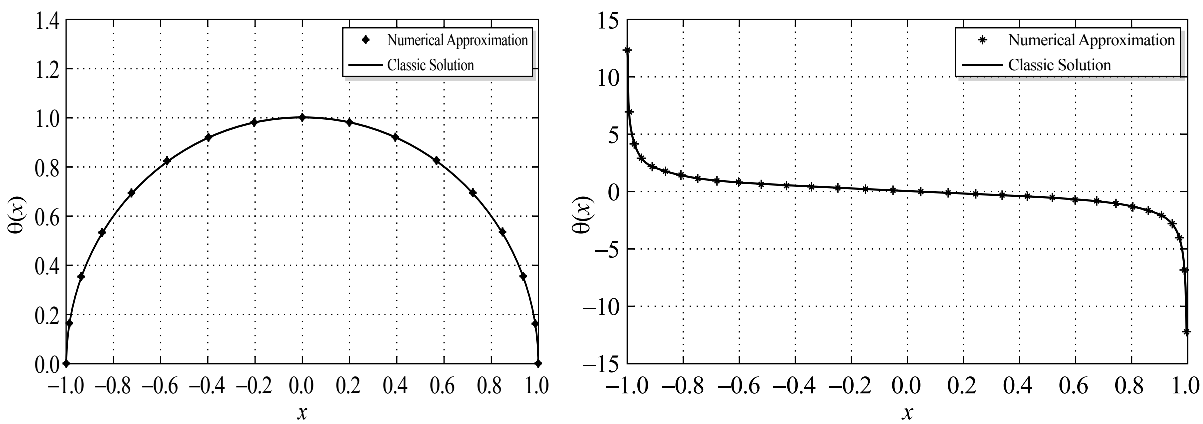

Figure 2. Plots of (5.2) and (5.3) compared with the numerical model.

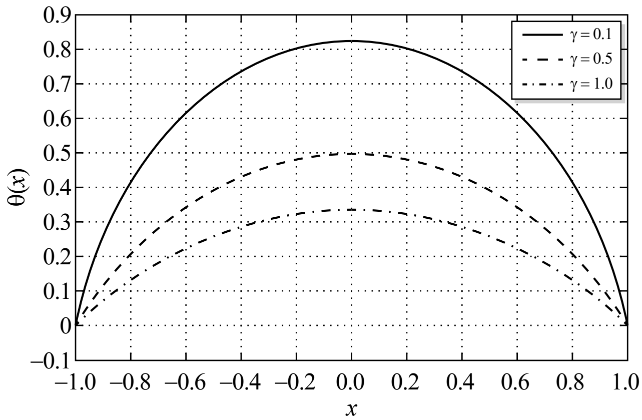

Figure 3. Effects of various surface parameters γ on the displacement of the (+) crack face.

effects of various surface parameters γ on the displacement of the (+) crack face. It is clear that the displacement is reduced on the crack face with increasing surface effect. We note also from Figure 4, left, that the stress θ_x is finite everywhere across the crack face, even at the crack tips. However, the unit stress θ_y, plotted on the right in Figure 4, remains singular at the crack tips. This result contradicts that presented in [Kim et al. 2010a; 2010b] but is not surprising. As mentioned above, the unique solution of (2.6) and (2.7) determines a priori the values of θ_x which are clearly never zero at the crack tips for any meaningful value of γ.

Figure 5 plots $\theta(0)$ ($\equiv \theta_{\max}$) with increasing N (the number of elements) using different values of the surface parameter (γ). These approximations demonstrate rapid convergence with less than 50 elements. Refinement convergence of the slope at the endpoints ($\theta_x(-1) \equiv \theta'_{\max}$) is shown in Figure 6. There is convergence here, however, more elements are required as the surface parameter decreases in value. This is not surprising since, as the surface parameter tends to zero (the classical model) the opposing stresses ($\tau_x \propto \theta_x$) at the crack tips become singular. Figure 7 shows the (decreased) stress in the bulk material

