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REFLECTION OF PLANE LONGITUDINAL WAVES FROM THE STRESS-FREE BOUNDARY OF A NONLOCAL, MICROPOLAR SOLID HALF-SPACE

AARTI KHURANA AND SUSHIL K. TOMAR

This work is concerned with plane waves propagating through an isotropic nonlocal micropolar solid. Two longitudinal waves and two sets of coupled transverse waves propagating with distinct speeds may travel in the medium. All these waves are found to be dispersive in nature. Reflection coefficients and energy ratios are presented for when a longitudinal displacement wave strikes at the stress-free boundary. The dispersion curves of various waves for a silicon crystal are computed numerically and depicted graphically. The effect of nonlocality on the reflection coefficients and energy ratios is observed. The energy balance law has been verified at each angle of incidence.

1. Introduction

The theory of nonlocal elasticity, developed in [Eringen 1972a; 1972b; 2002; Eringen and Edelen 1972], states that the nonlocal stress tensor at any reference point $\boldsymbol{x}$ of the body depends not only on the strain at the point $\boldsymbol{x}$ but also on the strains at all other points $\boldsymbol{x}'$ of the body. This observation is in accordance with the atomic theory of lattice dynamics and experimental observations on phonon dispersion [Chen et al. 2004]. In the limiting case, when the effects of strains at points other than $\boldsymbol{x}$ are neglected, one recovers the classical (local) theory of elasticity. The most general form of the constitutive relation in the nonlocal elasticity-type representation involves an integral over the entire region of interest. This integral contains a nonlocal kernel function, which describes the relative influence of the strains at various locations on the stress at a given location.

Polar theories, in principle, are nonlocal theories (see [Eringen 1999]) where the nonlocality is achieved through moment tensors associated with each point of the body. However, as the wavelengths of the waves transmitted become shorter, the number of moment tensors to be employed must be increased to provide sufficient accuracy in the prediction of the physical phenomena. The response of a body depends heavily on the ratio of the external characteristic length (L) to the internal characteristic length (l). In classical field theories, a ratio of $L/l \gg 1$ will yield reliable predictions. However, when $L/l \approx 1$, the classical field theories (local) fail and we must resort to nonclassical theories (nonlocal). One of the theories in which $L \approx l$ is the micropolar theory of elasticity. Some of the relevant papers on nonlocal theories are [Eringen 1984; Erbay et al. 1992; Wang and Dhaliwal 1993; Lazar and Kirchner 2006; Zeng et al. 2006; Najafi et al. 2012] among others.

In this paper, we have extended the work proposed in [Eringen 1984] by exploring the possibility of the propagation of plane elastic waves in a linear isotropic nonlocal micropolar solid. It is seen that there

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may exist four waves comprising of two coupled transverse waves and two independent longitudinal waves analogous to those existing in a local micropolar solid. In local micropolar elasticity, the longitudinal displacement wave is frequency independent, while the longitudinal microrotational wave and two sets of coupled transverse waves are frequency dependent. In nonlocal micropolar elasticity, all the longitudinal and transverse waves are frequency dependent. The reflection phenomena of a longitudinal displacement wave incident obliquely at a stress-free flat boundary of a nonlocal micropolar elastic half-space are investigated in detail. Reflection coefficients and their corresponding energy ratios are obtained analytically and depicted graphically for a silicon crystal against the angle of incidence. The reflection coefficients and energy ratios have been plotted for two values of the nonlocal parameter, namely $e_0 = 0$ and $e_0 = 0.39$. The parameter $e_0 = 0$ corresponds to the local micropolar medium. The sum of energy ratios is found to be unity at each angle of incidence which shows that there is no dissipation of energy during reflection at the free boundary surface of a nonlocal micropolar solid.

2. Basic equations and constitutive relations

For a linear anisotropic nonlocal micropolar solid, the strain energy density function W is given as [Eringen 2002]

$$W = \frac{1}{2}\iint \{A_{klmn}(\boldsymbol{x},\boldsymbol{x}')\epsilon_{kl}(\boldsymbol{x}')\epsilon_{mn}(\boldsymbol{x}) + B_{klmn}(\boldsymbol{x},\boldsymbol{x}')\gamma_{kl}(\boldsymbol{x}')\gamma_{mn}(\boldsymbol{x}) + C_{klmn}(\boldsymbol{x},\boldsymbol{x}')(\epsilon_{kl}(\boldsymbol{x}')\gamma_{mn}(\boldsymbol{x}) + \epsilon_{kl}(\boldsymbol{x})\gamma_{mn}(\boldsymbol{x}'))\}\, dv(\boldsymbol{x}')dv(\boldsymbol{x}), \tag{1}$$

where $\epsilon_{kl} = u_{l,k} - \varepsilon_{klm}\phi_m$ denotes the relative distortion tensor and $\gamma_{kl} = \phi_{k,l}$ is the curvature or wryness tensor. The nonlocal constitutive moduli possess the symmetries

$$A_{klmn}(\boldsymbol{x},\boldsymbol{x}') = A_{mnkl}(\boldsymbol{x}',\boldsymbol{x}) \quad \text{and} \quad B_{klmn}(\boldsymbol{x},\boldsymbol{x}') = B_{mnkl}(\boldsymbol{x}',\boldsymbol{x}).$$

In local micropolar elasticity, the force stress tensor ($t_{kl}(\boldsymbol{x})$) and couple stress tensor ($m_{kl}(\boldsymbol{x})$) are given in integral form by the nonlocal constitutive relations [Eringen 2002]

$$t_{kl}(\boldsymbol{x}) = \int \{A_{klmn}(\boldsymbol{x},\boldsymbol{x}')\epsilon_{mn}(\boldsymbol{x}') + C_{klmn}(\boldsymbol{x},\boldsymbol{x}')\gamma_{mn}(\boldsymbol{x}')\}\, dv(\boldsymbol{x}'), \tag{2}$$

$$m_{kl}(\boldsymbol{x}) = \int \{B_{lkmn}(\boldsymbol{x},\boldsymbol{x}')\gamma_{mn}(\boldsymbol{x}') + C_{mnlk}(\boldsymbol{x},\boldsymbol{x}')\epsilon_{mn}(\boldsymbol{x}')\}\, dv(\boldsymbol{x}'). \tag{3}$$

