Journal of
Mechanics of
Materials and Structures

TRANSIENT 3D SINGULAR SOLUTIONS FOR USE IN PROBLEMS OF PRESTRESSED HIGHLY ELASTIC SOLIDS

Louis Milton Brock

Volume 8, No. 1 January 2013

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 8, No. 1, 2013
dx.doi.org/10.2140/jomms.2013.8.37

msp

TRANSIENT 3D SINGULAR SOLUTIONS FOR USE IN PROBLEMS OF PRESTRESSED HIGHLY ELASTIC SOLIDS

LOUIS MILTON BROCK

The dynamic perturbation of a neo-Hookean solid in an initial equilibrium state of finite deformation can be viewed as the superposition of infinitesimal strains upon large ones. Three transient 3D singular problems whose solutions are useful in generating the former are studied. The first and second concern, respectively, a concentrated force and a point displacement discontinuity in an unbounded solid; the third problem involves a concentrated force applied at a point on the surface of a half-space. The governing equations resemble those for a linear anisotropic solid. Analytic solutions are obtained, as well as formulas and calculations for anisotropic wave speeds. Formulation of the problems is in terms of Cartesian coordinates, but expressions for the solutions and wave speeds make use of a quasipolar coordinate system.

1. Introduction

Loading of a prestressed highly elastic solid may produce incremental deformations that are infinitesimal in nature. Equations for infinitesimal deformation superimposed upon large can be developed [Green and Zerna 1968; Beatty and Usmani 1975] to describe this perturbation response. These equations are generally similar in form to those for anisotropic, linear elastic solids [Ting 1996], with elastic constants that depend on constitutive equations for the highly elastic solid and the prestress. As in isotropic elasticity, whether classical [Love 1944] or transient [Achenbach 1973], quasistatic [Willis 1965; Ting 1996] and transient [Wang and Achenbach 1992] singular solutions can serve as the basis for treating general dynamic loading. Moreover, the singular solutions themselves give insight into anisotropic behavior.

To illustrate the construction of such solutions this article considers a simple, neo-Hookean isotropic solid. The principal stress and principal material axes coincide, and the former is uniform. The anisotropic 3D equations of small deformation in an unbounded solid are solved for the cases of a concentrated force and a displacement discontinuity at a point. The 3D equations for the half-space subject to a concentrated surface force are then considered. Analytic solutions and formulas for body-wave and Rayleigh-wave speeds are provided. Sample calculations for the latter are also presented. The solution process involves integral transforms in Cartesian coordinates, coupled with inversions based on quasipolar and quasispherical coordinates [Brock 2012; 2013] and a standard method of de Hoop [1960]. The solution expressions are, therefore, in a hybrid but uncomplicated form.

Keywords: neo-Hookean, dynamic perturbation, 3D singular solution, transient.

2. Field equations

An elastic body $\Re$ is homogeneous and isotropic relative to an undisturbed reference configuration $\aleph_0$. Smooth motion $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X})$ takes $\Re$ to deformed equilibrium configuration $\aleph$. Cauchy stress $\boldsymbol{T}$ in $\aleph$ is [Green and Zerna 1968; Beatty and Usmani 1975]

$$\boldsymbol{T} = \alpha_0 \mathbf{1} + \alpha_1 \boldsymbol{B} + \alpha_2 \boldsymbol{B}^2, \quad \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T, \quad \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}. \tag{1}$$

Here $\mathbf{1}$ is the identity tensor, $(\alpha_0, \alpha_1, \alpha_2)$ are scalar functions of the principal invariant (I, II, J) of $\boldsymbol{B}$, and body forces are neglected. Inequalities based on the experiment [Truesdell and Noll 1965] support restrictions

$$\alpha_0 - II\alpha_2 \le 0, \quad \alpha_1 + I\alpha_2 > 0, \quad \alpha_2 \le 0. \tag{2}$$

An adjacent nonequilibrium deformed configuration $\aleph^*$ arises upon superposition of an infinitesimal displacement $\boldsymbol{u}$ that depends on $\boldsymbol{X}$ and time. This requires perturbation Cauchy stress $\boldsymbol{T}' = \boldsymbol{T}^* - \boldsymbol{T}$, where $\boldsymbol{T}^*$ is the Cauchy stress in $\aleph^*$. To the first order in $\nabla \boldsymbol{u}$ its components in the principal reference system, that is, $\boldsymbol{B} = \mathrm{diag}\{\lambda_1^2, \lambda_2^2, \lambda_3^2\}$ where $(\lambda_1, \lambda_2, \lambda_3)$ is the principal stretch, are

$$\begin{bmatrix} T'_{11} \\ T'_{22} \\ T'_{33} \end{bmatrix} = \begin{bmatrix} \lambda'_{11} + 2\mu'_{11} & \lambda'_{12} & \lambda'_{13} \\ \lambda'_{21} & \lambda'_{22} + 2\mu'_{22} & \lambda'_{23} \\ \lambda'_{31} & \lambda'_{32} & \lambda'_{33} + 2\mu'_{33} \end{bmatrix} \begin{bmatrix} \partial_1 u_1 \\ \partial_2 u_2 \\ \partial_3 u_3 \end{bmatrix}, \tag{3a}$$

$$T'_{ik} = \mu'_{ik} \partial_i u_k + \mu'_{ki} \partial_k u_i \quad (i \ne k). \tag{3b}$$

Here (i, k) take on values $(1, 2, 3)$ and (T'_{ik}, u_i, x_i) are scalar components of $(\boldsymbol{T}', \boldsymbol{u}, \boldsymbol{x})$. Operator ∂_i is the component of the gradient ∇ associated with coordinate x_i, and $(\lambda'_{ik}, \mu'_{ik})$ are generalized Lamé constants defined by

$$(\lambda'_{ik}, \mu'_{ik}) = (\lambda_{ik}, \mu_{ik})\lambda_k^2, \tag{4a}$$

$$\frac{1}{2}\lambda_{ik} = \frac{\partial \alpha_0}{\partial \lambda_k^2} + \lambda_i^2 \frac{\partial \alpha_1}{\partial \lambda_k^2} + \lambda_i^4 \frac{\partial \alpha_2}{\partial \lambda_k^2}, \tag{4b}$$

$$\mu_{ik} = \mu_{ki} = \alpha_1 + \alpha_2(\lambda_i^2 + \lambda_k^2). \tag{4c}$$