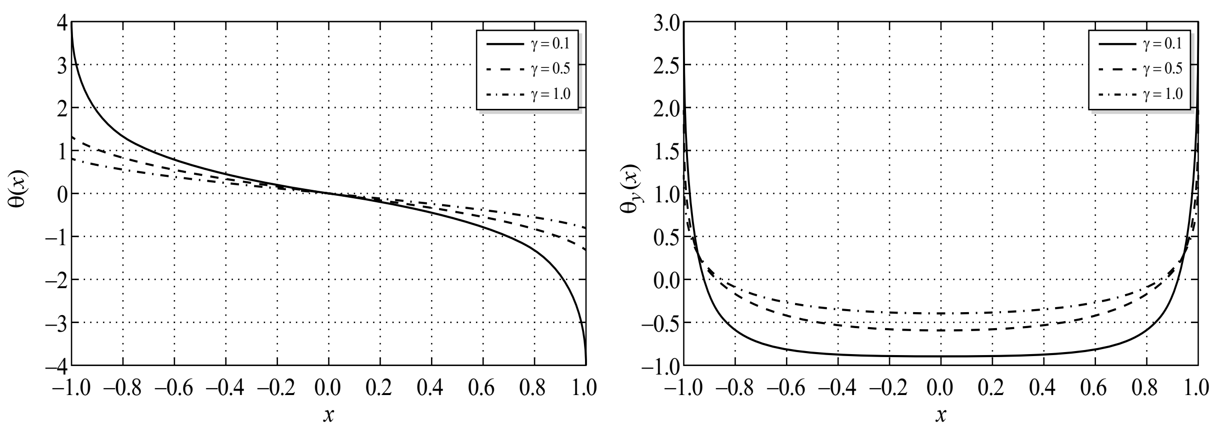

Figure 4. Stress θ_x (left) and unit stress θ_y (right) across the crack face.

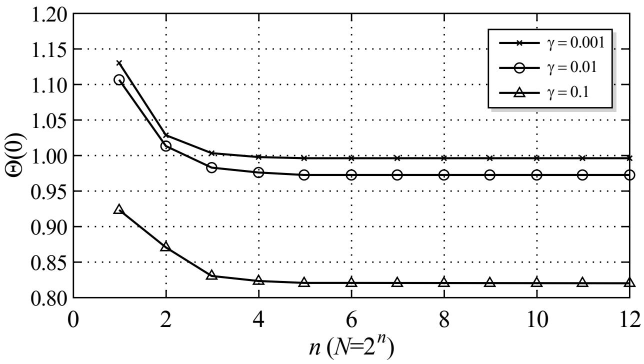

Figure 5. Plot of $\theta(0)$ ($\equiv \theta_{\max}$) with increasing number of elements, N, using different values of the surface parameter γ.

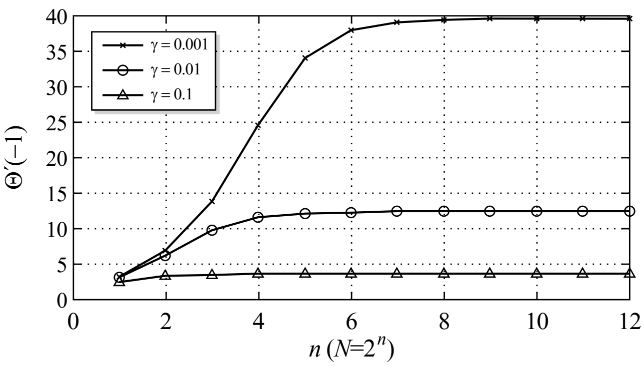

Figure 6. Refinement convergence of the slope at the endpoints ($\theta_x(-1) \equiv \theta'_{\max}$).

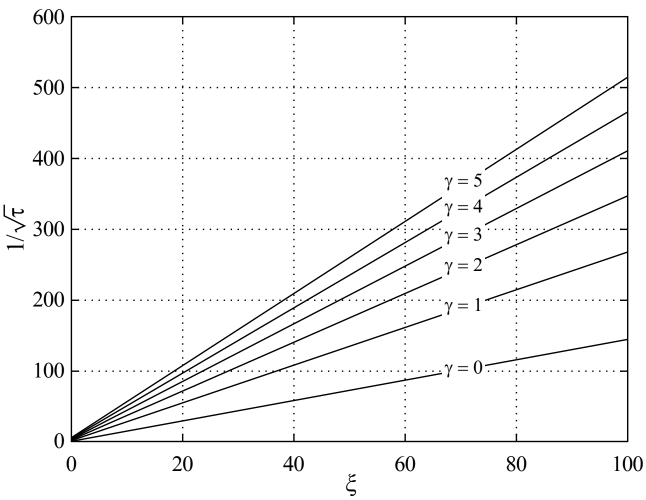

Figure 7. Stress in the bulk material outside the crack due to increasing surface parameter γ ($\xi(x) \equiv x - 1$, $x > 1$).

outside the crack due to an increasing surface parameter γ ($\xi(x) \equiv x - 1, x > 1$). Full details can be found in [Lengyel 2011].

5.1. ***Remark.*** Finally, we remark that the stability and rapid convergence of our numerical model allows us to obtain an approximate expression quantifying the effect of γ on the displacement of the crack face in the case of a constant stress input (relatively small crack length). In fact, using curve-fitting analysis, our numerical results suggest the following convenient approximation for the displacement field across the crack face in the case of a constant mode-III load input stress

$$w(x, 0^{\pm}) \simeq \pm a\left(\frac{P_0}{\mu}\right)\frac{(1-(x_1/a)^2)^{(27.67\gamma+1)/(27.67\gamma+2)}}{2\gamma+1}, \quad -a \le x \le a. \tag{5.5}$$

We find that the expression (5.5) agrees with the corresponding numerical results obtained using the complete finite element discretization method to an error of less than 5%.