For an isotropic micropolar solid, the nonlocal elastic moduli are [Eringen 2002]

$$A_{klmn}(\boldsymbol{x},\boldsymbol{x}') = \lambda\delta_{kl}\delta_{mn} + (\mu+K)\delta_{km}\delta_{ln} + \mu\delta_{kn}\delta_{lm},$$
$$B_{klmn}(\boldsymbol{x},\boldsymbol{x}') = \alpha\delta_{kl}\delta_{mn} + \gamma\delta_{km}\delta_{ln} + \beta\delta_{kn}\delta_{lm} \quad \text{and} \quad C_{klmn}(\boldsymbol{x},\boldsymbol{x}') = 0,$$

where the material moduli λ, μ, K, α, β, and γ depend on $\boldsymbol{x}$ and $\boldsymbol{x}'$ through $|\boldsymbol{x}-\boldsymbol{x}'|$, that is,

$$\{\lambda, \mu, K, \alpha, \beta, \gamma\} = \{\lambda', \mu', K', \alpha', \beta', \gamma'\}G(|\boldsymbol{x}-\boldsymbol{x}'|),$$

with λ', μ', K', α', β', and γ' as local micropolar elastic constants, of which λ' and μ' correspond to the classical Lamé constants, and $G(|\boldsymbol{x}-\boldsymbol{x}'|)$ as the nonlocal kernel. The function $G(|\boldsymbol{x}-\boldsymbol{x}'|)$ represents the effect of distant interactions of material points $\boldsymbol{x}'$ on the material point $\boldsymbol{x}$. Since the long-range effects quickly die out with distance, this function should attain its maximum at $\boldsymbol{x}' = \boldsymbol{x}$. Eringen has shown that the function G happens to be the Green's function for the infinite plane, that is, it satisfies [Eringen 1984]

$$(1-\epsilon^2\nabla^2)G = \delta(|\boldsymbol{x}'-\boldsymbol{x}|), \tag{4}$$

where $\epsilon = e_0 a$, a being the internal characteristic length (for example, the atomic lattice parameter in crystals, the average granular distance in granular solids, etc.), and e_0 is a material constant.

Using these expressions of the elastic moduli, the constitutive relations (2) and (3) become

$$t_{kl}(\boldsymbol{x}) = \int \{\lambda\delta_{kl}\epsilon_{rr}(\boldsymbol{x}') + (\mu+K)\epsilon_{kl}(\boldsymbol{x}') + \mu\epsilon_{lk}(\boldsymbol{x}')\}\, dv(\boldsymbol{x}'), \tag{5}$$

$$m_{kl}(\boldsymbol{x}) = \int \{\alpha\delta_{kl}\gamma_{rr}(\boldsymbol{x}') + \beta\gamma_{kl}(\boldsymbol{x}') + \gamma\gamma_{lk}(\boldsymbol{x}')\}\, dv(\boldsymbol{x}'). \tag{6}$$

The equations of motion for a nonlocal isotropic micropolar solid are given by [Eringen 2002]

$$t_{kl,k} + \rho(f_l - \ddot{u}_l) = 0, \tag{7}$$

$$m_{kl,k} + \varepsilon_{lmn}t_{mn} + \rho(l_l - j\ddot{\phi}_l) = 0. \tag{8}$$

Applying the operator $(1-\epsilon^2\nabla^2)$ to (5) and (6) and using the property (4) together with

$$\int f(x)\delta(x-a)\, dx = f(a), \tag{9}$$

we obtain

$$(1-\epsilon^2\nabla^2)t_{kl} = \sigma_{kl} = \lambda'\delta_{kl}\epsilon_{rr}(\boldsymbol{x}) + (\mu'+K')\epsilon_{kl}(\boldsymbol{x}) + \mu'\epsilon_{lk}(\boldsymbol{x}), \tag{10}$$

$$(1-\epsilon^2\nabla^2)m_{kl} = \mu_{kl} = \alpha'\delta_{kl}\gamma_{rr}(\boldsymbol{x}) + \beta'\gamma_{kl}(\boldsymbol{x}) + \gamma'\gamma_{lk}(\boldsymbol{x}). \tag{11}$$

We can see from the expressions of the above equations that σ_{kl} and μ_{kl} are the force stress and couple stress tensors of local micropolar elasticity.

Now, using (10) and (11) in the field equations (7) and (8), we obtain

$$(\lambda'+\mu')u_{k,kl} + (\mu'+K')u_{l,kk} + K'\varepsilon_{klm}\phi_{k,m} + (1-\epsilon^2\nabla^2)\rho(f_l - \ddot{u}_l) = 0, \tag{12}$$

$$(\alpha'+\beta')\phi_{k,kl} + \gamma'\phi_{l,kk} + K'\varepsilon_{lmn}u_{n,m} - 2K'\phi_l + (1-\epsilon^2\nabla^2)\rho(l_l - j\ddot{\phi}_l) = 0. \tag{13}$$

These are the equations of small motion in a nonlocal micropolar elastic medium. It is clear that in the absence of nonlocality, that is, when $e_0 = 0$, these equations reduce to the well-known equations of a uniform micropolar solid. Since $\epsilon = e_0 a$, the parameter ϵ may be called the nonlocal parameter.

3. Wave propagation

Introducing the scalar potentials (q, ξ) and vector potentials $(\boldsymbol{U}, \boldsymbol{\Pi})$ through the Helmholtz decomposition theorem as

$$\boldsymbol{u} = \nabla q + \nabla\times\boldsymbol{U}, \quad \boldsymbol{\phi} = \nabla\xi + \nabla\times\boldsymbol{\Pi}; \quad \nabla\cdot\boldsymbol{U} = \nabla\cdot\boldsymbol{\Pi} = 0, \tag{14}$$

and plugging them into (12) and (13), we obtain the following equations of motion, in the absence of body forces and body couples:

$$(\lambda' + 2\mu' + K')\nabla^2 q - \rho(1 - \epsilon^2\nabla^2)\ddot{q} = 0, \tag{15}$$

$$(\mu' + K')\nabla^2 \boldsymbol{U} + K'\nabla \times \boldsymbol{\Pi} - \rho(1 - \epsilon^2\nabla^2)\ddot{\boldsymbol{U}} = \boldsymbol{0}, \tag{16}$$

$$(\alpha' + \beta' + \gamma')\nabla^2 \xi - 2K'\xi - \rho j(1 - \epsilon^2\nabla^2)\ddot{\xi} = 0, \tag{17}$$

$$\gamma'\nabla^2 \boldsymbol{\Pi} + K'\nabla \times \boldsymbol{U} - 2K'\boldsymbol{\Pi} - \rho j(1 - \epsilon^2\nabla^2)\ddot{\boldsymbol{\Pi}} = \boldsymbol{0}. \tag{18}$$