Because configuration $\aleph_0$ is homogeneous, the perturbation balance of linear momentum in a Cartesian basis reduces to

$$\nabla \cdot \boldsymbol{T}' = \rho \ddot{\boldsymbol{u}} + \boldsymbol{Q}. \tag{5}$$

Here ρ is mass density, $\boldsymbol{Q}$ is a body force associated with $\boldsymbol{u}$, and a superposed dot signifies time differentiation. Perturbation traction on a surface in $\aleph^*$ with outwardly directed normal $\boldsymbol{n}$ is given by vector

$$\boldsymbol{t}'^{(n)} = \boldsymbol{T}'\boldsymbol{n} + \boldsymbol{T}\boldsymbol{n}[\boldsymbol{n}.(\nabla \boldsymbol{u})\boldsymbol{n}] - \boldsymbol{T}(\nabla \boldsymbol{u})^T \boldsymbol{n}. \tag{6}$$

In a principal basis, a Hadamard material can, in view of (1), be characterized by

$$\alpha_0 = 2J\frac{dG(J)}{dJ}, \qquad \alpha_1 = \frac{1}{\sqrt{J}}(a_0 - b_0 I), \qquad \alpha_2 = \frac{b_0}{\sqrt{J}}, \tag{7a}$$

$$I = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad II = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad J = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{7b}$$

Here $G(1) = 0$ and (a_0, b_0) are material constants, where $a_0 - b_0 = \mu$ and μ is the shear modulus for infinitesimal deformation. A simple model for the subclass of compressible isotropic neo-Hookean materials arises when [Brock 2001; Brock and Georgiadis 2001]

$$b_0 = 0, \quad G(J) = \mu\Big(\frac{1}{\sqrt{J}} - 1\Big). \tag{8}$$

Such a material for infinitesimal deformations exhibits a Poisson's ratio of 0.25. Use of (7) and (8) in (3) and (4) gives the constitutive forms

$$\frac{1}{\mu}T'_{11} = B_1\partial_1 u_1 + \Big(\frac{2}{J} - b_1\Big)(\partial_2 u_2 + \partial_3 u_3), \tag{9a}$$

$$\frac{1}{\mu}T'_{22} = B_2\partial_2 u_2 + \Big(\frac{2}{J} - b_2\Big)(\partial_3 u_3 + \partial_1 u_1), \tag{9b}$$

$$\frac{1}{\mu}T'_{33} = B_3\partial_3 u_3 + \Big(\frac{2}{J} - b_3\Big)(\partial_1 u_1 + \partial_2 u_2), \tag{9c}$$

$$T'_{ik} = T'_{ki} = \mu[(b\partial)_i u_k + (b\partial)_k u_i] \quad (i \neq k). \tag{9d}$$

In (9) dimensionless parameters

$$b_k = \frac{\lambda_k^2}{\sqrt{J}}, \quad B_k = \frac{2}{J} + b_k. \tag{10}$$

In (9d) and (10), (i, k) take on values $(1, 2, 3)$. Equations (6) and (9) for a surface in $\aleph^*$ with an outwardly directed normal in the negative x_3-direction give $t_3'^{(-3)} = -T'_{33}$ and, for $k = (1, 2)$,

$$t_k'^{(-3)} = -\mu\Big(\frac{1}{J}\partial_k u_3 + b_3\partial_3 u_k\Big). \tag{11}$$

In view of (1) and (7b) the principal stretches λ_k are obtained as functions of the homogeneous principal Cauchy stress T_k from the coupled nonlinear equations

$$\frac{T_k}{\mu} + \frac{1}{J} - b_k = 0 \quad \Big(\frac{T_k}{\mu} < \frac{1}{J}\Big). \tag{12}$$

The parenthetical restriction on tensile Cauchy stress guarantees that coefficients in (9) are nonnegative. Tensile stress of the same order of magnitude as μ is not precluded, for example, $(T_k)_{\max} < \mu/\sqrt{2}$ in plane strain [Brock 2001; Brock and Georgiadis 2001]. For convenience the temporal variable $\tau = v_0 \times$ time is introduced, where $v_0 = \sqrt{\mu/\rho}$ is the rotational wave speed for isotropic infinitesimal deformation. Thus $(\boldsymbol{u}, \boldsymbol{T}')$ are functions of $(\boldsymbol{x}, \tau)$, and (5) and (9) combine to give

$$\frac{2}{J}\nabla(\nabla\cdot\boldsymbol{u}) + (\nabla_S^2 - \partial^2)\boldsymbol{u} = \frac{\boldsymbol{Q}}{\mu} \qquad (\tau > 0), \tag{13a}$$

$$(\boldsymbol{u}, \boldsymbol{T}', \boldsymbol{Q}) \equiv 0 \qquad (\tau \leq 0). \tag{13b}$$

Here ∂ signifies differentiation with respect to τ. We now introduce operators

$$\nabla_S^2 = b_k\partial_k^2, \quad \nabla_D^2 = B_k\partial_k^2 = \nabla_S^2 + \frac{2}{J}\nabla^2. \tag{14}$$

The summation convention applies and ∇^2 is the Laplacian. For the homogeneous case (13a) lends itself to a decomposition of Helmholtz type [Achenbach 1973]:

$$\boldsymbol{u} = \boldsymbol{u}_D + \boldsymbol{u}_S, \tag{15a}$$

$$(\nabla_S^2 - \partial^2)\boldsymbol{u}_S = 0, \qquad \nabla \cdot \boldsymbol{u}_S = 0, \tag{15b}$$

$$(\nabla_D^2 - \partial^2)\boldsymbol{u}_D = 0, \qquad \boldsymbol{u}_D = \nabla \boldsymbol{u}_D. \tag{15c}$$

3. Concentrated force

Convolution of the concentrated force solution can be used to study the effect of dynamically induced body forces on equilibrium configuration $\aleph$. In this case $\Re$ is unbounded, and nonequilibrium configuration $\aleph^*$ arises due to imposition for $\tau > 0$ of a concentrated force at $\boldsymbol{x} = 0$, that is,

$$\boldsymbol{Q} = \boldsymbol{P}(\tau)\delta(x_1)\delta(x_2)\delta(x_3). \tag{16}$$