6. Conclusions

The incorporation of surface effects allows for the assumption of energy dispersion on the boundary which reduces the input shear stress (Figure 4, right). With the exception of the region in the vicinity of the crack tips, the adjusted input stress magnitude shows a reduction when γ increases. From this reduction of input shear stress, the displacement (u) and opposing stress (τ_x) are reduced and remain finite across the crack face (including at the crack tips). Figures 3 and 4, left, demonstrate this conclusively for both displacement ($\theta \propto u$) and slope ($\theta' \propto \tau_x$). The displacements and stress magnitudes in the bulk are also reduced accordingly.

Given the convergence demonstrated in Figures 5 and 6, it is apparent that the function $\theta(x)$ ($\propto u$) along with it's derivatives $\theta_x(x)$ ($\propto \tau_x$) and $\theta''(x)$ ($\propto \tau_y$) are convergent across the crack face. The function $\theta''(x)$ does not converge near the crack tips, however all three functions showed a reduction and convergence across the crack tip when γ increased. Consequently, the incorporation of surface effects leads to a more stable and convergent numerical model. This additional stability and convergence is directly related to the effect of γ on the displacement profile. As the solution transitions from a semicircle (when $\gamma \to 0$) to a quadratic function (when $\gamma \to \infty$), the stability of the displacement (u) and stress (τ_x and τ_y) increases.

7. Acknowledgements

Schiavone acknowledges the support of the Natural Sciences and Engineering Research Council of Canada. The authors would like to thank an anonymous reviewer for his detailed comments which have led to improvements in the paper.

References

[Altenbach et al. 2009] H. Altenbach, V. A. Eremeyev, and N. F. Morozov, "Linear theory of shells taking into account surface stresses", *Dokl. Phys.* **54** (2009), 531–535.

[Altenbach et al. 2010a] H. Altenbach, V. A. Eremeyev, and L. P. Lebedev, "On the existence of solution in the linear elasticity with surface stresses", *Z. Angew. Math. Mech.* **90**:3 (2010), 231–240.

[Altenbach et al. 2010b] H. Altenbach, V. A. Eremeyev, and N. F. Morozov, "On equations of the linear theory of shells with surface stresses taken into account", *Mech. Solids* **45** (2010), 331–342.

[Altenbach et al. 2011] H. Altenbach, V. A. Eremeyev, and L. P. Lebedev, "On the spectrum and stiffness of an elastic body with surface stresses", *Z. Angew. Math. Mech.* **91**:9 (2011), 699–710.

[Antipov and Schiavone 2011] Y. A. Antipov and P. Schiavone, "Integro-differential equation for a finite crack in a strip with surface effects", *Q. J. Mech. Appl. Math.* **64**:1 (2011), 87–106.

[Benveniste and Aboudi 1984] Y. Benveniste and J. Aboudi, "Continuum model for fiber reinforced materials with debonding", *Int. J. Solids Struct.* **20**:11-12 (1984), 935–951.

[Broberg 1998] K. Broberg, "Crack expanding with constant velocity in an anisotropic solid under anti-plane strain", *Int. J. Fract.* **93** (1998), 1–12.

[Cammarata 1994] R. C. Cammarata, "Surface and interface stress effects in thin films", *Prog. Surf. Sci* **46**:1 (1994), 1–38.

[Chan and Larché 1982] J. W. Chan and F. Larché, "Surface stress and chemical equilibrium of small crystals, II: Solid particles embedded in a solid matrix", *Acta Metall.* **30** (1982), 51–56.

[Gibbs 1906] J. W. Gibbs, *The scientific papers of J. Willard Gibbs, I*, Longmans Green, London, 1906.

[Gurtin and Murdoch 1975] M. E. Gurtin and A. I. Murdoch, "A continuum theory of elastic material surfaces", *Arch. Ration. Mech. Anal.* **57**:4 (1975), 291–323.

[Gurtin and Murdoch 1978] M. E. Gurtin and A. I. Murdoch, "Surface stress in solids", *Int. J. Solids Struct.* **14** (1978), 431–440.

[Kim et al. 2010a] C. I. Kim, C.-Q. Ru, and P. Schiavone, "Analysis of a mode-III crack in the presence of surface elasticity and a prescribed non-uniform surface traction", *Z. Angew. Math. Phys.* **61**:3 (2010), 555–564.

[Kim et al. 2010b] C. I. Kim, C.-Q. Ru, and P. Schiavone, "The effects of surface elasticity on an elastic solid with mode-III crack: complete solution", *J. Appl. Mech.* (*ASME*) **77**:2 (2010), Article ID #021011.