It can be seen that (16) and (18) are coupled in vector potentials $\boldsymbol{U}$ and $\boldsymbol{\Pi}$ and (15) and (17) are independent in scalar potentials q and ξ. It is also noted that in the absence of nonlocality, that is, when $\epsilon = 0$, (15)–(18) reduce completely to the wave equations of a linear micropolar solid.

For plane waves propagating in the positive direction of a unit vector $\boldsymbol{n}$, we have

$$\{q, \xi, \boldsymbol{U}, \boldsymbol{\Pi}\} = \{a, b, \boldsymbol{A}, \boldsymbol{B}\} \exp\{\iota k(\boldsymbol{n} \cdot \boldsymbol{r} - Vt)\}, \tag{19}$$

where a and b are scalar constants, $\boldsymbol{A}$ and $\boldsymbol{B}$ are vector constants, and V is the phase speed. The circular frequency ω is defined by $\omega = kV$, k being the wavenumber. Inserting the expression of q from (19) into (15), we obtain

$$V_1^2 = (\lambda' + 2\mu' + K')\rho^{-1} - \epsilon^2\omega^2. \tag{20}$$

This is the speed of the longitudinal displacement wave representing the longitudinal acoustic branch. We see that the speed of the longitudinal displacement wave in the nonlocal micropolar solid is equal to the speed of the longitudinal wave in the local micropolar solid decreased by an amount $\epsilon^2\omega^2$. Next, inserting the expression of ξ from (19) into (17), we obtain

$$V_2^2 = \left(\frac{\alpha' + \beta' + \gamma'}{\rho j} - \epsilon^2\omega^2\right)\left(1 - \frac{2K'}{\rho j \omega^2}\right)^{-1}. \tag{21}$$

This is the speed of the longitudinal microrotational wave representing the longitudinal optic branch. Similarly, inserting the expressions of $\boldsymbol{U}$ and $\boldsymbol{\Pi}$ from (19) into (16) and (18), we obtain

$$\boldsymbol{A}\{(\mu' + K')k^2 - \rho\omega^2 - \rho\omega^2\epsilon^2k^2\} - \iota k K'\boldsymbol{n} \times \boldsymbol{B} = \boldsymbol{0}, \tag{22}$$

$$\iota k K'\boldsymbol{n} \times \boldsymbol{A} - \boldsymbol{B}\{k^2\gamma' + 2K' - \rho j\omega^2 - \rho j\omega^2\epsilon^2k^2\} = \boldsymbol{0}. \tag{23}$$

Elimination of $\boldsymbol{A}$ or $\boldsymbol{B}$ from (22) and (23) yields a quadratic equation in V^2 given by

$$AV^4 + BV^2 + C = 0. \tag{24}$$

The roots of this equation are given by

$$V_3^2 = \frac{1}{2A}\left(-B + \sqrt{B^2 - 4AC}\right), \quad V_4^2 = \frac{1}{2A}\left(-B - \sqrt{B^2 - 4AC}\right), \tag{25}$$

where

$$A = 1 - \Omega, \quad B = \omega^2\epsilon^2 - c_4^2 - \tfrac{1}{2}c_3^2\Omega + (1 - \Omega)(\omega^2\epsilon^2 - c_2^2 - c_3^2), \quad C = (\omega^2\epsilon^2 - c_2^2 - c_3^2)(\omega^2\epsilon^2 - c_4^2),$$

$$\Omega = \frac{2\omega_0^2}{\omega^2}, \quad \omega_0^2 = \frac{K'}{\rho j}, \quad c_2^2 = \frac{\mu'}{\rho}, \quad c_3^2 = \frac{K'}{\rho}, \quad c_4^2 = \frac{\gamma'}{\rho j}.$$

When $\epsilon = 0$, the expressions of the coefficients A, B, and C exactly match with those obtained in [Parfitt and Eringen 1969] for micropolar elasticity. These authors showed:

(i) For $A > 0$, and keeping in mind the restrictions imposed on elastic moduli, the quantity B is always negative and the quantity C is always positive. Thus, the discriminant is $B^2 - 4AC > 0$ and hence the value of V_3^2 is finite and positive.

(ii) For $A < 0$, the quantity $B^2 - 4AC$ is finite and positive, since C is already a positive quantity. Also, $\sqrt{B^2 - 4AC} > |-B|$, which makes the value of V_3^2 negative. This shows that $V_3^2 < 0$ or > 0 for $A < 0$ or > 0, respectively.

(iii) V_4^2 is a finite and positive quantity for $A > 0$ as well as for $A < 0$. Thus, a wave propagating with phase speed V_4 exists for all values of ω.

In the present case, that is, when $\epsilon \neq 0$ and $A > 0$, the quantity C will have a negative value if ω/ω_0 lies between $(c_4/c_3)\big(\sqrt{j}/\epsilon\big)$ and $\sqrt{(1 + c_2^2/c_3^2)(j/\epsilon^2)}$. Outside this range, we find that the quantity $C > 0$. Thus, it is seen that V_3^2 is finite and positive provided $C < 0$, that is, when

$$\min\left\{\sqrt{\left(1 + \frac{c_2^2}{c_3^2}\right)\frac{j}{\epsilon^2}}, \frac{c_4}{c_3}\frac{\sqrt{j}}{\epsilon}\right\} < \frac{\omega}{\omega_0} < \max\left\{\sqrt{\left(1 + \frac{c_2^2}{c_3^2}\right)\frac{j}{\epsilon^2}}, \frac{c_4}{c_3}\frac{\sqrt{j}}{\epsilon}\right\}.$$

Even for $A < 0$, one can show that the value of V_3^2 is finite and positive, provided $C < 0$. Thus, a wave propagating with phase speed V_3 exists only when $C < 0$. This means that the value of ω/ω_0 must lie between $(c_4/c_3)\big(\sqrt{j}/\epsilon\big)$ and $\sqrt{(1 + c_2^2/c_3^2)(j/\epsilon^2)}$ for the existence of a wave propagating with speed V_3.

