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JIAN DENG, WEI-CHAU XIE AND MAHESH D. PANDEY

The moment and almost-sure stochastic stability of two-degree-of-freedom coupled viscoelastic systems, under parametric excitation of white noise, are investigated through moment Lyapunov exponents and Lyapunov exponents, respectively. The system of stochastic differential equations of motion is first decoupled by using the method of stochastic averaging for dynamic systems with small damping and weak excitations. Then a new scheme for determining the moment Lyapunov exponents is proposed for a coupled viscoelastic system. The largest Lyapunov exponent is calculated through its relation with moment Lyapunov exponent. The moment and almost-sure stability boundaries and critical excitation are obtained analytically. These analytical results are confirmed by numerical simulation. As an application example, the stochastic stability of flexural-torsional viscoelastic beam is studied. It is found that, under white noise excitation, the parameters of damping β and the viscoelastic intensity γ have stabilizing effects on the moment and almost-sure stability. However, viscosity parameter η plays a destabilizing role. The stability index decreases from positive to negative values with the increase of the amplitude of noise power spectrum, which suggests that the noise destabilize the system. These results are useful in engineering applications.

1. Introduction

Dynamic responses of many engineering structures are governed in general by an equation of motion of the form

$$\ddot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{\xi}(t)), \tag{1-1}$$

where the superscript dot denotes the derivative with respect to time, $\boldsymbol{x}(t)$ is the response vector, and $\boldsymbol{\xi}(t)$ is a vector of dynamic loadings. These dynamic loadings, such as those arising from earthquakes, wind, and blasting, can be characterized satisfactorily only by probabilistic models, which leads to the fact that (1-1) is actually a stochastic differential equation. Oseledec [1968] showed that for continuous dynamical systems, both deterministic and stochastic, there exist deterministic real numbers characterizing the average exponential rates of growth or decay of the solution for large time and called them Lyapunov exponents. The sample or almost sure stability of stochastic system (1-1) is governed by this Lyapunov exponent which is defined as (see [Xie 2006])

$$\lambda_{\boldsymbol{x}} = \lim_{t\to\infty} \frac{1}{t} \log \|\boldsymbol{x}\|, \tag{1-2}$$

where $\|\boldsymbol{x}\| = (\boldsymbol{x}^T \boldsymbol{x})^{1/2}$ is the Euclidean norm. If the largest Lyapunov exponent is negative, the trivial solution of system (1-1) is stable with probability 1; otherwise, it is unstable almost surely. Hence, the

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vanishing of the largest Lyapunov exponent indicates the almost-sure stability boundaries in parameter space.

On the other hand, the stability of the p-th moment $\mathsf{E}[\|\boldsymbol{x}\|^p]$ of the solution of system (1-1) is governed by the p-th moment Lyapunov exponent defined by (see [Xie 2006])

$$\Lambda_{\boldsymbol{x}}(p) = \lim_{t\to\infty} \frac{1}{t} \log \mathsf{E}[\|\boldsymbol{x}\|^p], \tag{1-3}$$

where $\mathsf{E}[\,\cdot\,]$ denotes the expected value. If $\Lambda_{\boldsymbol{x}}(p)$ is negative, then the p-th moment is stable; otherwise, it is unstable. Hence, the vanishing of the p-th moment Lyapunov exponent indicates the p-th moment stability boundaries in parameter space.

These stability properties of system (1-1) with stochastic differential equations are difficult to obtain exactly. The method of stochastic averaging, originally formulated in [Stratonovich 1963] and mathematically proved in [Khasminskii 1966], has been widely used to solve approximately stochastic differential equations containing a small parameter. Under certain conditions stochastic averaging can reduce the dimension of some problems, which greatly simplifies the solution [Xie 2006]. The popularity of stochastic averaging can be felt from the large number of papers in the literature, such as [Roberts and Spanos 1986; Ariaratnam 1996; Sri Namachchivaya and Ariaratnam 1987]. Larionov [1969] justified an averaging method in a rigorous manner for integrodifferential equations of both deterministic and stochastic systems.

For single-degree-of-freedom (SDOF) systems, Arnold et al. [1997] constructed an approximation for the moment Lyapunov exponents of a two-dimensional linear system driven by real or white noise, by using a perturbation approach. At the same time, asymptotic expansion series for the moment Lyapunov exponent and stability index are constructed and justified for the two-dimensional linear stochastic system [Khasminskii and Moshchuk 1998]. Xie [2006] systematically studied moment Lyapunov exponents, Lyapunov exponents, and the stability index of SDOF systems under various noise excitation in terms of the small fluctuation parameter. In the same book, Xie proposed a Monte Carlo simulation method for determining the moment Lyapunov exponents of stochastic systems. For two-degree-of-freedom (2DOF) coupled systems, Namachchivaya and van Roessel [2001] studied the moment Lyapunov exponents of two coupled elastic oscillators under real noise excitation, by using stochastic averaging method and asymptotic expansion method. More recently, in their serial papers, Kozic el al. [2012] investigated the Lyapunov exponent and moment Lyapunov exponents of 2DOF linear elastic systems subjected to a white noise parametric excitation.

The stability of elastic systems has been studied extensively [Bolotin 1964]. However, in engineering applications more and more viscoelastic materials, such as polymers and composite materials, are employed. For these materials, stress is not a function of instantaneous strain but depends on the past time history of strain and vice versa, which is quite different from linear elastic materials. In the case of a structural column, the viscoelastic property has a great impact on its dynamic behavior, particularly the dynamic stability [Potapov 1994]. When viscoelasticity is properly treated, the equation of motion becomes more complicated and turns out to be an integrodifferential equation rather than an ordinary differential equation as in the elastic case.

The deterministic dynamic stability of viscoelastic systems has been investigated by many authors [Ahmadi and Glocker 1983]. Ariaratnam was among the first who studied the stochastic almost-sure stability of viscoelastic systems under wide band random fluctuations in the stiffness parameter [Ariaratnam 1993] and under bounded noise excitation [Ariaratnam 1996], by evaluating the largest Lyapunov exponent and the rotation number using the method of stochastic averaging. Abdelrahman [2002] systematically investigated stochastic stability of coupled systems and gyroscopic systems and extended Ariaratnam's method from SDOF to 2DOF systems, but still only the Lyapunov exponent was calculated. Sufficient conditions for almost-sure stability were obtained for both elastic and viscoelastic columns under the excitation of a random wide band stationary process using Lyapunov's direct method [Potapov 1994]. Later, Potapov [1997] described the behavior of stochastic viscoelastic systems by numerically evaluating Lyapunov exponents of linear integrodifferential equations.

It is known that the almost-sure stability cannot assure the moment stability. To have a complete picture of the dynamical stability of a stochastic system, it is important to study both the sample and moment stability and to determine both the Lyapunov exponents and the moment Lyapunov exponents, because moment Lyapunov exponents give not only the moment stability but also the almost-sure stability. Xie was the first to deal with the determination of small noise expansion of the moment Lyapunov exponent of a SDOF viscoelastic column under bounded noise excitation [Xie 2003]. Moment Lyapunov exponents of such a viscoelastic system under the excitation of a wide band noise was further investigated by using the averaging method of both first order and second order [Huang and Xie 2008].