Here δ is the Dirac function and $\boldsymbol{P} \equiv 0$ $(\tau < 0)$. After [Stakgold 1967] we treat $\Re$ as half-spaces $x_3 > 0$ and $x_3 < 0$. The homogeneous form of (13a) is the field equation in each half-space, and (16) and the welding of the half-spaces are manifested as conditions for $\tau > 0$ on interface $x_3 = 0$:

$$[\boldsymbol{u}]_-^+ = 0, \tag{17a}$$

$$b_3[\partial_3 u_k]_-^+ + \frac{2}{J}[\partial_k u_3]_-^+ = \frac{1}{\mu}P_k(\tau)\delta(x_1)\delta(x_2), \quad k = (1, 2), \tag{17b}$$

$$\frac{2}{J}[\partial_1 u_1]_-^+ + \frac{2}{J}[\partial_2 u_2]_-^+ + B_3[\partial_3 u_3]_-^+ = \frac{1}{\mu}P_3(\tau)\delta(x_1)\delta(x_2). \tag{17c}$$

Here $[f]_-^+$ signifies a jump in quantity f as the interface is crossed from half-space $x_3 < 0$ to half-space $x_3 > 0$. Solutions to (13)–(15) and (17) must be bounded above as $|\boldsymbol{x}| \to \infty$ for finite $\tau > 0$.

To obtain these solutions, unilateral and multiple bilateral Laplace transforms are introduced [van der Pol and Bremmer 1950; Sneddon 1972]:

$$\hat{f} = \int f(\tau)\exp(-p\tau)\,d\tau, \tag{18a}$$

$$f^* = \iint \hat{f}(x_1, x_2)\exp p(-q_1 x_1 - q_2 x_2)\,dx_1\,dx_2. \tag{18b}$$

For $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(q_1, q_2) = 0$ integration can be taken over the positive τ-axis for (18a), and over the entire x_1- and x_2-axes for (18b). Application of (18) to the homogeneous form of (13a) in view of (13b) and (15) gives for $x_3 > 0(+)$ and $x_3 < 0(-)$

$$\boldsymbol{u}_S^* = (U_1^\pm, U_2^\pm, U_3^\pm)\exp(-p\omega_S|x_3|), \tag{19a}$$

$$q_1 U_1^\pm + q_2 U_2^\pm \mp \omega_S U_3^\pm = 0, \tag{19b}$$

$$\boldsymbol{u}_D^* = p(q_1, q_2, \mp\omega_D)U_D^\pm \exp(-p\omega_D|x_3|). \tag{19c}$$

In (19) (ω_S, ω_D) are radicals:

$$\omega_S = \frac{1}{\sqrt{b_3}}\sqrt{1 - b_1 q_1^2 - b_2 q_2^2}, \qquad \mathrm{Re}(\omega_S) \geq 0, \tag{20a}$$

$$\omega_D = \frac{1}{\sqrt{B_3}}\sqrt{1 - B_1 q_1^2 - B_2 q_2^2}, \qquad \mathrm{Re}(\omega_D) \geq 0. \tag{20b}$$

Application of (18) to (17) and substitution of (19) gives the results

$$p^2 U_D^{\pm} = \frac{1}{2\mu\Delta}\Big[\pm\hat{P}_3 - \frac{1}{\omega_S}(q_1\hat{P}_1 + q_2\hat{P}_2)\Big], \tag{21a}$$

$$pU_k^{\pm} = \frac{-\hat{P}_k}{2\mu b_3 \omega_S} - p^2 q_k U_D^{\pm}, \quad k = (1, 2). \tag{21b}$$

Here $k = (1, 2)$ and

$$\Delta = 1 + (b_1 - b_3)q_1^2 + (b_2 - b_3)q_2^2. \tag{22}$$

4. Transform inversion

The inversion operation for (18b) is [van der Pol and Bremmer 1950; Sneddon 1972]

$$\hat{f}(x_1, x_2) = \Big(\frac{p}{2\pi i}\Big)^2 \iint f^* \exp p(q_1 x_1 + q_2 x_2)\, dq_1\, dq_2. \tag{23}$$

If there are no branch points or poles there, integration can be taken along the entire $\mathrm{Im}(q_1)$ and $\mathrm{Im}(q_2)$-axes. In view of (19) and (21), f^* for any contribution to $(\hat{\boldsymbol{u}}_S, \hat{\boldsymbol{u}}_D)$ exhibits one of the following forms:

$$f^* = \frac{1}{\Delta}\Big(q_k, \frac{q_k^2}{\omega}, \omega\Big)\exp(-p\omega|x_3|), \tag{24a}$$

$$f^* = \frac{1}{\omega}\exp(-p\omega|x_3|). \tag{24b}$$

Here $k = (1, 2)$ and $\omega = (\omega_S, \omega_D)$. Thus for $x_3 > 0(+)$ and $x_3 < 0(-)$ the right-hand side of (23) reduces to the operations

$$(\mp\partial_3\partial_k, \partial_k^2, \partial_3^2)\Big(\frac{1}{2\pi i}\Big)^2 \iint \exp\big(p(q_1 x_1 + q_2 x_2 - \omega|x_3|\big)\frac{dq_1\, dq_2}{\Delta\omega}, \tag{25a}$$

$$\Big(\frac{1}{2\pi i}\Big)^2 \iint \exp\big(p(q_1 x_1 + q_2 x_2 - \omega|x_3|)\big)\frac{dq_1\, dq_2}{\omega}. \tag{25b}$$

For the integration procedure, results in [Brock 2012; 2013] suggest transformations

$$q_1 = q\cos\psi, \quad q_2 = q\sin\psi, \tag{26a}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \tag{26b}$$

Here $|\psi| < \pi/2$ and $|x, y, \mathrm{Im}(q)| < \infty$, and (q, ψ) and $(y = 0, x, \psi)$ form quasipolar coordinate systems.