The quantity V_4^2 will be finite and positive for $A < 0$, that is, when $\omega < \sqrt{2}\omega_0$. Beyond this critical value of ω, a wave propagating with phase speed V_4 will degenerate into distance-decaying sinusoidal vibrations. Thus, we conclude that the nonlocality in an isotropic micropolar medium results in the wave speed V_3 behaving like the wave speed V_4 of local micropolar elasticity (see [Parfitt and Eringen 1969]), but oppositely/adversely.

The quantity V_3 is the speed of a set of coupled transverse waves and represents the transverse acoustic branch, while the quantity V_4 is also the speed of another set of coupled transverse waves and represents the transverse optic branch.

In the absence of nonlocality, all the phase speeds of longitudinal and transverse waves of a linear isotropic micropolar solid are recovered.

4. Reflection phenomena

Let $M = \{(x, z); -\infty < x < \infty, -\infty < z \leq 0\}$ be the region occupied by an isotropic nonlocal micropolar solid. Let $z = 0$ be the plane boundary surface of M that is assumed to be free from stresses. We discuss a two-dimensional problem in the x-z plane, so we take

$$\boldsymbol{u} = (u_1, 0, u_3), \quad \boldsymbol{\phi} = (0, \phi_2, 0), \quad \frac{\partial}{\partial y} \equiv 0.$$

From (14), we have

$$u_1 = q_{,x} - U_{2,z}, \quad u_3 = q_{,z} + U_{2,x}, \quad \phi_2 = \Pi_{1,z} - \Pi_{3,x},$$

where U_2 is the y-component of $\boldsymbol{U}$ and Π_1 and Π_3 are the x and z-components of $\boldsymbol{\Pi}$.

Let a train of longitudinal displacement waves having amplitude A_0 and speed V_1 be made incident at an angle θ_0 on the free surface $z = 0$. We postulate the existence of the following reflected waves to satisfy the boundary conditions at the free plane surface:

(i) a longitudinal wave of amplitude A_1 with speed V_1, making an angle θ_1 with the normal,

(ii) a set of coupled transverse waves of amplitude A_{3y} propagating with speed V_3, making an angle θ_3 with the normal, and

(iii) a similar set of coupled transverse waves of amplitude A_{4y} propagating with speed V_4, making an angle θ_4 with the normal.

The complete geometry of the problem is shown in Figure 1. Thus, the total wave field is given by

$$q = A_0 \exp\{\iota k_1(\sin\theta_0 x + \cos\theta_0 z) - \iota\omega_1 t\} + A_1 \exp\{\iota k_1(\sin\theta_1 x - \cos\theta_1 z) - \iota\omega_1 t\}, \tag{26}$$

$$\boldsymbol{U} = \sum_{p=3,4} A_{py}\hat{\boldsymbol{e}}_y \exp\{\iota k_p(\sin\theta_p x - \cos\theta_p z) - \iota\omega_p t\}, \tag{27}$$

$$\boldsymbol{\Pi} = \sum_{p=3,4} (B_{px}\hat{\boldsymbol{e}}_x + B_{pz}\hat{\boldsymbol{e}}_z) \exp\{\iota k_p(\sin\theta_p x - \cos\theta_p z) - \iota\omega_p t\}, \tag{28}$$

where $\omega_l = k_l V_l$ $(l = 1, 3, 4)$ have been defined earlier and $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$, and $\hat{\boldsymbol{e}}_z$ are the Cartesian unit base vectors along the x, y, and z directions, respectively.

Comparing the x and z components of (18) and then using (27) and (28), we obtain

$$\boldsymbol{B}_p = \frac{\iota\omega_0^2}{k_p(c_4^2 + 2\omega_0^2/k_p^2 - V_p^2 - \epsilon^2\omega_p^2)}(\cos\theta_p\hat{\boldsymbol{e}}_x + \sin\theta_p\hat{\boldsymbol{e}}_z)A_{py}. \tag{29}$$

This gives us the relation between the coefficients $\boldsymbol{A}_p$ and $\boldsymbol{B}_p$.

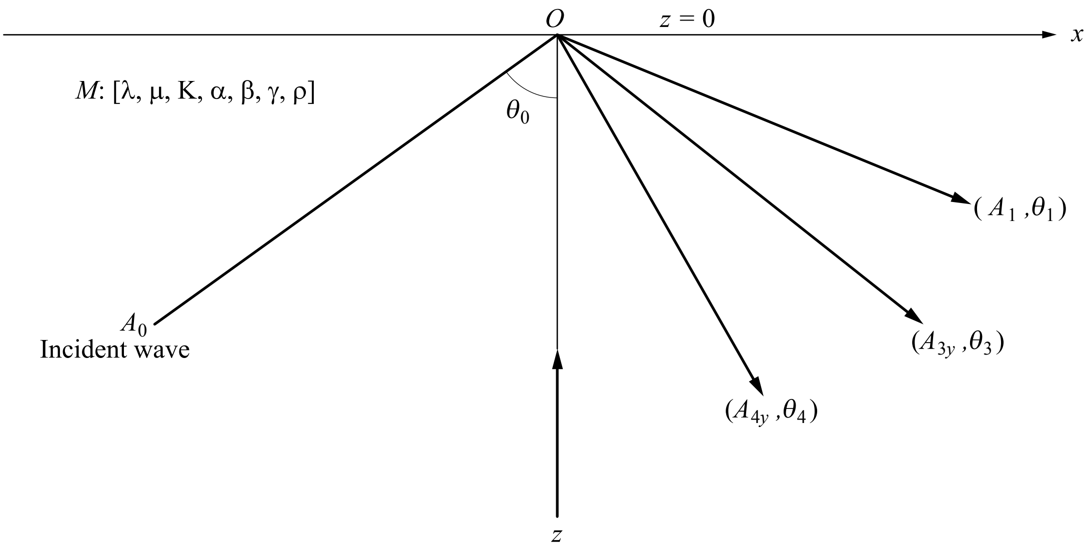

Figure 1. The geometry of the problem.