The objective of this paper is to study the moment and almost sure stability of 2DOF coupled viscoelastic systems driven by white noise using the method of stochastic averaging. A new scheme for determining the moment Lyapunov exponents is presented for 2DOF coupled viscoelastic systems, which is an extension of [Ariaratnam et al. 1991] from elastic systems to viscoelastic and from almost-sure stability to moment stability characterized by moment Lyapunov exponents. This paper is also different from [Sri Namachchivaya and van Roessel 2001] where the Girsanov theorem and Feynman–Kac formula were used and viscoelasticity was not considered. Furthermore, this study carries out Monte Carlo simulation of moment Lyapunov exponents for coupled systems. This research is motivated by problems in the dynamic stability of viscoelastic systems subjected to stochastically fluctuating loads. Examples and numerical results are provided for illustration.

2. Formulation

Consider the coupled nongyroscopic stochastic system

$$\begin{aligned} \ddot{q}_1 + 2\varepsilon\beta_1 \dot{q}_1 + \omega_1^2(1-\varepsilon\mathscr{H})q_1 + \varepsilon^{1/2}\omega_1(k_{11} q_1 + k_{12} q_2)\xi(t) = 0, \\ \ddot{q}_2 + 2\varepsilon\beta_2 \dot{q}_2 + \omega_2^2(1-\varepsilon\mathscr{H})q_2 + \varepsilon^{1/2}\omega_2(k_{21} q_1 + k_{22} q_2)\xi(t) = 0, \end{aligned} \tag{2-1}$$

where q_1, q_2 are state coordinates, β_1, β_2 are damping coefficients, ω_1, ω_2 are natural frequencies, ε is a small parameter introduced to make the analysis more convenient. and the k_{ij}, $i, j = 1, 2$ are constants. We call k_{12} and k_{21} the coupling parameters; the case of $k_{12} = k_{21} = k$ is called symmetric coupling and that of $k_{12} = -k_{21} = k$ skew-symmetric coupling. $\mathscr{H}$ is a linear viscoelastic operator given by

$$\mathscr{H}[\psi(t)] = \int_0^t \mathcal{H}(t-\tau)\psi(\tau)\,\mathrm{d}\tau, \quad 0 \le \int_0^\infty \mathcal{H}(\theta)\,\mathrm{d}\theta < 1, \tag{2-2}$$

where $\mathcal{H}(\theta)$ is the relaxation kernel. $\xi(t)$ is the stochastic loads imposing on the system.

Equation (2-1) is a typical system of 2DOF coupled Stratonovich stochastic differential equations, which is extremely difficult to directly determine its stability property. The stochastic averaging method was often used to approximate the original Stratonovich stochastic system by an averaged Itô stochastic system, which is presumably easier to study, and infer properties of the dynamics of the original system by the understanding of the dynamics of the averaged system [Ariaratnam et al. 1991]. To apply the averaging method, one may first consider the unperturbed system, i.e., $\varepsilon = 0$ and $\xi(t) = 0$, which is of the form: $\ddot{q}_i + \omega_i^2 q_i = 0$, $i = 1, 2$. The stable solutions for the unperturbed system are found to be

$$q_i = a_i \cos\Phi_i, \quad \dot{q}_i = -\omega_i a_i \sin\Phi_i, \quad \Phi_i = \omega_i t + \phi_i, \quad i = 1, 2. \tag{2-3}$$

Then the method of variation of parameters is used. Differentiating the first equation of (2-3) and comparing with the second equation lead to

$$\dot{a}_i \cos\Phi_i - a_i \dot{\phi}_i \sin\Phi_i = 0. \tag{2-4}$$

Substituting (2-3) into (2-1) results in

$$\dot{a}_i \sin\Phi_i + a_i \dot{\phi}_i \cos\Phi_i = G_i, \tag{2-5}$$

where

$$G_i = -\varepsilon\big(2\beta_i\, \omega_i a_i \sin\Phi_i - \omega_i^2 \mathscr{H}[a_i \cos\Phi_i]\big) + \varepsilon^{1/2}\xi(t)\omega_i(k_{ii}\, a_i \cos\Phi_i + k_{ij}\, a_j \cos\Phi_j).$$

By solving (2-4) and (2-5), the equations in (2-1) can be written in amplitude, a_i, and phase, ϕ_i, as

$$\dot{a}_i = \varepsilon F_{a,i}^{(1)} + \varepsilon^{1/2} F_{a,i}^{(0)}, \quad \dot{\phi}_i = \varepsilon F_{\phi,i}^{(1)} + \varepsilon^{1/2} F_{\phi,i}^{(0)}, \tag{2-6}$$

where

$$\begin{aligned}
F_{a,i}^{(0)} &= \xi(t)\big(\tfrac{1}{2}k_{ii}a_i \sin 2\Phi_i + \tfrac{1}{2}k_{ij}a_j \cos\Phi_j \sin\Phi_i\big),\\
F_{a,i}^{(1)} &= -\beta_i a_i + \beta_i a_i \cos 2\Phi_i - \sin\Phi_i \omega_i \tau_\varepsilon \mathscr{H}(a_i \cos\Phi_i),\\
F_{\phi,i}^{(0)} &= \xi(t)\Big(k_{ii}\cos^2\Phi_i + \frac{a_j}{a_i}k_{ij}\cos\Phi_i \cos\Phi_j\Big),\\
F_{\phi,i}^{(1)} &= -\beta_i \sin 2\Phi_i - \frac{1}{a_i(t)}\cos\Phi_i \omega_i \tau_\varepsilon \mathscr{H}(a_i(s)\cos\Phi_i).
\end{aligned} \tag{2-7}$$

If the correlation function $R(\tau)$ of the noise $\xi(t)$ decays sufficiently quickly to zero as τ increases, then the processes a_i and ϕ_i converge weakly on a time interval of order $1/\varepsilon$ to a Itô stochastic differential equation for the averaged amplitudes $\bar{a}_i$ and phase angles $\bar{\phi}_i$, whose solutions provide a uniformly valid first-order approximation to the exact values

$$\mathrm{d}a_i = \varepsilon m_i^a\, \mathrm{d}t + \varepsilon^{1/2} \sum_{j=1}^{2} \sigma_{ij}^a\, \mathrm{d}W_j^a, \quad i = 1, 2, \tag{2-8}$$

$$\mathrm{d}\phi_i = \varepsilon m_i^\phi\, \mathrm{d}t + \varepsilon^{1/2} \sum_{j=1}^{2} \sigma_{ij}^\phi\, \mathrm{d}W_j^\phi, \quad i = 1, 2, \tag{2-9}$$

where the overbar is dropped for simplicity of presentation and W_i^a, W_i^ϕ, $i = 1, 2$ are independent standard Wiener processes. The drift coefficients εm_i^a, εm_i^ϕ and the 2×2 diffusion matrices $\varepsilon \boldsymbol{b}^a = \varepsilon \boldsymbol{\sigma}^a(\boldsymbol{\sigma}^a)^T$, $\varepsilon \boldsymbol{b}^\phi = \varepsilon \boldsymbol{\sigma}^\phi(\boldsymbol{\sigma}^\phi)^T$, in which $\boldsymbol{\sigma}^a = [\sigma_{ij}^a]$, $\boldsymbol{\sigma}^\phi = [\sigma_{ij}^\phi]$, $\boldsymbol{b}^a = [b_{ij}^a]$, $\boldsymbol{b}^\phi = [b_{ij}^\phi]$, are given by