Thus the integration procedures in (25a) and (25b) respectively become

$$\frac{1}{i\pi}\int_{\Psi} d\psi \frac{1}{2\pi i}\int \frac{|q|}{\Delta\omega} \exp p(qx-\omega|x_3|)\,dq, \tag{27a}$$

$$\frac{1}{i\pi}\int_{\Psi} \frac{|q|}{\omega} \exp(qx-\omega|x_3|)\,dq. \tag{27b}$$

Here Ψ signifies integration over the range $|\psi| < \pi/2$, and integration with respect to q is over the entire $\mathrm{Im}(q)$-axis. In (27)

$$\Delta = 1+(b_3-b)q^2, \tag{28a}$$

$$\omega_S = \frac{1}{\sqrt{b_3}}\sqrt{1-bq^2}, \qquad b = b_1\cos^2\psi + b_2\sin^2\psi, \tag{28b}$$

$$\omega_D = \frac{1}{\sqrt{B_3}}\sqrt{1-Bq^2}, \quad B = \frac{2}{J}+b. \tag{28c}$$

Equation (28) shows that conditions $\mathrm{Re}(\omega_S) \geq 0$ and $\mathrm{Re}(\omega_D) \geq 0$ hold in the q-plane with cuts $\mathrm{Im}(q) = 0$, $|\mathrm{Re}(q)| > 1/\sqrt{b}$ and $\mathrm{Im}(q) = 0$, $|\mathrm{Re}(q)| > 1/\sqrt{B}$, respectively. When $b_3 > b$, Δ in (27a) exhibits roots $q = \pm 1/\sqrt{b-b_3}$. Because $\sqrt{B} > \sqrt{b} > \sqrt{b-b_3}$ these lie on the branch cuts of (ω_S, ω_D). For $b > b_3$ roots $q = \pm i/\sqrt{b_3-b}$ lie on the $\mathrm{Im}(q)$-axis, so that deformations in the integration contour in (27a) are required. Instead the de Hoop method [de Hoop 1960] is employed to change the integration contour to a path in the q-plane parametrized by the positive real variable t. Thus (27a) gives

$$\frac{1}{i\pi}\int_{\Psi} d\psi \frac{1}{i\pi}\,\mathrm{Re}\int \exp(-pt)\frac{q_+ q_+'\,dt}{\Delta(q_+)\omega(q_+)}. \tag{29}$$

For the case ω_S, t-integration is over the range (S, ∞) and

$$\sqrt{b}S^2 q_+ = -\frac{tx}{\sqrt{b}} + i\frac{|x_3|}{\sqrt{b_3}}\sqrt{t^2-S^2}, \quad q_+' = i\sqrt{\frac{b_3}{b}}\frac{\omega_S(q_+)}{\sqrt{t^2-S^2}}. \tag{30a}$$

For the case ω_D, t-integration is over the range (D, ∞) and

$$\sqrt{B}D^2 q_+ = -\frac{tx}{\sqrt{B}} + i\frac{|x_3|}{\sqrt{B_3}}\sqrt{t^2-D^2}, \quad q_+' = i\sqrt{\frac{B_3}{B}}\frac{\omega_D(q_+)}{\sqrt{t^2-D^2}}. \tag{30b}$$

Parameters (S, D) are given by

$$S = \sqrt{\frac{x^2}{b}+\frac{x_3^2}{b_3}}, \quad D = \sqrt{\frac{x^2}{B}+\frac{x_3^2}{B_3}}. \tag{31}$$

Taking the real part of the integrand in (29) gives for (ω_S, ω_D), respectively,

$$-\frac{|x_3|}{\pi^2}\int_{\Psi}\frac{d\psi}{S^2}\int_S^\infty \frac{N_S}{M}\exp(-pt)\,dt, \quad -\frac{|x_3|}{\pi^2}\int_{\Psi}\frac{d\psi}{D^2}\int_D^\infty \frac{N_D}{M}\exp(-pt)\,dt. \tag{32a}$$

The analogous result for (27b) is

$$-\frac{|x_3|}{\pi^2 p}\int_{\Psi}\frac{d\psi}{bS^2}\exp(-pS), \quad -\frac{|x_3|}{\pi^2 p}\int_{\Psi}\frac{d\psi}{BD^2}\exp(-pD). \tag{32b}$$

In (32a) (M, N_S, N_D) are given by

$$M = [(b - b_3)t^2 - r^2]^2 + 2t^2(b - b_3)x_3^2, \tag{33a}$$

$$N_S = (b_3 - b)\left[\frac{2x^2}{b} + (b - b_3)\left(\frac{x^2}{b} - \frac{x_3^2}{b_3}\right)\right]t^2 - r^2 S^2, \tag{33b}$$

$$N_D = (b_3 - b)\left[\frac{2x^2}{B} + (b - b_3)\left(\frac{x^2}{B} - \frac{x_3^2}{B_3}\right)\right]t^2 - r^2 D^2. \tag{33c}$$

In (30)–(33) $r = \sqrt{x^2 + x_3^2}$, where $x = x_1 \cos\psi + x_2 \sin\psi$. Equations (21), (25), and (32) show that explicit dependence of $(\hat{\boldsymbol{u}}_S, \hat{\boldsymbol{u}}_D)$ on transform parameter p is confined to products $\hat{P}_k \exp(-pt)$. Inversion of $(\hat{\boldsymbol{u}}_S, \hat{\boldsymbol{u}}_D)$ can then be, in view of (18a) and (24), performed by inspection. For the case of common temporal load behavior $\boldsymbol{P}(\tau) = \boldsymbol{P}\delta(\tau)$ (15a) gives

$$\boldsymbol{u} = \frac{1}{2\pi^2}(\boldsymbol{P}\cdot\nabla)\nabla|x_3| \int_\Psi d\psi \left[\int_D^\tau \frac{N_D}{MD^2}\,dt - \int_S^\tau \frac{N_S}{MS^2}\,dt\right] + \frac{1}{2\pi^2}\boldsymbol{P}\nabla^2|x_3| \int_\Psi \frac{d\psi}{bS^2}(\tau - S). \tag{34}$$

Wave speeds (v_S, v_D) associated with the integration terms in (34) can be obtained by introducing the quasispherical coordinate system

$$x_1 = X\cos\theta\sin\phi, \quad x_2 = X\sin\theta\sin\phi, \quad x_3 = X\cos\phi. \tag{35}$$