Since the boundary of the half-space M is mechanically stress-free, the appropriate boundary conditions are the vanishing of the force stress and the couple stress. Mathematically, these boundary conditions can be written as:

$$\sigma_{33} = \sigma_{31} = \mu_{32} = 0 \quad \text{at} \quad z = 0. \tag{30}$$

The requisite components of stresses are given by

$$\begin{gathered} \sigma_{33} = \lambda' q_{,xx} + (\lambda' + 2\mu' + K') q_{,zz} + (2\mu' + K') U_{2,xz}, \\ \sigma_{31} = (2\mu' + K') q_{,xz} + \mu' U_{2,xx} - (\mu' + K') U_{2,zz} - K'\phi_2, \qquad \mu_{32} = \gamma' \phi_{2,z}. \end{gathered} \tag{31}$$

We shall also assume that at the boundary surface, all frequencies are equal, that is, $\omega_1 = \omega_3 = \omega_4 = \omega$, say, and Snell's law holds, which gives $k_1 \sin\theta_0 = k_1 \sin\theta_1 = k_3 \sin\theta_3 = k_4 \sin\theta_4$. The potentials given in (26)–(28) will satisfy the above boundary conditions (30) at $z = 0$, if

$$\sum_{p=0,1} [\lambda' + (2\mu' + K')\cos^2\theta_p] k_1^2 A_p - (2\mu' + K') \sum_{p=3,4} \sin\theta_p \cos\theta_p k_p^2 A_{py} = 0, \tag{32}$$

$$\begin{aligned} &(2\mu' + K')\sin\theta_0\cos\theta_0 k_1^2 A_0 - (2\mu' + K')\sin\theta_1\cos\theta_1 k_1^2 A_1 \\ &\quad - \sum_{p=3,4}\left[\mu'\cos 2\theta_p + K'\cos^2\theta_p - K'\omega_0^2 k_p^{-2}\left(c_4^2 + \frac{2\omega_0^2}{k_p^2} - \epsilon^2\omega_p^2 - V_p^2\right)^{-1}\right] k_p^2 A_{py} = 0, \end{aligned} \tag{33}$$

$$\sum_{p=3,4} \gamma'\omega_0^2 \cos\theta_p k_p A_{py}\left(c_4^2 + \frac{2\omega_0^2}{k_p^2} - \epsilon^2\omega_p^2 - V_p^2\right)^{-1} = 0. \tag{34}$$

These equations enable us to provide the amplitude ratios of various reflected waves. Equations (32)–(34) can be written in matrix form as

$$[a_{ij}][Z] = [M], \tag{35}$$

where $[a_{ij}]$ is a 3×3 matrix, $[Z] = [Z_1, Z_3, Z_4]^t$ is a column matrix (where superscript t denotes the transpose), and $Z_1 = A_1/A_0$ and $Z_p = A_{py}/A_0$ ($p = 3, 4$) are the reflection coefficients. All the entries of the matrix $[a_{ij}]$ together with the column matrix $[M]$ are given in the Appendix. Following [Achenbach 1973], the rate of energy transmission per unit area is given by

$$P^* = \sigma_{33}\dot{u}_3 + \sigma_{31}\dot{u}_1 + \mu_{32}\dot{\phi}_2. \tag{36}$$

The expressions of the energy ratios E_i ($i = 1, 3, 4$) corresponding to various reflected waves are

$$E_1 = -Z_1^2, \qquad E_p = \frac{1}{P_1}\left[\mu' + K' - \frac{\omega_0^2}{k_p^2 D_p}\left(K' + \frac{\gamma'\omega_0^2}{D_p}\right)\right] k_p^3 \cos\theta_p Z_p^2, \qquad p = 3, 4,$$

where $P_1 = -k_1^3 \cos\theta_0 (\lambda' + 2\mu' + K')$ and $D_p = c_4^2 + 2\omega_0^2/k_p^2 - \epsilon^2\omega_p^2 - V_p^2$.

5. Numerical results and discussion

For a silicon crystal, the following values of the relevant parameters are taken for an isotropic nonlocal micropolar solid [Zeng et al. 2006]: $\lambda = 0.1055 \times 10^{13}$ dyne/cm^2, $\mu = 0.2518 \times 10^{12}$ dyne/cm^2, $K = 0.1 \times 10^{12}$ dyne/cm^2, $e_0 = 0.39$, $j = 9.21 \times 10^{-12}$ cm^2, $\rho = 2.330$ gm/cm^3, $\gamma = 0.1423 \times 10^{13}$ dyne, and

$a = 0.5 \times 10^{-7}$ cm, while for a local micropolar solid, $\lambda = 0.7431 \times 10^{12}$ dyne/cm^2, $\mu = 0.1373 \times 10^{12}$ dyne/cm^2, $K = 0.1 \times 10^{12}$ dyne/cm^2, $j = 9.21 \times 10^{-12}$ cm^2, $\rho = 2.330$ gm/cm^3, $\gamma = 0.1275 \times 10^{13}$ dyne, $a = 0.5 \times 10^{-7}$ cm, and $e_0 = 0$.