$$\begin{aligned}
m_i^a &= \mathop{\mathscr{M}}_t\left\{F_{a,i}^{(1)} + \int_{-\infty}^{0} \mathsf{E}\left[\sum_{j=1}^{2}\left(\frac{\partial F_{a,i}^{(0)}}{\partial a_j}F_{a,j\tau}^{(0)} + \frac{\partial F_{a,i}^{(0)}}{\partial \phi_j}F_{\phi,j\tau}^{(0)}\right)\right]\mathrm{d}\tau\right\}\\
&= a_i\left[-\beta_i - \omega_i\tau_\varepsilon \mathop{\mathscr{M}}_t\left(I_i^{sc}\right) + \tfrac{3}{16}k_{ii}^2 S_0\right] + \frac{1}{8}\frac{a_j^2}{a_i}k_{ij}^2 S_0,\\
m_i^\phi &= \mathop{\mathscr{M}}_t\left\{F_{\phi,i}^{(1)} + \int_{-\infty}^{0} \mathsf{E}\left[\sum_{j=1}^{2}\left(\frac{\partial F_{\phi,i}^{(0)}}{\partial a_j}F_{\phi,j\tau}^{(0)} + \frac{\partial F_{\phi,i}^{(0)}}{\partial \phi_j}F_{\phi,j\tau}^{(0)}\right)\right]\mathrm{d}\tau\right\} = -\omega_i\tau_\varepsilon \mathop{\mathscr{M}}_t\left(I_i^{cc}\right),\\
b_{ii}^a &= \mathop{\mathscr{M}}_t\left\{\int_{-\infty}^{\infty}\mathsf{E}\left[F_{a,i}^{(0)}F_{a,i\tau}^{(0)}\right]\mathrm{d}\tau\right\} = \tfrac{1}{8}\left(k_{ii}^2a_i^2 + 2k_{ij}^2a_j^2\right)S_0,\\
b_{ij}^a &= \mathop{\mathscr{M}}_t\left\{\int_{-\infty}^{\infty}\mathsf{E}\left[F_{a,i}^{(0)}F_{a,j\tau}^{(0)}\right]\mathrm{d}\tau\right\} = 0,\\
b_{ii}^\phi &= \mathop{\mathscr{M}}_t\left\{\int_{-\infty}^{\infty}\mathsf{E}\left[F_{\phi,i}^{(0)}F_{\phi,i\tau}^{(0)}\right]\mathrm{d}\tau\right\} = \tfrac{3}{8}k_{ii}^2 S_0 + \frac{1}{4}\frac{a_j^2}{a_i^2}k_{ij}^2 S_0,\\
b_{ij}^\phi &= \mathop{\mathscr{M}}_t\left\{\int_{-\infty}^{\infty}\mathsf{E}\left[F_{\phi,i}^{(0)}F_{\phi,j\tau}^{(0)}\right]\mathrm{d}\tau\right\} = \tfrac{1}{4}\left(k_{ii}k_{jj} + k_{ij}k_{ji}\right)S_0,\\
I_i^{\mathrm{cc}} &= \cos\Phi_i(t)\int_0^t \mathcal{H}(t-s)\cos\Phi_i(s)\,\mathrm{d}s,\\
I_i^{\mathrm{sc}} &= \sin\Phi_i(t)\int_0^t \mathcal{H}(t-s)\cos\Phi_i(s)\,\mathrm{d}s,\\
F_{j\tau}^{(0)} &= F_j^{(0)}\left(a, \phi, \xi(t+\tau), t+\tau\right), \quad i, j = 1,\ 2,
\end{aligned} \tag{2-10}$$

where the coupled oscillators are assumed to have noncommensurable frequencies, i.e., $\omega_1 \neq \omega_2$. A Gaussian white noise process is a weakly stationary process that is delta-correlated and has mean zero. This process is formally the derivative of the Wiener process given by $\xi(t) = \sqrt{S_0}\dot{W}(t)$, with constant cosine power spectral density $S(\omega) = S_0$ and sine power spectral density $\Psi(\omega) = 0$ over the entire frequency range.

The averaging operator is defined as

$$\mathop{\mathscr{M}}_t(\cdot) = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}(\cdot)\,\mathrm{d}t.$$

When applying the averaging operator, the integration is performed over explicitly appearing t only. The term containing viscoelastic operator is averaged according to the method given in [Larionov 1969]. Applying the transformation $s = t - \tau$ and changing the order of integration lead to

$$
\begin{aligned}
\mathscr{M}_t\left\{I_i^{\mathrm{sc}}\right\} &= \lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\int_{s=0}^{t}\mathcal{H}(t-s)\cos\Phi_i(s)\sin\Phi_i(t)\,\mathrm{d}s\,\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{t=0}^{T}\int_{\tau=0}^{t}\mathcal{H}(\tau)\sin\Phi_i(t)\cos\Phi_i(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{\tau=0}^{T}\int_{t=\tau}^{T}\mathcal{H}(\tau)[\sin(2\omega_i t-\omega_i\tau+2\bar{\varphi})+\sin\omega_i\tau]\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \frac{1}{2}\int_0^{\infty}\mathcal{H}(\tau)\sin\omega_i\tau\,\mathrm{d}\tau = \tfrac{1}{2}\mathcal{H}^s(\omega_i).
\end{aligned}
\tag{2-11}
$$

Similarly, one obtains

$$
\mathscr{M}_t\left\{I_i^{\mathrm{cc}}\right\} = \tfrac{1}{2}\mathcal{H}^c(\omega_i), \tag{2-12}
$$

where

$$
\mathcal{H}^s(\omega) = \int_0^{\infty}\mathcal{H}(\tau)\sin\omega\tau\,\mathrm{d}\tau, \quad \mathcal{H}^c(\omega) = \int_0^{\infty}\mathcal{H}(\tau)\cos\omega\tau\,\mathrm{d}\tau \tag{2-13}
$$

are the sine and cosine transformations of the viscoelastic kernel function $\mathcal{H}(t)$, respectively. The term

$$
E_i = \beta_i + \omega_i\tau_\varepsilon\mathscr{M}_t(I_i^{sc}) = \beta_i + \tfrac{1}{2}\omega_i\tau_\varepsilon\frac{\omega_i\gamma}{\eta^2+\omega_i^2}, \quad i=1,2, \tag{2-14}
$$

may be called pseudodamping, because it plays the role of damping but includes viscoelasticity as well.

In this study, the viscoelastic kernel function is supposed to follow ordinary Maxwell model

$$
\mathcal{H}(t) = \gamma\mathrm{e}^{-\eta t}, \tag{2-15}
$$

which can be used as an approximation to most linear viscoelastic behavior as closely as possible if enough number of Maxwell units are arranged in parallel. Its sine and cosine transformations in (2-13) are given by

$$
\mathcal{H}^s(\omega) = \frac{\omega\gamma}{\eta^2+\omega^2}, \qquad \mathcal{H}^c(\omega) = \frac{\gamma\eta}{\eta^2+\omega^2}. \tag{2-16}
$$

It is of importance to note that both the averaged amplitude a_i and phase angle equations ϕ_i do not involve the phase angles and then the amplitude equations are advantageously decoupled from the phase angle equations. Hence, the averaged amplitude vector (a_1, a_2) is a two dimensional diffusion process.