Here $|X| < \infty$, $|\theta| < \pi/2$, $0 < \phi < \pi/2$, and in view of (31),

$$v_S = c_S v_0, \qquad v_D = c_D v_0, \tag{36a}$$

$$c_S = \frac{\sqrt{b_3 b}}{\sqrt{b_3\sin^2\phi + b\cos^2\phi}}, \qquad b = b_1\cos^2\theta + b_2\sin^2\theta, \tag{36b}$$

$$c_D = \frac{\sqrt{B_3 B}}{\sqrt{B_3\sin^2\phi + B\cos^2\phi}}, \qquad B = \frac{2}{J} + b. \tag{36c}$$

5. Displacement discontinuity

In this instance configuration $\aleph^*$ arises in unbounded $\Re$ due to the existence for $\tau > 0$ of a finite discontinuity in displacement at $\boldsymbol{x} = 0$. Convolution of the solution for this problem can be used to study dynamic perturbation of $\aleph$ by formation of a crack on the plane $x_3 = 0$. We again treat two half-spaces $x_3 > 0(+)$ and $x_3 < 0(-)$. Equations (13)–(15) are valid for $x_3 \neq 0$, with $\boldsymbol{Q} \equiv 0$, but Equation (17) for $x_3 = 0$, $\tau > 0$ is replaced with

$$[\boldsymbol{u}]_-^+ = \boldsymbol{U}^C(\tau)\delta(x_1)\delta(x_2), \qquad \boldsymbol{t}'^{(3)} + \boldsymbol{t}'^{(-3)} = 0. \tag{37}$$

Here $\boldsymbol{U}^C$ vanishes for $\tau \le 0$, but for $\tau > 0$ it is continuous and bounded above. The transform operation (18) produces (19), (20), and (22), with $(U_D^\pm, U_k^\pm)$ now given by

$$pU_D^{\pm} = -\frac{\beta_3}{2}\hat{U}_3^C - \frac{1}{2\Delta}\Big[\Big(\frac{1}{J}+b_3\Big)\frac{\omega_D}{b_3}\hat{U}_3^C \mp 2(q_1\hat{U}_1^C + q_2\hat{U}_2^C)\Big], \tag{38a}$$

$$U_k^{\pm} = \pm\frac{1}{2}\hat{U}_k^C + \frac{q_k}{2\Delta}\Big[\Big(\frac{1}{J}+b_3\Big)\frac{\omega_S}{b_3}\hat{U}_3^C \mp 2(q_1\hat{U}_1^C + q_2\hat{U}_2^C)\Big], \tag{38b}$$

$$\beta_3 = \frac{1}{(Jb_3)^2}\Big(\frac{2}{2+Jb_3} - b_3\Big). \tag{38c}$$

In (38) $k = (1, 2)$, and comparison of (21) and (38) indicates that (25) is replaced by

$$(\mp\partial_3\partial_k^2, \partial_k\partial_3^2)\Big(\frac{1}{2\pi i}\Big)^2 \iint \exp p(q_1x_1 + q_2x_2 - \omega|x_3|)\frac{dq_1\,dq_2}{p\Delta\omega}, \tag{39a}$$

$$(\mp\partial_3, \partial_k)\Big(\frac{1}{2\pi i}\Big)^2 \iint \exp p(q_1x_1 + q_2x_2 - \omega|x_3|)\,p\frac{dq_1\,dq_2}{\omega}. \tag{39b}$$

The transform inversion process involving (19), (20), (22), and (39) is similar to that for the concentrated force problem. For the case $\boldsymbol{U}^C(\tau) = \boldsymbol{U}^C$ the results are

$$\boldsymbol{u} = \boldsymbol{u}^D + \boldsymbol{u}^S + \frac{U_k^C}{2\pi^2}\nabla\partial_3\partial_k|x_3| \int_\Psi d\psi\Big[\int_D^\tau \frac{N_D}{MD^2}(\tau - t)\,dt - \int_S^\tau \frac{N_S}{MS^2}(\tau - t)\,dt\Big]$$
$$+ \frac{U_3^C}{2\pi^2}\Big(1 + \frac{1}{Jb_3}\Big)\nabla\partial_3^2|x_3| \int_\Psi d\psi\Big[\int_D^\tau \frac{N_D}{MD^2}(\tau - t)\,dt - \int_S^\tau \frac{N_S}{MS^2}(\tau - t)\,dt\Big], \tag{40a}$$

$$\boldsymbol{u}^D = \frac{U_3^C}{2\pi^2}\nabla|x_3| \int_\Psi \frac{\beta_3}{BD^2}\,d\psi\,H(\tau - D), \tag{40b}$$

$$u_k^S = \frac{U_k^C}{2\pi^2}\partial_3|x_3| \int_\Psi \frac{d\psi}{bS^2}H(\tau - S), \tag{40c}$$

$$u_3^S = \frac{1}{2\pi^2}\Big[U_k^C\partial_k + U_3^C\Big(1 + \frac{1}{Jb_3}\Big)\partial_3\Big]|x_3| \int_\Psi \frac{d\psi}{bS^2}H(\tau - S). \tag{40d}$$

Here $\boldsymbol{u}^D \neq \boldsymbol{u}_D$ and $\boldsymbol{u}^S \neq \boldsymbol{u}_S$, $k = (1, 2)$, the summation convention holds, and H is the unit step function.

6. Behavior on a principal plane: concentrated surface force

Convolution of the solution for this problem can serve as the basis for study of dynamic perturbation by dynamic contact. Thus $\Re$ is the half-space $x_3 > 0$ with traction-free surface $x_3 = 0$. Configuration $\aleph^*$ arises due to imposing for $\tau > 0$ the surface load

$$t_k'^{(-3)} = -P_{3k}(\tau)\delta(x_1)\delta(x_2). \tag{41}$$

Here $k = (1, 2, 3)$ and $P_{3k} \equiv 0$ $(\tau < 0)$ and is bounded above for $\tau > 0$. Because $\Re$ in $\aleph$ has no surface traction, $T_3 \equiv 0$ and (9c) and (11) give

$$b_3 = \frac{1}{J}, \quad B_3 = \frac{3}{J}, \quad \frac{1}{b_1b_2} = \sqrt{J}\sqrt{\frac{b_1}{b_2} - \frac{T_1}{\mu}} = \sqrt{J}\sqrt{\frac{b_2}{b_1} - \frac{T_2}{\mu}}. \tag{42}$$