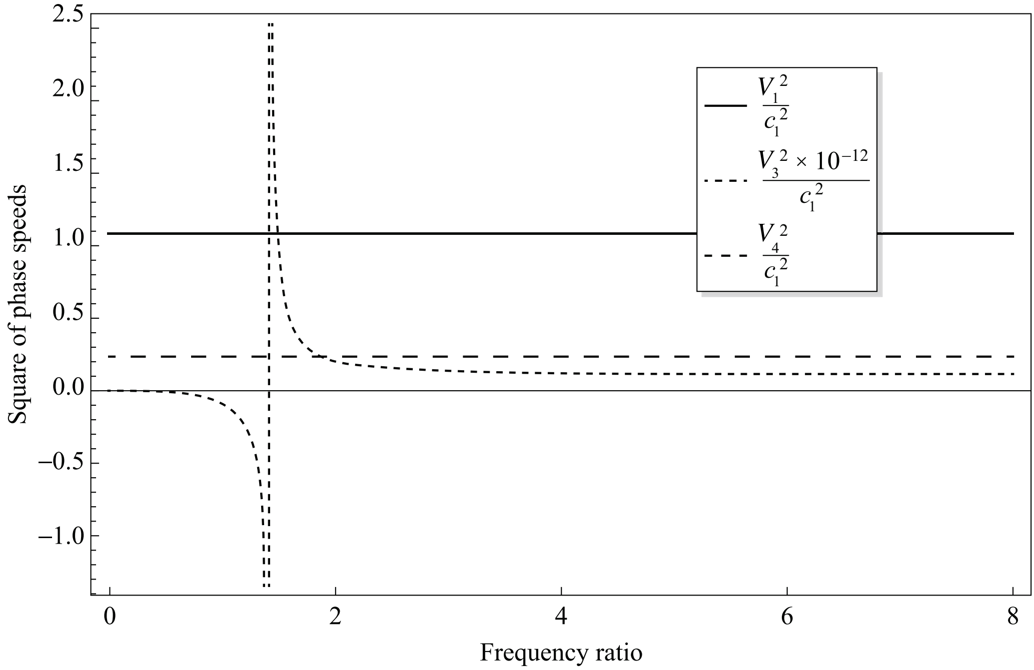

Figure 2. Variation of square of phase speeds V_i^2/c_1^2 ($i = 1, 3, 4$) with frequency ratio.

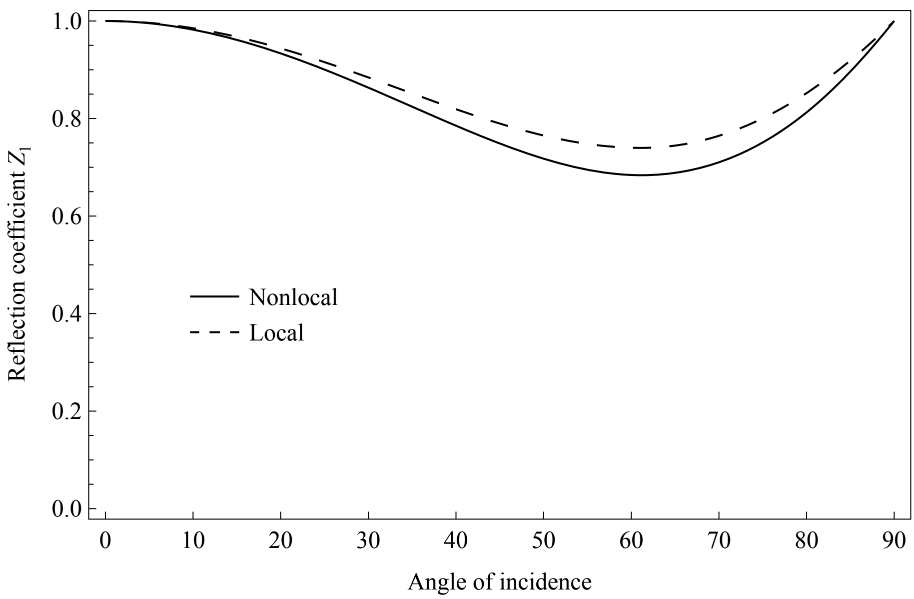

Figure 3. Variation of modulus of reflection coefficient Z_1 with angle of incidence.

In Figure 2, we have shown the variation of the square of phase speeds (nondimensional) V_i^2/c_1^2 $(i = 1, 3, 4)$ in the nonlocal micropolar solid, with the frequency ratio (ω/ω_0). It is seen that the values of V_1^2/c_1^2 and V_4^2/c_1^2 remain almost constant in the considered range $0 \le \omega/\omega_0 \le 8$. We have plotted the curve of V_3^2/c_1^2 after magnifying it by a factor of 10^{-12} as its value was large enough in comparison

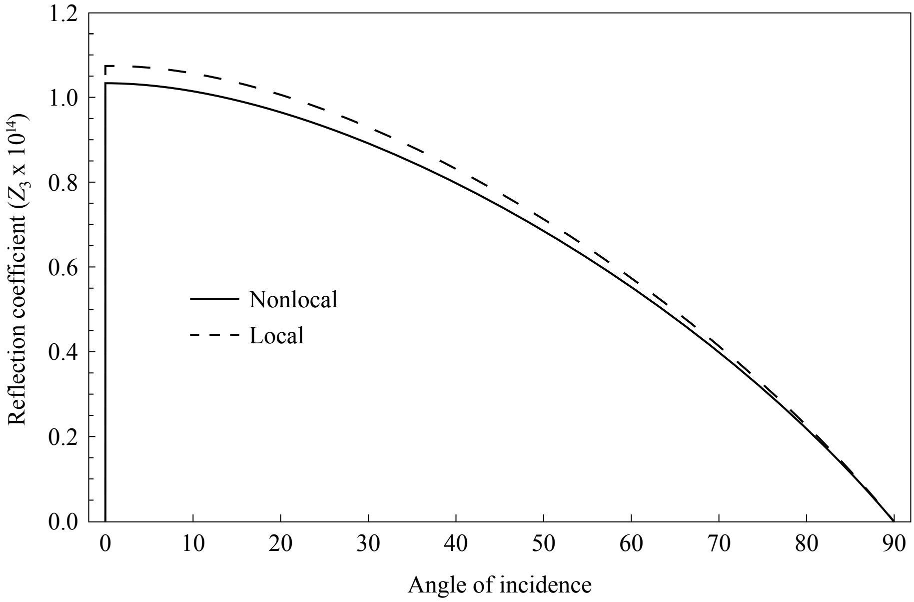

Figure 4. Variation of modulus of reflection coefficient Z_3 with angle of incidence.