3. Moment Lyapunov exponents and Lyapunov exponents

To obtain moment Lyapunov exponents from (2-8), one may transform the Itô stochastic differential equations for the amplitudes by using Khasminskii's transformation [1966]

$$
r = \sqrt{a_1^2+a_2^2}, \quad \varphi = \tan^{-1}\frac{a_2}{a_1}, \quad a_1 = r\cos\varphi, \quad a_2 = r\sin\varphi, \quad P = r^p, \tag{3-1}
$$

and then the moment Lyapunov exponent is given by

$$
\Lambda = \lim_{t\to\infty}\frac{1}{t}\log\mathsf{E}[P]. \tag{3-2}
$$

The Itô equations for P can be obtained using Itô's lemma [Xie 2006]:

$$\mathrm{d}P = m_P(P,\varphi)\,\mathrm{d}t + \sigma_{P1}\,\mathrm{d}W_1 + \sigma_{P2}\,\mathrm{d}W_2, \tag{3-3}$$

where

$$\sigma_{P1} = \varepsilon^{1/2}\left(\sigma_{11}^a\frac{\partial P}{\partial a_1} + \sigma_{21}^a\frac{\partial P}{\partial a_2}\right), \quad \sigma_{P2} = \varepsilon^{1/2}\left(\sigma_{12}^a\frac{\partial P}{\partial a_1} + \sigma_{22}^a\frac{\partial P}{\partial a_2}\right), \tag{3-4}$$

and

$$\begin{aligned} m_P(P,\varphi) &= \varepsilon\left\{m_1^a\frac{\partial P}{\partial a_1} + m_2^a\frac{\partial P}{\partial a_2} + \frac{1}{2}\left[b_{11}^a\frac{\partial^2 P}{\partial a_1^2} + (b_{12}^a + b_{21}^a)\frac{\partial^2 P}{\partial a_1\partial a_2} + b_{22}^a\frac{\partial^2 P}{\partial a_2^2}\right]\right\}, \\ &= \varepsilon pP\left\{\frac{m_1^a}{\sqrt{a_1^2+a_2^2}}\cos\varphi + \frac{m_2^a}{\sqrt{a_1^2+a_2^2}}\sin\varphi + \frac{1}{2}\left[\frac{b_{11}^a}{a_1^2+a_2^2}\big((p-2)\cos^2\varphi+1\big)\right.\right. \\ &\qquad \left.\left. + \frac{(b_{12}^a+b_{21}^a)}{2(a_1^2+a_2^2)}(p-2)\sin 2\varphi + \frac{b_{22}^a}{a_1^2+a_2^2}\big((p-2)\sin^2\varphi+1\big)\right]\right\}. \end{aligned} \tag{3-5}$$

To determine the diffusion terms, one may use the replacement as

$$\Sigma_P\,\mathrm{d}W = \sigma_{P1}\,\mathrm{d}W_1 + \sigma_{P2}\,\mathrm{d}W_2,$$

which yields the equation

$$\begin{aligned} \Sigma_P^2 = \sigma_{P1}^2+\sigma_{P2}^2 &= \varepsilon\left\{\sigma_{11}^a\frac{\partial P}{\partial a_1}+\sigma_{21}^a\frac{\partial P}{\partial a_2}\;\; \sigma_{12}^a\frac{\partial P}{\partial a_1}+\sigma_{22}^a\frac{\partial P}{\partial a_2}\right\}\left\{\begin{matrix}\sigma_{11}^a\,\partial P/\partial a_1+\sigma_{21}^a\,\partial P/\partial a_2\\ \sigma_{12}^a\,\partial P/\partial a_1+\sigma_{22}^a\,\partial P/\partial a_2\end{matrix}\right\} \\ &= \varepsilon\left[((\sigma_{11}^a)^2+(\sigma_{12}^a)^2)\left(\frac{\partial P}{\partial a_1}\right)^2+2(\sigma_{11}^a\sigma_{21}^a+\sigma_{12}^a\sigma_{22}^a)\frac{\partial P}{\partial a_1}\frac{\partial P}{\partial a_2}+((\sigma_{21}^a)^2+(\sigma_{22}^a)^2)\left(\frac{\partial P}{\partial a_2}\right)^2\right] \\ &= \varepsilon\left[b_{11}^a\left(\frac{\partial P}{\partial a_1}\right)^2+(b_{12}^a+b_{21}^a)\frac{\partial P}{\partial a_1}\frac{\partial P}{\partial a_2}+b_{22}^a\left(\frac{\partial P}{\partial a_2}\right)^2\right]. \end{aligned} \tag{3-6}$$

Hence,

$$\mathrm{d}P = m_P(P,\varphi)\,\mathrm{d}t + \sigma_{P1}\,\mathrm{d}W_1 + \sigma_{P2}\,\mathrm{d}W_2 = m_P(P,\varphi)\,\mathrm{d}t + \Sigma_P(P,\varphi)\,\mathrm{d}W. \tag{3-7}$$

Similarly, one can obtain the Itô equations for φ,

$$\mathrm{d}\varphi = m_\varphi(\varphi)\,\mathrm{d}t + \sigma_{\varphi 1}\,\mathrm{d}W_1 + \sigma_{\varphi 2}\,\mathrm{d}W_2 = m_\varphi(\varphi)\,\mathrm{d}t + \Sigma_\varphi(\varphi)\,\mathrm{d}W, \tag{3-8}$$

where

$$\sigma_{\varphi 1} = \varepsilon^{1/2}\left(\sigma_{11}^a\frac{\partial\varphi}{\partial a_1} + \sigma_{21}^a\frac{\partial\varphi}{\partial a_2}\right), \quad \sigma_{\varphi 2} = \varepsilon^{1/2}\left(\sigma_{12}^a\frac{\partial\varphi}{\partial a_1} + \sigma_{22}^a\frac{\partial\varphi}{\partial a_2}\right), \tag{3-9}$$

$$m_\varphi(\varphi) = \varepsilon\left\{m_1^a\frac{\partial\varphi}{\partial a_1} + m_2^a\frac{\partial\varphi}{\partial a_2} + \frac{1}{2}\left[b_{11}^a\frac{\partial^2\varphi}{\partial a_1^2} + (b_{12}^a + b_{21}^a)\frac{\partial^2\varphi}{\partial a_1\partial a_2} + b_{22}^a\frac{\partial^2\varphi}{\partial a_2^2}\right]\right\}$$

$$= \varepsilon\bigg\{ -\frac{m_1^a}{\sqrt{a_1^2+a_2^2}}\sin\varphi + \frac{m_2^a}{\sqrt{a_1^2+a_2^2}}\cos\varphi + \frac{1}{2}\bigg[\frac{b_{11}^a}{a_1^2+a_2^2}\sin 2\varphi - \frac{(b_{12}^a+b_{21}^a)}{a_1^2+a_2^2}(2\cos^2\varphi - 1) - \frac{b_{22}^a}{a_1^2+a_2^2}\sin 2\varphi\bigg]\bigg\}, \tag{3-10}$$

and

$$\Sigma_\varphi^2 = \sigma_{\varphi1}^2 + \sigma_{\varphi2}^2 = \varepsilon\bigg[b_{11}^a\Big(\frac{\partial\varphi}{\partial a_1}\Big)^2 + (b_{12}^a+b_{21}^a)\frac{\partial\varphi}{\partial a_1}\frac{\partial\varphi}{\partial a_2} + b_{22}^a\Big(\frac{\partial\varphi}{\partial a_2}\Big)^2\bigg]$$
$$= \varepsilon\bigg[\frac{b_{11}^a}{a_1^2+a_2^2}\sin^2\varphi - \frac{(b_{12}^a+b_{21}^a)}{a_1^2+a_2^2}\sin\varphi\cos\varphi + \frac{b_{22}^a}{a_1^2+a_2^2}\cos^2\varphi\bigg]. \tag{3-11}$$