Equations (13)–(15), with $\boldsymbol{Q} \equiv 0$, govern for $(x_3, \tau) > 0$. Application of (18) in view of boundary condition (41) leads to expressions for (U_D^+, U_k^+). Of interest here is displacement $\boldsymbol{u}^0$ on the half-space surface $x_3 = 0$, and use of these expressions and variables defined by (26) in (23) give the transform

$$\hat{\boldsymbol{u}}^0 = \frac{1}{i\pi}\int_\Psi J\,d\psi \frac{1}{2\pi i}\int \frac{|q|\,dq}{\mu p R}\boldsymbol{N}\cdot\hat{\boldsymbol{t}}'^{(-3)}\exp(pqx), \tag{43a}$$

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \frac{1}{\omega_S}\begin{bmatrix} N_1 & N_{12} \\ N_{12} & N_2 \end{bmatrix}, \quad \begin{bmatrix} N_{31} \\ N_{32} \end{bmatrix} = -\begin{bmatrix} N_{13} \\ N_{23} \end{bmatrix} = qN\begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix}, \quad N_{33} = -\omega_D\Delta. \tag{43b}$$

Again $x = x_1\cos\psi + x_2\sin\psi$, but the definition of terms in (43) is influenced by (42):

$$\omega_S = \sqrt{q^2 - T}, \quad \omega_D = \frac{1}{\sqrt{3}}\sqrt{-q^2 - T}, \quad T = (1+Jb)q^2 - J, \tag{44a}$$

$$N = 2\omega_S\omega_D - T, \quad R = 4q^2\omega_S\omega_D + T^2, \tag{44b}$$

$$(N_1, N_2) = Mq^2(\cos^2\psi, \sin^2\psi) - R, \quad N_{12} = Mq^2\sin\psi\cos\psi, \tag{44c}$$

$$M = q^2 + \omega_S(2\omega_D - 3\omega_S), \quad \Delta = J + q^2(1 - Jb). \tag{44d}$$

Integration in (43a) is along the entire $\mathrm{Im}(q)$-axis but, as a special case of (30), the contour can be changed to a path around branch cuts on the $\mathrm{Re}(q)$-axis. In view of (44a) the branch points are defined by $q = \pm(1/\sqrt{b}, 1/\sqrt{B})$. In addition, the Rayleigh function R exhibits real roots $q = \pm q_R$, where $q_R > 1/\sqrt{b}$ and is defined by

$$q_R = \frac{\sqrt{J}}{\sqrt{Jb + 1 - 2/\sqrt{3}}}. \tag{45}$$

These values of q define the body wave speeds (v_S, v_D) — see (36) — and the Rayleigh speed v_R in the principal plane $x_3 = 0$:

$$v_S = c_S v_0, \quad v_D = c_D v_0, \quad v_R = c_R v_0, \tag{46a}$$

$$c_S = \sqrt{b}, \quad c_D = \sqrt{B}, \quad c_R = \sqrt{b + \frac{1}{J}\left(1 - \frac{2}{\sqrt{3}}\right)}, \tag{46b}$$

$$b = b_1\cos^2\theta + b_2\sin^2\theta, \quad B = \frac{2}{J} + b. \tag{46c}$$

In (46c) quasipolar coordinate $|\theta| < \pi/2$ is measured with respect to the x_1-direction in the principal plane. Changing the integration path by means of the Cauchy theorem gives expressions that can be inverted by inspection. For the step function $P_{3k}(\tau) = P_{3k}H(\tau)$,

$$u_1^0 = J\int_\Psi U_0\cos\psi\,d\psi + \frac{JP_{31}}{\mu\pi^2}\int_\Psi \frac{q\,d\psi}{\Omega_S x}H\left(\tau - \frac{|x|}{c_S}\right), \tag{47a}$$

$$u_2^0 = J\int_\Psi U_0\sin\psi\,d\psi + \frac{JP_{32}}{\mu\pi^2}\int_\Psi \frac{q\,d\psi}{\Omega_S x}H\left(\tau - \frac{|x|}{c_S}\right), \tag{47b}$$

$$u_3^0 = J\int_\Psi U_3\,d\psi. \tag{47c}$$

Equation (47) involves the definitions $P_{12} = P_{31}\cos\psi + P_{32}\sin\psi$ and

$$U_0 = \frac{P_{33}}{\mu\pi^2 x}\frac{6q^2 T\Omega_D\omega_S}{(3T^2-4q^4)(T+2q^2)}H\left(\tau-\frac{|x|}{c_D}\right)H\left(\frac{|x|}{c_S}-\tau\right)$$
$$-\frac{JP_{33}}{\mu\pi c_R}\frac{(\sqrt{3}-1)\,\mathrm{sgn}(x)}{8-3\sqrt{3}(1+Jb)}\delta\left(\tau-\frac{|x|}{c_R}\right)$$
$$-\frac{P_{12}}{\mu\pi^2 x}\frac{3q^3}{(3T^2-4q^4)(T+2q^2)}\times 4\Omega_D\Omega_S H\left(\tau-\frac{|x|}{c_D}\right)+\frac{16q^2\Omega_D\Omega_S^2-3T^3}{\Omega_S(T-2q^2)}H\left(\tau-\frac{|x|}{c_S}\right), \quad (48a)$$

$$U_3 = -\frac{P_{12}}{\mu\pi^2 x}\frac{6q^2 T\Omega_D\omega_S}{(3T^2-4q^4)(T+2q^2)}H\left(\tau-\frac{|x|}{c_D}\right)H\left(\frac{|x|}{c_S}-\tau\right)$$
$$+\frac{JP_{12}}{\mu\pi c_R}\frac{(\sqrt{3}-1)\,\mathrm{sgn}(x)}{8-3\sqrt{3}(1+Jb)}\delta\left(\tau-\frac{|x|}{c_R}\right)$$
$$+\frac{P_{33}}{\mu\pi^2 x}\frac{3qT\Omega_D}{(3T^2-4q^4)(T^2-4q^4)}\times T^2 H\left(\tau-\frac{|x|}{c_D}\right)+4q^2\Omega_S\Omega_D H\left(\tau-\frac{|x|}{c_S}\right), \quad (48b)$$

$$\Omega_S = \sqrt{T-q^2}, \quad \Omega_D = \frac{1}{\sqrt{3}}\sqrt{T+q^2}, \quad q = \frac{\tau}{x}. \quad (48c)$$