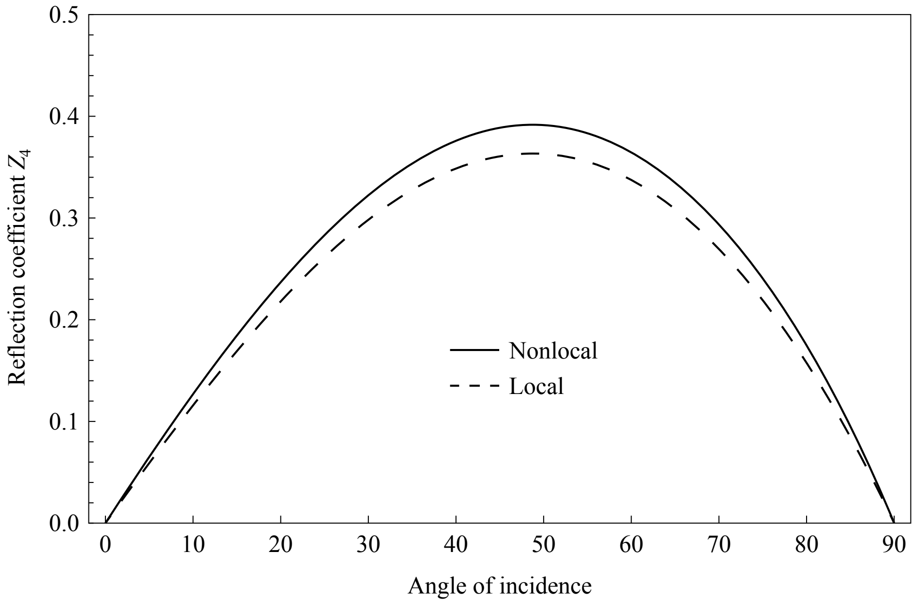

Figure 5. Variation of modulus of reflection coefficient Z_4 with angle of incidence.

with the values of other quantities. It can be seen through this figure that $\omega_c = \sqrt{2}\omega_0$ is the critical frequency for a wave propagating with phase speed V_3, that is, before $\omega_c = \sqrt{2}\omega_0$ the quantity V_3^2/c_1^2 has negative values while later on it remains positive. Moreover, it has also been observed that the effect of the nonlocality parameter on phase speed depends heavily on the value of characteristic length a. At higher values of a, the phase speeds of waves are found to be more dispersive.

Figures 3–5 depict the comparison between the nonlocal and local micropolar solids for modulus values of reflection coefficients Z_1, Z_3, and Z_4 with angle of incidence of the longitudinal displacement wave propagating with phase speed V_1. The solid curve is for the nonlocal micropolar solid, that is, when $e_0 = 0.39$, while the dotted curve is for the local micropolar solid.

In Figure 3, the modulus value of the reflection coefficient Z_1 has a maximum value equal to unity at normal incidence in both the cases. Then, its value decreases till $\theta_0 = 58°$. Thereafter it increases with the increase of θ_0 to attain its maximum value at grazing incidence. The pattern is similar for the nonlocal micropolar solid and the local micropolar solid.

In Figure 4, we have plotted the variation of modulus values of the reflection coefficient $Z_3 \times 10^{14}$ as the value of Z_3 is negligibly small. The value of the reflection coefficient Z_3 is maximal at normal incidence. This value decreases with an increase of θ_0 throughout the range and approaches zero as θ_0 approaches 90°. Figure 5 depicts the variation of the absolute values of the reflection coefficient Z_4. It is seen that the value increases with increase in θ_0 in the range $0° \le \theta_0 \le 52°$, and thereafter it decreases and vanishes at $\theta_0 = 90°$.

In Figures 3–5, we have seen that at each angle of incidence the modulus value of the reflection coefficients Z_1 and Z_3 for the local micropolar solid is bigger than the corresponding values for the nonlocal micropolar solid. However, the value of Z_4 for the nonlocal micropolar solid is bigger at each angle of incidence than the value for the local micropolar solid.

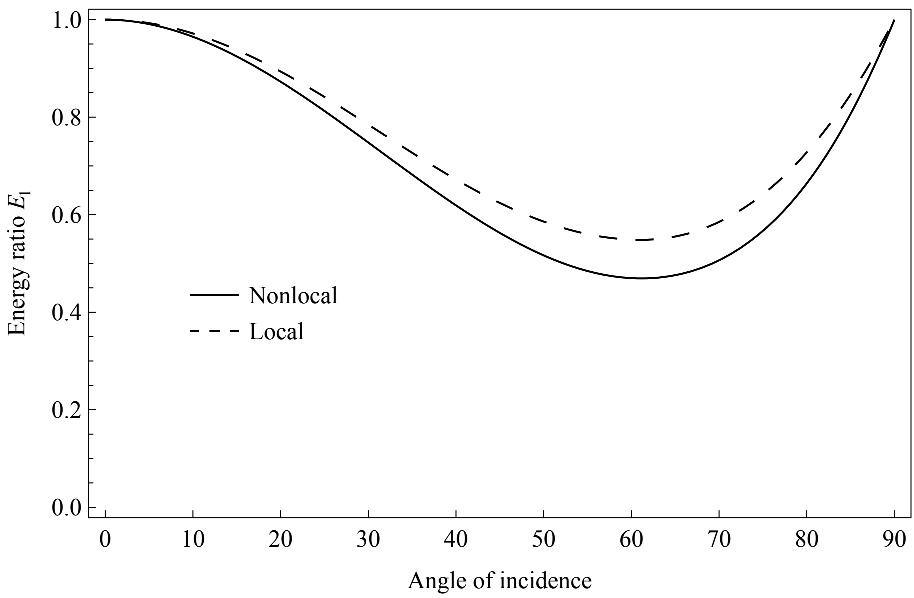

Figure 6. Variation of modulus of energy ratio E_1 with angle of incidence.

Figures 6–8 depict the variation of the modulus values of the energy ratios with the angle of incidence. At the free boundary surface, the sum of the energy ratios is equal to unity during reflection at each angle of incidence of a longitudinal displacement wave propagating with speed V_1. The formulae for reflection coefficients and their corresponding energy ratios are obtained analytically and numerically. This shows that there is no dissipation of energy at the free boundary surface.