[Kim et al. 2013] C. I. Kim, C.-Q. Ru, and P. Schiavone, "A clarification of the role of crack-tip conditions in linear elasticity with surface effects", *Math. Mech. Solids* **18**:1 (2013), 59–66.

[Knowles and Pucik 1973] J. Knowles and T. A. Pucik, "Uniqueness for plane crack problems in linear elastostatics", *J. Elasticity* **3** (1973), 155–160.

[Kress 1999] R. Kress, *Linear integral equations*, 2nd ed., Applied Mathematical Sciences **82**, Springer, New York, 1999.

[Lengyel 2011] T. H. Lengyel, "Analysis of displacement in an elastic solid with a mode-III crack in the presence of surface elasticity", Masters thesis, University of Alberta, 2011, available at http://hdl.handle.net/10402/era.27830.

[Milne-Thompson 1962] L. M. Milne-Thompson, *Anti-plane elastic systems*, Academic Press, New York, 1962.

[Muskhelishvili 1963] N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity: fundamental equations, plane theory of elasticity, torsion and bending*, 2nd ed., Noordhoff, Groningen, 1963.

[Orowan 1970] E. Orowan, "Surface energy and surface tension in solids and fluids", *Proc. R. Soc. Lond. A* **316** (1970), 473–491.

[Ru 2010] C.-Q. Ru, "Simple geometrical explanation of Gurtin–Murdoch model of surface elasticity with clarification of its related versions", *Sci. China Phys. Mech. Astronom.* **53**:3 (2010), 536–544.

[Sih 1965] G. C. Sih, "Boundary problems for longitudinal shear cracks", pp. 117–130 in *Developments in theoretical and applied mechanics* (Atlanta, GA, 1964), vol. 2, edited by W. A. Shaw, Pergamon, Oxford, 1965.

[Steigmann and Ogden 1997a] D. J. Steigmann and R. W. Ogden, "A necessary condition for energy-minimizing plane deformations of elastic solids with intrinsic boundary elasticity", *Math. Mech. Solids* **2**:1 (1997), 3–16.

[Steigmann and Ogden 1997b] D. J. Steigmann and R. W. Ogden, "Plane deformations of elastic solids with intrinsic boundary elasticity", *Proc. R. Soc. Lond. A* **453**:1959 (1997), 853–877.

[Thomson and Chuang 1986] R. Thomson and T. J. Chuang, "The role of surface stress in fracture", *Acta Metall.* **34**:6 (1986), 1133–1143.

[Walton 2012] J. R. Walton, "A note on fracture models incorporating surface elasticity", *J. Elasticity* **109**:1 (2012), 95–102.

[Wang et al. 2007] G. F. Wang, X. Q. Feng, and S. W. Yu, "Surface buckling of a bending microbeam due to surface elasticity", *Europhys. Lett.* **77**:4 (2007), Article ID #44002.

[Wang et al. 2010a] J. S. Wang, Y. H. Cui, X. Q. Feng, G. F. Wang, and Q. H. Qin, “Surface effects on the elasticity of nanosprings”, *Europhys. Lett.* **92** (2010), 16002–1–6.

[Wang et al. 2010b] Z. Q. Wang, Y. P. Zhao, and Z. P. Huang, “The effects of surface tension on the elastic properties of nanostructures”, *Int. J. Eng. Sci.* **48** (2010), 140–150.

[Zhu 2008] H. X. Zhu, “The effects of surface and initial stresses on the bending stiffness of nanowires”, *Nanotechnology* **19**:40 (2008), Article ID #405703.

Received 17 May 2012. Revised 30 Jul 2012. Accepted 2 Aug 2012.

TAMRAN H. LENGYEL: tamran.lengyel@gmail.com
Department of Mechanical Engineering, University of Alberta, 4-9 Mechanical Engineering Building, Edmonton, AB T6G 2G8, Canada

PETER SCHIAVONE: p.schiavone@ualberta.ca
Department of Mechanical Engineering, University of Alberta, 4-9 Mechanical Engineering Building, Edmonton, AB T6G 2G8, Canada
http://www.mece.ualberta.ca/~schiavone/schiavon.htm

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

jomms.net

Founded by Charles R. Steele and Marie-Louise Steele

Cover design: Alex Scorpan

See http://jomms.net for submission guidelines.

JoMMS (ISSN 1559-3959) is published in 10 issues a year. The subscription price for 2012 is US $555/year for the electronic version, and $735/year (+ $60 shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720–3840.

JoMMS peer-review and production is managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
http://msp.org/
A NON-PROFIT CORPORATION
Typeset in LaTeX

Journal of Mechanics of Materials and Structures

Volume 7, No. 8-9 October 2012

1559-3959(2012)7:8;1-6