7. Dimensionless speed values

The $\Re$ considered here is an idealized isotropic neo-Hookean solid with an effective Poisson's ratio of 0.25 for infinitesimal strain. It is chosen for purposes of illustration and, therefore, sample values of dimensionless speeds (c_S, c_D) in (36) are given for the spherical octant $0 \le (\theta, \phi) \le 90°(\pi/2)$ in Tables 1 and 2. Configuration $\aleph$ results from the plane strain defined by $\lambda_3 = 1$ and $T_1 + T_2 = 0$, so that (9c) and (11) give

$$b_1 = \sqrt{\frac{\chi}{\sqrt{J}}}, \quad b_2 = \frac{1}{\sqrt{\chi\sqrt{J}}}, \quad b_3 = \frac{1}{\sqrt{J}}, \quad B_k = \frac{2}{J} + b_k, \quad (49a)$$

$$J = \frac{2\chi}{1+\chi^2}, \quad \chi = \frac{T_1}{\mu} + \sqrt{1+\left(\frac{T_1}{\mu}\right)^2}, \quad \frac{T_3}{\mu} = \frac{1}{\sqrt{J}} - \frac{1}{J}. \quad (49b)$$

	$\phi = 0°$	15°	30°	45°	60°	75°	90°
$\theta = 0°$	1.04811	1.04847	1.04945	1.05080	1.05215	1.05314	1.05351
15°	1.04811	1.04813	1.04817	1.04824	1.04830	1.04835	1.04837
30°	1.04811	1.04070	1.04459	1.04110	1.03765	1.03515	1.03423
45°	1.04811	1.04578	1.03942	1.03095	1.02268	1.01675	1.01460
60°	1.04811	1.04425	1.03392	1.02030	1.00720	0.99779	0.99459
75°	1.04811	1.04307	1.29664	1.01216	0.99552	0.98385	0.97917
90°	1.04811	1.04262	1.02805	1.00911	0.99117	0.97863	0.97416

Table 1. Dimensionless speed c_S in spherical octant $0 \le (\theta, \phi) \le 90°$ for $T_1/\mu = 0.2$.

	$\phi = 0°$	15°	30°	45°	60°	75°	90°
$\theta = 0°$	1.77148	1.77169	1.77228	1.77308	1.77388	1.77446	1.77467
15°	1.77148	1.77149	1.77152	1.77156	1.77160	1.77162	1.77163
30°	1.77148	1.77093	1.76943	1.76738	1.76534	1.76385	1.76331
45°	1.77148	1.77014	1.76651	1.76159	1.75671	1.75316	1.75186
60°	1.77148	1.76934	1.76354	1.75571	1.74798	1.74238	1.74035
75°	1.77148	1.76874	1.76132	1.75134	1.74152	1.73444	1.73187
90°	1.77148	1.76852	1.76050	1.74973	1.73915	1.73152	1.72876

Table 2. Dimensionless speed c_D in spherical octant $0 \le (\theta, \phi) \le 90°$ for $T_1/\mu = 0.2$.

$T_1/\mu =$	-0.2	-0.1	0.1	0.2
$\theta = 0°$	0.94068	0.97015	1.03015	1.01227
15°	0.92778	0.97360	1.02689	1.05418
30°	0.96691	0.98296	1.01794	1.03667
45°	0.99243	0.99560	1.00558	1.01226
60°	1.01731	1.00808	0.99307	0.98725
75°	1.03515	1.01712	0.98380	0.96855
90°	1.04160	1.02041	0.98039	0.96159

Table 3. Dimensionless speed c_S in circular quadrant $0 \le \theta \le 90°$ for various T_1.

$T_1/\mu =$	-0.2	-0.1	0.1	0.2
$\theta = 0°$	1.74779	1.73887	1.72729	1.72453
15°	1.75162	1.74079	1.72535	1.72064
30°	1.76204	1.74604	1.72004	1.70997
45°	1.77618	1.75319	1.71275	1.69528
60°	1.79020	1.76031	1.70544	1.68047
75°	1.80039	1.76550	1.70006	1.66955
90°	1.80411	1.76740	1.69809	1.66552

Table 4. Dimensionless speed c_D in circular quadrant $0 \le \theta \le 90°$ for various T_1.

Because (49b) gives $J < 1$, any finite $|T_1|$ satisfies the parenthetical restriction in (11). In similar fashion dimensionless speeds (c_S, c_D, c_R) associated with principal plane $x_3 = 0$ in (45) are given for the circular quadrant $0 \le \theta \le 90°(\pi/2)$ in Tables 3, 4, and 5. Here $\aleph$ is induced by uniaxial loading $(T_2, T_3) = 0$. Equations (9), (11), and (42) give

$$b_1 = J^{3/2}, \quad (b_2, b_3) = \frac{1}{J}, \quad J^{5/2} - \frac{T_1}{\mu} J - 1 = 0 \quad \left(\frac{T_1}{\mu} < \frac{1}{2^{0.4}}\right). \tag{50}$$

In this case the restriction in (11) imposes a tension limit on T_1. Equations (9)–(11) show that a nonhydrostatic principal stress in $\Re$ gives configuration $\aleph$, whose infinitesimal perturbation response is anisotropic. Even for the simple nonhydrostatic cases chosen, the data in Tables 1–5 clearly illustrate this behavior.

$T_1/\mu =$	-0.2	-0.1	0.1	0.2
$\theta = 0°$	0.84679	0.88324	0.95526	0.99078
15°	0.85467	0.88703	0.95174	0.98400
30°	0.87582	0.89729	0.94207	0.96521
45°	0.90392	0.91112	0.92871	0.93895
60°	0.93118	0.92474	0.91514	0.91193
75°	0.95063	0.93458	0.90508	0.89165
90°	0.95765	0.93816	0.90138	0.88409

Table 5. Dimensionless speed c_R in circular quadrant $0 \le \theta \le 90°$ for various T_1.