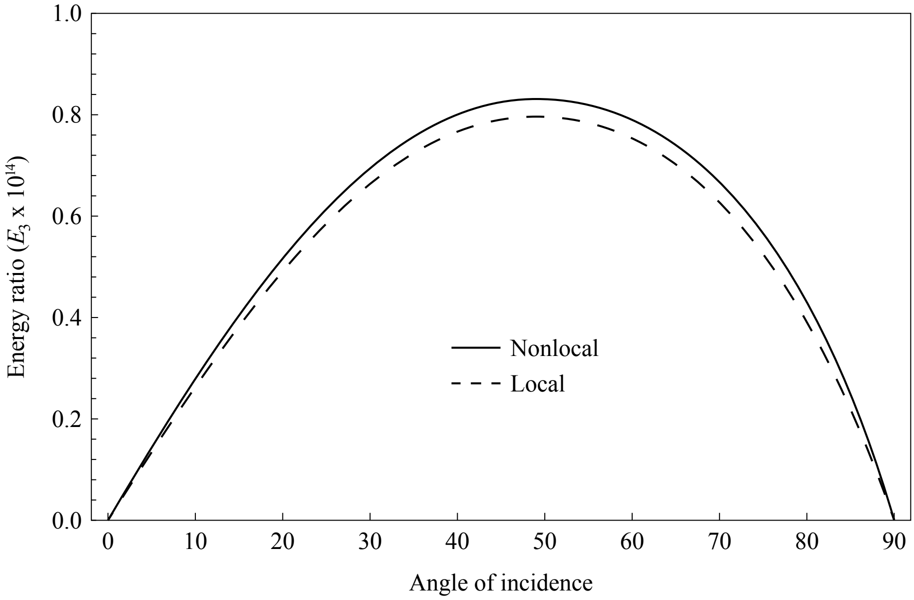

Figure 7. Variation of modulus of energy ratio E_3 with angle of incidence.

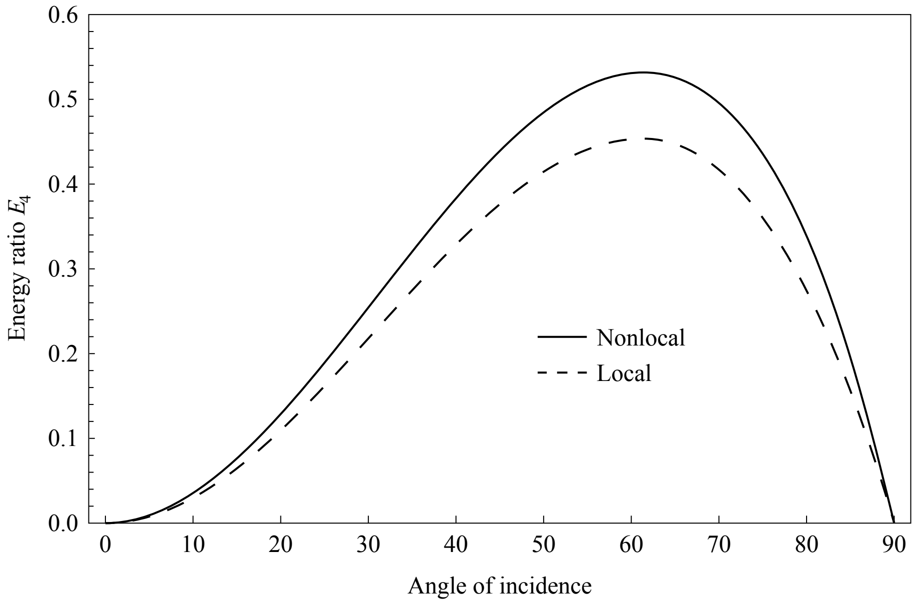

Figure 8. Variation of modulus of energy ratio E_4 with angle of incidence.

6. Conclusions

This paper deals with the possibility of plane-wave propagation in an isotropic nonlocal micropolar solid. The reflection phenomena of a plane wave striking obliquely at the free boundary surface is also discussed. The following is concluded.

(i) Four waves may travel with distinct speeds in a nonlocal micropolar solid: a longitudinal displacement wave, a longitudinal microrotational wave, and two sets of coupled transverse waves. All the waves are dispersive in nature.

(ii) The effect of the nonlocality parameter on the reflection coefficients (as well as on the energy ratios) is found to be maximal at some intermediate angle of incidence of a longitudinal displacement wave. However, there is no significant difference seen at the grazing incidence and the normal incidence on the reflection coefficients and energy ratios.

(iii) The balance of energy law has been verified at each angle of incidence of a longitudinal displacement wave at the free boundary surface.

Appendix

Entries of the matrices $[a_{ij}]$ and $[M]$:

$$a_{11} = -1, \quad a_{12} = \frac{(2\mu' + K')\sin\theta_0}{[\lambda' + (2\mu' + K')\cos^2\theta_0]v_{31}}\sqrt{1 - v_{31}^2\sin^2\theta_0},$$

$$a_{13} = \frac{(2\mu' + K')\sin\theta_0}{[\lambda' + (2\mu' + K')\cos^2\theta_0]v_{41}}\sqrt{1 - v_{41}^2\sin^2\theta_0}, \quad a_{21} = \sin\theta_0\cos\theta_0,$$

$$a_{22} = \frac{1}{(2\mu' + K')v_{31}^2}\left[\mu'(1 - 2v_{31}^2\sin^2\theta_0) + K'(1 - v_{31}^2\sin^2\theta_0) - \frac{K'\omega_0^2}{k_3^2 D_3}\right],$$

$$a_{23} = \frac{1}{(2\mu' + K')v_{41}^2}\left[\mu'(1 - 2v_{41}^2\sin^2\theta_0) + K'(1 - v_{41}^2\sin^2\theta_0) - \frac{K'\omega_0^2}{k_4^2 D_4}\right],$$

$$a_{31} = 0, \quad a_{32} = \frac{\Omega\sqrt{1 - v_{31}^2\sin^2\theta_0}}{2v_{31}^3\left(1 + \frac{\epsilon^2\omega_3^2}{V_3^2} - \frac{2\omega_0^2}{\omega_3^2} - \frac{c_4^2}{V_3^2}\right)}, \quad a_{33} = \frac{\Omega\sqrt{1 - v_{41}^2\sin^2\theta_0}}{2v_{41}^3\left(1 + \frac{\epsilon^2\omega_4^2}{V_4^2} - \frac{2\omega_0^2}{\omega_4^2} - \frac{c_4^2}{V_4^2}\right)},$$

$$M_1 = 1, \quad M_2 = \sin\theta_0\cos\theta_0, \quad M_3 = 0,$$

where $v_{31} = \dfrac{V_3}{V_1}$ and $v_{41} = \dfrac{V_4}{V_1}$.

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