It is noted that the coefficients of the right-hand side terms of amplitude equations in (2-8), such as m_i^a, $i = 1, 2$, are homogeneous of degree one in a_1 and a_2, which results in the fact that the diffusion term $\boldsymbol{b}^a$ in (2-10) are homogeneous of degree two in a_1 and a_2. Therefore, substituting $a_1 = r\cos\varphi$ and $a_2 = r\sin\varphi$ from (3-1) into (3-5), one finds that the drift $m_P(P,\varphi)$ and diffusion term $\Sigma_P(P,\varphi)$ are functions of P of degree one and φ. However, the drift $m_\varphi(\varphi)$ in (3-10) and diffusion term $\Sigma_\varphi(\varphi)$ in (3-11) are functions of φ only, which shows $P(t)$ in (3-7) and $\varphi(t)$ in (3-8) are coupled, although $\varphi(t)$ is itself a diffusion process.

To obtain the moment Lyapunov exponent, a linear stochastic transformation is adopted:

$$S = T(\varphi)\,P, \quad P = T^{-1}(\varphi)\,S, \quad 0 \le \varphi < \pi,$$

from which one obtains

$$\frac{\partial S}{\partial P} = T(\varphi), \quad \frac{\partial S}{\partial\varphi} = T_\varphi' P, \quad \frac{\partial^2 S}{\partial P^2} = 0, \quad \frac{\partial^2 S}{\partial P\partial\varphi} = T_\varphi', \quad \frac{\partial^2 S}{\partial^2\varphi} = PT_{\varphi\varphi}'', \tag{3-12}$$

where T_φ' and $T_{\varphi\varphi}''$ denote the first and second derivative of $T(\varphi)$ with respect to φ, respectively.

The Itô equation for the transformed p-th norm process S can also be derived using Itô's Lemma

$$\mathrm{d}S = m_S\,\mathrm{d}t + \Big(\sigma_{P1}\frac{\partial S}{\partial P}\,\mathrm{d}W_1 + \sigma_{P2}\frac{\partial S}{\partial P}\,\mathrm{d}W_2 + \sigma_{\varphi1}\frac{\partial S}{\partial\varphi}\,\mathrm{d}W_1 + \sigma_{\varphi2}\frac{\partial S}{\partial\varphi}\,\mathrm{d}W_2\Big). \tag{3-13}$$

The drift coefficient is given by

$$m_S = m_P\frac{\partial S}{\partial P} + m_\varphi\frac{\partial S}{\partial\varphi} + \frac{1}{2}\Big(b_{11}^S\frac{\partial^2 S}{\partial P^2} + (b_{12}^S+b_{21}^S)\frac{\partial^2 S}{\partial P\partial\varphi} + b_{22}^S\frac{\partial^2 S}{\partial\varphi^2}\Big)$$
$$= \tfrac{1}{2}P(\sigma_{\varphi1}^2+\sigma_{\varphi2}^2)T_{\varphi\varphi}'' + (m_\varphi P + \sigma_{P1}\sigma_{\varphi1} + \sigma_{P2}\sigma_{\varphi2})T_\varphi' + m_P T, \tag{3-14}$$

and the diffusion terms have the relation

$$\boldsymbol{b}^S = [\boldsymbol{\sigma}^S(\boldsymbol{\sigma}^S)^T], \quad \sigma^S = \begin{bmatrix}\sigma_{P1} & \sigma_{P2}\\ \sigma_{\varphi1} & \sigma_{\varphi2}\end{bmatrix}. \tag{3-15}$$

Substituting σ_{P1}, σ_{P2} from (3-4) and $\sigma_{\varphi 1}, \sigma_{\varphi 2}$ from (3-9) into (3-15) lead to

$$
\begin{aligned}
b^S_{11} &= \sigma^2_{P1} + \sigma^2_{P2} = \Sigma^2_P,\ b^S_{22} = \sigma^2_{\varphi 1} + \sigma^2_{\varphi 2} = \Sigma^2_\varphi,\\
b^S_{12} &= b^S_{21} = \sigma_{P1}\sigma_{\varphi 1} + \sigma_{P2}\sigma_{\varphi 2}\\
&= \varepsilon\big((\sigma^a_{11})^2+(\sigma^a_{12})^2\big)\frac{\partial P}{\partial a_1}\frac{\partial \varphi}{\partial a_1}+\varepsilon(\sigma^a_{11}\sigma^a_{21}+\sigma^a_{12}\sigma^a_{22})\left(\frac{\partial P}{\partial a_1}\frac{\partial \varphi}{\partial a_2}+\frac{\partial P}{\partial a_2}\frac{\partial \varphi}{\partial a_1}\right)+\varepsilon\big((\sigma^a_{21})^2+(\sigma^a_{22})^2\big)\frac{\partial P}{\partial a_2}\frac{\partial \varphi}{\partial a_2}\\
&= \varepsilon\left\{b^a_{11}\frac{\partial P}{\partial a_1}\frac{\partial \varphi}{\partial a_1} + b^a_{12}\left(\frac{\partial P}{\partial a_1}\frac{\partial \varphi}{\partial a_2} + \frac{\partial P}{\partial a_2}\frac{\partial \varphi}{\partial a_1}\right) + b^a_{22}\frac{\partial P}{\partial a_2}\frac{\partial \varphi}{\partial a_2}\right\}\\
&= \varepsilon p P\left(-\frac{b^a_{11}}{a_1^2+a_2^2}\cos\varphi\sin\varphi + \frac{b^a_{12}}{a_1^2+a_2^2}(\cos^2\varphi - \sin^2\varphi) + \frac{b^a_{22}}{a_1^2+a_2^2}\cos\varphi\sin\varphi\right). \qquad (3\text{-}16)
\end{aligned}
$$

For bounded and nonsingular transformation $T(\varphi)$, both processes P and S are expected to have the same stability behavior. Therefore, $T(\varphi)$ is chosen so that the drift term of the Itô differential Equation (3-13) is independent of the phase process φ so that

$$
\mathrm{d}S = \varepsilon\Lambda S\ \mathrm{d}t + \varepsilon^{1/2}(\sigma_{S1}\ \mathrm{d}W_1 + \sigma_{S2}\ \mathrm{d}W_2). \qquad (3\text{-}17)
$$

Comparing the drift terms of equations (3-13) and (3-17), one can find that such a transformation $T(\varphi)$ is given by the equation

$$
\tfrac{1}{2}P(\sigma^2_{\varphi 1} + \sigma^2_{\varphi 2})T''_{\varphi\varphi} + \big(m_\varphi P + \sigma_{P1}\sigma_{\varphi 1} + \sigma_{P2}\sigma_{\varphi 2}\big)T'_\varphi + m_P T = \varepsilon\Lambda S, \quad 0 \le \varphi < \pi, \qquad (3\text{-}18)
$$

in which $T(\varphi)$ is a periodic function in φ of period π. Equation (3-18) defines an eigenvalue problem of a second-order differential operator with Λ being the eigenvalue and $T(\varphi)$ the associated eigenfunction. Taking the expected value of both sides of (3-17), one can find that $\mathsf{E}[\sigma_{S1}\ \mathrm{d}W_1 + \sigma_{S2}\ \mathrm{d}W_2] = 0$ [Xie 2006]. Hence, one obtains $\mathsf{E}[S] = \varepsilon\Lambda\mathsf{E}[S]\,\mathrm{d}t$, from which the eigenvalue Λ is seen to be the Lyapunov exponent of the p-th moment of the system (2-1), that is, $\Lambda = \Lambda_{q(t)}(p)$. It is noted that both processes P and S have the same stability behavior.