8. Comments

Equations (34), (40), and (47) express exact transient solutions as integrals with respect to quasipolar angle measure $|\psi| < \pi/2$. Equations (34) and (47) also exhibit integration with respect to a temporal variable t. This integration can, in fact, be performed with use of standard tables [Pierce and Foster 1956; Gradshteyn and Ryzhik 1980], but the results are cumbersome; for example, in (34), for $b > b_3$:

$$|x_3| \int_S^\tau \frac{N_S}{M}\,dt = \frac{1}{2\sqrt{r^2+x^2}} \operatorname{Im} \frac{r^2S^2 + A_S T}{\sqrt{(b-b_3)T}} \times \ln \frac{\sqrt{T} - \tau\sqrt{b-b_3}}{\sqrt{T} - S\sqrt{b-b_3}} \frac{\sqrt{T} + S\sqrt{b-b_3}}{\sqrt{T} + \tau\sqrt{b-b_3}}, \tag{51a}$$

$$A_S = 2\frac{x^2}{b} + (b-b_3)\left(\frac{x^2}{b} - \frac{x_3^2}{b_3}\right), \qquad T = x^2 + i|x_3|\sqrt{r^2+x^2}. \tag{51b}$$

Here $\sqrt{b-b_3} \to i\sqrt{b_3-b}$ when $b_3 > b$. Use of a quasipolar measure, therefore, renders (34), (40), and (47) as a "hybrid" of quasipolar and principal Cartesian coordinates, as well as general operators $(\nabla, \nabla^2, \boldsymbol{f} \cdot \boldsymbol{q})$. Moreover, dimensionless speeds (c_S, c_D) associated with (34) and (40) involve quasi-spherical measure (θ, ϕ), and (c_S, c_D, c_R) involve quasipolar measure θ. Nevertheless, the form of (34), (40), and (47) is not complicated and is similar to corresponding results for the linear isotropic case, compare (34) and [Achenbach 1973, Equation (3.92)].

This type of similarity is well known for linear quasistatic anisotropic results [Ting 1996]. Indeed, in light of results such as [Ting 1996; Jones 1999] it is often useful to categorize superimposed infinitesimal deformation equations in accordance with classes of anisotropic materials, for example, [Green and Zerna 1968; Beatty and Usmani 1975]. Due to the idealized neo-Hookean $\mathfrak{R}$ treated, this article took an ad hoc solution approach. However, the approach itself may be useful in generation of singular transient solutions in linear anisotropic solids [Willis 1965; Wang and Achenbach 1992], and such efforts are underway.

References

[Achenbach 1973] J. D. Achenbach, *Wave propagation in elastic solids*, North-Holland, Amsterdam, 1973.

[Beatty and Usmani 1975] M. F. Beatty and S. A. Usmani, "On the indentation of a highly elastic half-space", *Quart. J. Mech. Appl. Math.* **28** (1975), 47–62.

[Brock 2001] L. M. Brock, "Rapid sliding contact on a highly elastic pre-stressed material", *Int. J. Non-Linear Mech.* **36** (2001), 433–442.

[Brock 2012] L. M. Brock, "Two cases of rapid contact on an elastic half-space: sliding ellipsoidal die, rolling sphere", *J. Mech. Mater. Struct.* **7**:5 (2012), 469–483.

[Brock 2013] L. M. Brock, "Rapid contact on a pre-stressed highly elastic half-space: the sliding ellipsoid and rolling sphere", *J. Appl. Mech.* (*ASME*) **80**:2 (2013), Art. ID # 021023.

[Brock and Georgiadis 2001] L. M. Brock and H. G. Georgiadis, "An illustration of sliding contact at any constant speed on highly elastic half-spaces", *IMA J. Appl. Math.* **66**:6 (2001), 551–566.

[Gradshteyn and Ryzhik 1980] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*, Academic, New York, 1980.

[Green and Zerna 1968] A. E. Green and W. Zerna, *Theoretical elasticity*, 2nd ed., Clarendon, Oxford, 1968.

[de Hoop 1960] A. T. de Hoop, "A modification of Cagniard's method for solving seismic pulse problems", *Appl. Sci. Res. B* **8**:1 (1960), 349–356.

[Jones 1999] R. M. Jones, *Mechanics of composite materials*, 2nd ed., Taylor and Francis, New York, 1999.

[Love 1944] A. E. H. Love, *A treatise on the mathematical theory of elasticity*, 4th ed., Dover, New York, 1944.

[Pierce and Foster 1956] B. O. Pierce and R. M. Foster, *A short table of integrals*, 4th ed., Ginn, Waltham, MA, 1956.

[van der Pol and Bremmer 1950] B. van der Pol and H. Bremmer, *Operational calculus: based on the two-sided Laplace integral*, Cambridge University Press, Cambridge, 1950.

[Sneddon 1972] I. N. Sneddon, *The use of integral transforms*, McGraw-Hill, New York, 1972.

[Stakgold 1967] I. Stakgold, *Boundary value problems of mathematical physics*, vol. 1, MacMillan, New York, 1967.

[Ting 1996] T. C. T. Ting, *Anisotropic elasticity: theory and applications*, Oxford Engineering Science Series **45**, Oxford University Press, New York, 1996.

[Truesdell and Noll 1965] C. T. Truesdell and W. Noll, *The non-linear field theories of mechanics*, edited by W. Flugge, Handbuch der Physik **III/3**, Springer, Berlin, 1965.

[Wang and Achenbach 1992] C.-Y. Wang and J. D. Achenbach, "A new look at 2-D time-domain elastodynamic Green's functions for general anisotropic solids", *Wave Motion* **16**:4 (1992), 389–405.

[Willis 1965] J. R. Willis, "The elastic interaction energy of dislocation loops in anisotropic media", *Quart. J. Mech. Appl. Math.* **18** (1965), 419–433.

Received 7 Aug 2012. Revised 19 Nov 2012. Accepted 22 Nov 2012.

LOUIS MILTON BROCK: brock@engr.uky.edu
Department of Mechanical Engineering, University of Kentucky, 265 Ralph G. Anderson Building, Lexington KY 40506-0503, United States

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

msp.org/jomms

Founded by Charles R. Steele and Marie-Louise Steele

Cover photo: Ev Shafrir

See msp.org/jomms for submission guidelines.

JoMMS (ISSN 1559-3959) at Mathematical Sciences Publishers, 798 Evans Hall #6840, c/o University of California, Berkeley, CA 94720-3840, is published in 10 issues a year. The subscription price for 2013 is US $555/year for the electronic version, and $705/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to MSP.

JoMMS peer-review and production is managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Journal of Mechanics of Materials and Structures

Volume 8, No. 1 January 2013