Substituting (3-16) into (3-18) yields

$$
\mathcal{L}(p)[T] = \frac{1}{P}\big(\tfrac{1}{2}\Sigma^2_\varphi P T''_{\varphi\varphi} + [m_\varphi P + b^S_{12}]T'_\varphi + m_P T\big) = \varepsilon\Lambda T, \quad 0 \le \varphi < \pi, \qquad (3\text{-}19)
$$

where Σ^2_φ, m_φ, b^S_{12}, and m_P are given in (3-11), (3-10), (3-16), and (3-5), respectively. Substituting these equations into (3-19) yields

$$
\mathcal{L}(p)[T] = \lambda_2 T''_{\varphi\varphi} + \lambda_1 T'_\varphi + \lambda_0 T = \Lambda T, \quad 0 \le \varphi < \pi, \qquad (3\text{-}20)
$$

where

$$
\lambda_2 = \frac{1}{2}\left[\frac{b^a_{11}}{a_1^2+a_2^2}\sin^2\varphi - \frac{(b^a_{12}+b^a_{21})}{a_1^2+a_2^2}\sin\varphi\cos\varphi + \frac{b^a_{22}}{a_1^2+a_2^2}\cos^2\varphi\right],
$$

$$\lambda_1 = \left\{ \left[\frac{m_2^a \cos\varphi}{\sqrt{a_1^2+a_2^2}} - \frac{m_1^a \sin\varphi}{\sqrt{a_1^2+a_2^2}} + \frac{b_{11}^a \sin 2\varphi}{2(a_1^2+a_2^2)} - \frac{(b_{12}^a+b_{21}^a)(2\cos^2\varphi-1)}{2(a_1^2+a_2^2)} - \frac{b_{22}^a \sin 2\varphi}{2(a_1^2+a_2^2)} \right] \right.$$
$$\left. + p\left(-\frac{b_{11}^a}{a_1^2+a_2^2}\cos\varphi\sin\varphi + \frac{b_{12}^a}{a_1^2+a_2^2}(\cos^2\varphi-\sin^2\varphi) + \frac{b_{22}^a}{a_1^2+a_2^2}\cos\varphi\sin\varphi \right) \right\},$$

$$\lambda_0 = p\left\{ \frac{m_1^a}{\sqrt{a_1^2+a_2^2}}\cos\varphi + \frac{m_2^a}{\sqrt{a_1^2+a_2^2}}\sin\varphi \right.$$
$$\left. + \frac{1}{2}\left(\frac{b_{11}^a}{a_1^2+a_2^2}\big((p-2)\cos^2\varphi+1\big) + \frac{b_{12}^a+b_{21}^a}{2(a_1^2+a_2^2)}(p-2)\sin 2\varphi + \frac{b_{22}^a}{a_1^2+a_2^2}\big((p-2)\sin^2\varphi+1\big) \right) \right\}.$$

Substituting $a_1 = r\cos\varphi$, $a_2 = r\sin\varphi$ into these equations, one finds that the coefficients λ_0, λ_1, and λ_2 are functions of φ and p only. Solving Equation (3-20) yields the moment Lyapunov exponent Λ.

This idea was first applied in [Wedig 1988] to derive the eigenvalue problem for the moment Lyapunov exponent of a SDOF linear Itô stochastic system. This paper extends to 2DOF coupled systems and considered viscoelasticity.

Determination of moment Lyapunov exponents. It is found that the coefficients in (3-20) are periodic with period π, it is reasonable to consider a Fourier cosine series expansion of the eigenfunction $T(\varphi)$ in the form

$$T(\varphi) = \sum_{i=0}^{K} C_i \cos(2i\varphi). \tag{3-21}$$

Here only cosine functions are adopted because sometimes the eigenvalue problems in (3-20) contains $1/\sin(2\varphi)$. This Fourier expansion method is actually a method for solving partial differential equations and has been used in [Wedig 1988; Bolotin 1964; Sri Namachchivaya and van Roessel 2001], and elsewhere.

Substituting this expansion and $a_1 = r\cos\varphi$, $a_2 = r\sin\varphi$ into eigenvalue problem (3-20), multiplying both sides by $\cos(2j\varphi)$, $j = 0, 1, \ldots, K$, and performing integration with respect to φ from 0 to $\pi/2$ yield a set of equations for the unknown coefficients C_i, $i = 0, 1, \ldots, K$:

$$\sum_{i=0}^{K} a_{ij} C_i = \Lambda C_j, \quad j = 0, 1, 2, \ldots, \tag{3-22}$$

where

$$a_{ij} = \frac{4}{\pi}\int_0^{\pi/2} \mathcal{L}(p)[\cos(2i\varphi)]\cos(2j\varphi)\,\mathrm{d}\varphi, \quad j = 0, 1, \ldots, K, \tag{3-23}$$

and

$$\int_0^{\pi/2} \cos(2i\varphi)\cos(2j\varphi)\,\mathrm{d}\varphi = \begin{cases} \pi/4 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The eigenvalue can be obtained by solution of a polynomial equation as follows. Rearranging (3-22) leads to

$$\begin{bmatrix} a_{00}-\hat{\Lambda} & a_{01} & a_{02} & a_{03} & \cdots \\ a_{10} & a_{11}-\Lambda & a_{12} & a_{13} & \cdots \\ a_{20} & a_{21} & a_{22}-\Lambda & a_{23} & \cdots \\ a_{30} & a_{31} & a_{32} & a_{33}-\Lambda & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ \cdots \end{Bmatrix} = \mathbf{0}. \tag{3-24}$$

The third-order submatrix is listed here. For convenience, Λ is inserted in a_{ij}.

$$\begin{aligned}
a_{00} &= \tfrac{1}{128}\big(3k_{11}^2+3k_{22}^2+8k^2\big)S_0p^2+\tfrac{1}{64}\big(5k_{11}^2S_0+5k_{22}^2S_0+24k^2S_0-32(E_1+E_2)\big)p-\Lambda,\\
a_{01} &= \tfrac{1}{64}\big(k_{11}^2-k_{22}^2\big)S_0p^2+\tfrac{1}{32}\big(k_{11}^2S_0-k_{22}^2S_0-8(E_1-E_2)\big)p,\\
a_{02} &= \tfrac{1}{256}\big(k_{11}^2+k_{22}^2-8k^2\big)S_0p^2-\tfrac{1}{128}\big(k_{11}^2+k_{22}^2-16k^2\big)S_0p,\\
a_{10} &= \tfrac{1}{64}\big(k_{11}^2-k_{22}^2\big)S_0p^2+\tfrac{1}{16}\big(k_{11}^2S_0-k_{22}^2S_0-4(E_1-E_2)\big)p+\tfrac{1}{16}\big(k_{11}^2S_0-k_{22}^2S_0-8(E_1-E_2)\big),\\
a_{11} &= \tfrac{1}{512}\big(7k_{11}^2+7k_{22}^2+8k^2\big)S_0p^2+\tfrac{1}{256}\big(11k_{11}^2S_0+11k_{22}^2S_0+40k^2S_0-64(E_1+E_2)\big)p\\
&\qquad -\tfrac{1}{64}\big(k_{11}^2+k_{22}^2+24k^2\big)S_0-\tfrac{1}{2}\Lambda, \qquad (3\text{-}25)\\
a_{12} &= \tfrac{1}{128}\big(k_{11}^2-k_{22}^2\big)S_0p^2-\tfrac{1}{8}(E_1-E_2)p-\tfrac{1}{32}\big(k_{11}^2S_0-k_{22}^2S_0-8(E_1-E_2)\big),\\
a_{20} &= \tfrac{1}{256}\big(k_{11}^2+k_{22}^2-8k^2\big)S_0p^2+\tfrac{3}{128}\big(k_{11}^2+k_{22}^2-8k^2\big)S_0p+\tfrac{1}{32}\big(k_{11}^2+k_{22}^2-40k^2\big)S_0,\\
a_{21} &= \tfrac{1}{128}\big(k_{11}^2-k_{22}^2\big)S_0p^2+\tfrac{1}{64}\big(3k_{11}^2S_0-3k_{22}^2S_0-8(E_1-E_2)\big)p+\tfrac{1}{16}\big(k_{11}^2S_0-k_{22}^2S_0-8(E_1-E_2)\big),\\
a_{22} &= = \tfrac{1}{256}\big(3k_{11}^2+3k_{22}^2+8k^2\big)S_0p^2+\tfrac{1}{128}\big(5k_{11}^2S_0+5k_{22}^2S_0+24k^2S_0-32(E_1+E_2)\big)p\\
&\qquad -\tfrac{1}{16}\big(k_{11}^2+k_{22}^2+16k^2\big)S_0-\tfrac{1}{2}\Lambda,
\end{aligned}$$

where E_i is the pseudodamping defined in (2-14).

To have a nontrivial solution of the C_k, it is required that the determination of the coefficient matrix of (3-24) equal zero, from which the eigenvalue $\Lambda(p)$ can be obtained,

$$e_{K+1}^{(K)}[\Lambda^{(K)}]^{K+1}+e_K^{(K)}[\Lambda^{(K)}]^K+\cdots+e_1^{(K)}[\Lambda^{(K)}]^1+e_0^{(K)}=0, \tag{3-26}$$

where $\Lambda^{(K)}$ denotes the approximate moment Lyapunov exponent under the assumption that the expansion of eigenfunction $T(\varphi)$ is up to K-th order Fourier cosine series. The set of approximate eigenvalues obtained by this procedure converges to the corresponding true eigenvalues as $K\to\infty$. However, as shown in Figure 1, the approximate eigenvalues converges so quickly that the approximations almost coincide after the order $K\geq 1$. One may approximate the moment Lyapunov exponent of the system by

$$\Lambda_{q(t)}(p)\approx\Lambda^{(K)}(p). \tag{3-27}$$

Determination of Lyapunov exponents. The p-th moment Lyapunov exponent $\Lambda_{q(t)}(p)$ is a convex analytic function in p that passes through the origin and the slope at the origin is equal to the largest

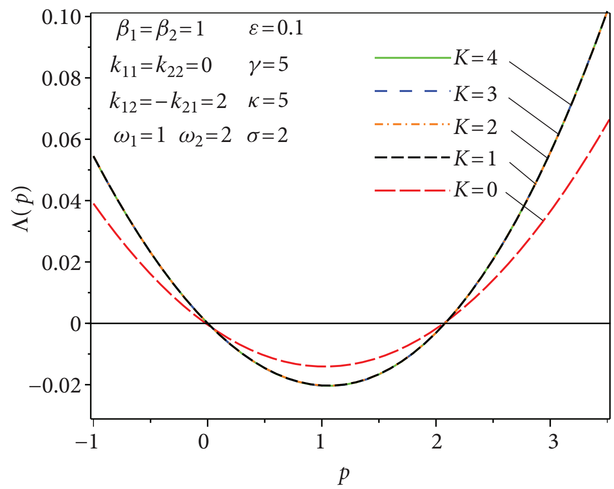

Figure 1. Moment Lyapunov exponents for various K-th order Fourier expansion.

Lyapunov exponent $\lambda_{q(t)}$, i.e.,

$$\lambda_{q(t)}(p) = \lim_{p\to 0} \frac{\Lambda^{(K)}(p)}{p} = -\lim_{p\to 0} \frac{e_0^{(K)}}{p e_1^{(K)}}, \tag{3-28}$$

which is obtained directly from (3-26).

For comparison, the largest Lyapunov exponent for system (2-8) can be directly derived from invariant probability density by solving a Fokker–Plank equation [Xie 2006].

(1) If $k_{11}^2 + k_{22}^2 > 4\,|k_{12}k_{21}|$, i.e., $\Delta_0 > 0$, we have

$$\lambda = \frac{1}{2}\left[(\lambda_1+\lambda_2) + (\lambda_1-\lambda_2)\coth\left(\frac{\lambda_1-\lambda_2}{\sqrt{\Delta_0}}\alpha\right)\right], \tag{3-29}$$

where $\alpha = \cosh^{-1} \dfrac{Q}{4\,|k_{12}k_{21}|\,S_0}$.

(2) If $k_{11}^2 + k_{22}^2 < 4\,|k_{12}k_{21}|$, i.e., $\Delta_0 < 0$, we have

$$\lambda = \frac{1}{2}\left[(\lambda_1+\lambda_2) + (\lambda_1-\lambda_2)\coth\left(\frac{\lambda_1-\lambda_2}{\sqrt{-\Delta_0}}\alpha\right)\right], \tag{3-30}$$

where $\alpha = \cos^{-1} \dfrac{Q}{4\,|k_{12}k_{21}|\,S_0}$.

(3) If $k_{11}^2 + k_{22}^2 = 4\,|k_{12}k_{21}|$, i.e., $\Delta_0 = 0$, we have

$$\lambda = \frac{1}{2}\left((\lambda_1+\lambda_2) + (\lambda_1-\lambda_2)\coth\frac{2(\lambda_1-\lambda_2)}{|k_{12}k_{21}|\,S_0}\right). \tag{3-31}$$

The constants Q and Δ_0 are defined as

$$\begin{aligned} Q &= \big(k_{11}^2 + k_{22}^2 - 4k_{12}k_{21}\big)S_0, & \Delta_0 &= \tfrac{1}{64}(Q^2 - 16k_{12}^2k_{21}^2S_0), \\ \lambda_1 &= -\beta_1 - \omega_1\tau_\varepsilon \underset{t}{\mathscr{M}}(I_1^{sc}) + \tfrac{1}{8}k_{11}^2S_0, & \lambda_2 &= -\beta_2 - \omega_2\tau_\varepsilon \underset{t}{\mathscr{M}}(I_2^{sc}) + \tfrac{1}{8}k_{22}^2S_0. \end{aligned} \tag{3-32}$$

4. Stability boundary

Moment Lyapunov exponents can be numerically determined from (3-26). When $K = 0$, the eigenfunction in (3-21) is $T(\varphi) = C_0$; then, from $a_{00} - \Lambda^{(0)} = 0$, the moment Lyapunov exponent is defined as

$$\Lambda_{q(t)}(p) \approx \Lambda^{(0)}(p) = \tfrac{1}{128}\big(3k_{11}^2 + 3k_{22}^2 + 8k^2\big)S_0p^2 + \tfrac{1}{64}[5k_{11}^2S_0 + 5k_{22}^2S_0 + 24k^2S_0 - 32(E_1 + E_2)]p. \tag{4-1}$$

If viscoelasticity is not considered, i.e., $\tau_\varepsilon = 0$ and $E_i = \beta_i$, then (4-1) reduces to the moment Lyapunov exponent for 2DOF linear systems subjected to white noise parametric excitation which was reported in Equation (3.19) in [Janevski et al. 2012], where the perturbation method was applied.

The moment stability boundary is then obtained as

$$\big(3k_{11}^2 + 3k_{22}^2 + 8k^2\big)S_0p + 2\big[5k_{11}^2S_0 + 5k_{22}^2S_0 + 24k^2S_0 - 32(E_1 + E_2)\big] = 0. \tag{4-2}$$

Again, if viscoelasticity and coupling parameter ($k = 0$) are not considered, (4-2) can be reduced to that obtained by other approximate methods such as asymptotic expansion of integrals and stochastic averaging; see Equation (40) of [Ariaratnam and Xie 1993].

The Lyapunov exponent is given by

$$\lambda^{(0)}(p) = \lim_{p\to 0}\frac{\Lambda^{(0)}(p)}{p} = \tfrac{1}{64}\big[5k_{11}^2S_0 + 5k_{22}^2S_0 + 24k^2S_0 - 32(E_1 + E_2)\big]. \tag{4-3}$$

The almost-sure stability region is found to be

$$5k_{11}^2S_0 + 5k_{22}^2S_0 + 24k^2S_0 - 32(E_1 + E_2) < 0. \tag{4-4}$$

From (4-2) and (4-4), the almost-sure and the moment stability boundary are both a straight line in the coordinates with E_1 and E_2.

When $K = 1$, the eigenfunction is $T(\varphi) = C_0 + C_1\cos 2\varphi$, the moment Lyapunov exponent can be solved from

$$\begin{vmatrix} a_{00} - \Lambda^{(1)} & a_{01} \\ a_{10} & a_{11} - \Lambda^{(1)} \end{vmatrix} = \mathbf{0}. \tag{4-5}$$

Expanding we get an equation, $(\Lambda^{(1)})^2 + e_1^{(1)}\Lambda^{(1)} + e_0^{(1)} = 0$, leading to an analytical expression for the moment Lyapunov exponent. The coefficients for systems with white noise ($S(\omega)=S_0$) are given by

$$
\begin{aligned}
e_1^{(1)} &= -\tfrac{1}{256}(13p^2+42p-8)S_0(k_{11}^2+k_{22}^2)-\tfrac{1}{64}(3p^2+22P-56)S_0k^2+p(E_1+E_2),\\
e_0^{(1)} &= S_0^2(k_{11}^4+k_{22}^4)\big(\tfrac{5}{32768}p^4+\tfrac{5}{4096}p^3+\tfrac{1}{8192}p^2-\tfrac{13}{2048}p\big)\\
&\quad+S_0^2k_{11}^2k_{22}^2\big(\tfrac{37}{16384}p^4+\tfrac{29}{2048}p^3+\tfrac{97}{4096}p^2+\tfrac{3}{1024}p\big)+S_0^2k^4\big(\tfrac{1}{2048}p^4+\tfrac{1}{128}p^3+\tfrac{1}{512}p^2-\tfrac{21}{128}p\big)\\
&\quad+S_0^2(k_{11}^2+k_{22}^2)k^2\big(\tfrac{5}{4096}p^4+\tfrac{13}{1024}p^3+\tfrac{7}{1024}p^2-\tfrac{19}{256}p\big)\\
&\quad+S_0k_{11}^2\big[-\big(\tfrac{5}{512}E_1+\tfrac{21}{512}E_2\big)p^3-\big(\tfrac{5}{256}E_1+\tfrac{37}{256}E_2\big)p^2+\big(\tfrac{5}{64}E_1-\tfrac{3}{64}E_2\big)p\big]\\
&\quad+S_0k_{22}^2\big[-\big(\tfrac{5}{512}E_2+\tfrac{21}{512}E_1\big)p^3-\big(\tfrac{5}{256}E_2+\tfrac{37}{256}E_1\big)p^2+\big(\tfrac{5}{64}E_2-\tfrac{3}{64}E_1\big)p\big]\\
&\quad+S_0k^2(E_1+E_2)\big(-\tfrac{3}{128}p^3-\tfrac{11}{64}p^2+\tfrac{7}{16}p\big)+\tfrac{1}{8}p^2(E_1^2+E_2^2+6E_1E_2)-\tfrac{1}{4}p(E_1-E_2)^2. \qquad (4\text{-}6)
\end{aligned}
$$

It can be found that on the condition of white noise, our case reduced to the results of [Janevski et al. 2012]. When $k_{11}=k_{22}=0$, the moment Lyapunov exponent is given by

$$
\begin{aligned}
\Lambda^{(1)}(p) &= \big(\tfrac{3}{128}p^2+\tfrac{11}{64}p-\tfrac{7}{16}\big)k^2S_0-\tfrac{1}{2}p(E_1+E_2)\\
&\quad+\tfrac{1}{128}\big[\big(3136+224p+116p^2+4p^3+p^4\big)k^4S_0^2+2048p(p+2)(E_1-E_2)^2\big]^{1/2}. \qquad (4\text{-}7)
\end{aligned}
$$

The moment stability boundary is then obtained as $\Lambda^{(1)}(p)=0$. From this boundary, one can establish the relation between E_1 and E_2, and the critical excitation S_0, which will be shown in application part.

The corresponding largest Lyapunov exponents are

$$\lambda^{(1)}(p)=\frac{1}{112}\frac{21k^4S_0^2-56k^2S_0(E_1+E_2)+32(E_1-E_2)^2}{k^2S_0}. \qquad (4\text{-}8)$$

The almost-sure stability region is found to be

$$21k^4S_0^2-56k^2S_0(E_1+E_2)+32(E_1-E_2)^2<0. \qquad (4\text{-}9)$$

The analytical accurate results of Lyapunov exponents from (3-29) and numerical results from (3-28) are compared in Figure 2, left, which shows the two results almost overlap. Care should be taken that for $S(\omega)\to 0$, the numerical results may become unstable due to roundoff errors. Figure 2, left, illustrates that Lyapunov moments converge quickly with the increase of expansion order K. It is noted that although the results in Figure 2 overlap, the analytical expressions from (3-29) to (3-31) seem not to agree with the expression given in (4-8). This can be explained by the discrepancy between exact solutions and a sequence of approximations.

When $K=2$, the eigenfunction is $T(\varphi)=C_0+C_1\cos 2\varphi+C_2\cos 4\varphi$, and the moment Lyapunov exponent can be solved from

$$\begin{vmatrix} a_{00}-\Lambda^{(2)} & a_{01} & a_{02}\\ a_{10} & a_{11}-\Lambda^{(2)} & a_{12}\\ a_{20} & a_{21} & a_{22}-\Lambda^{(2)}\end{vmatrix}=\mathbf{0}. \qquad (4\text{-}10)$$

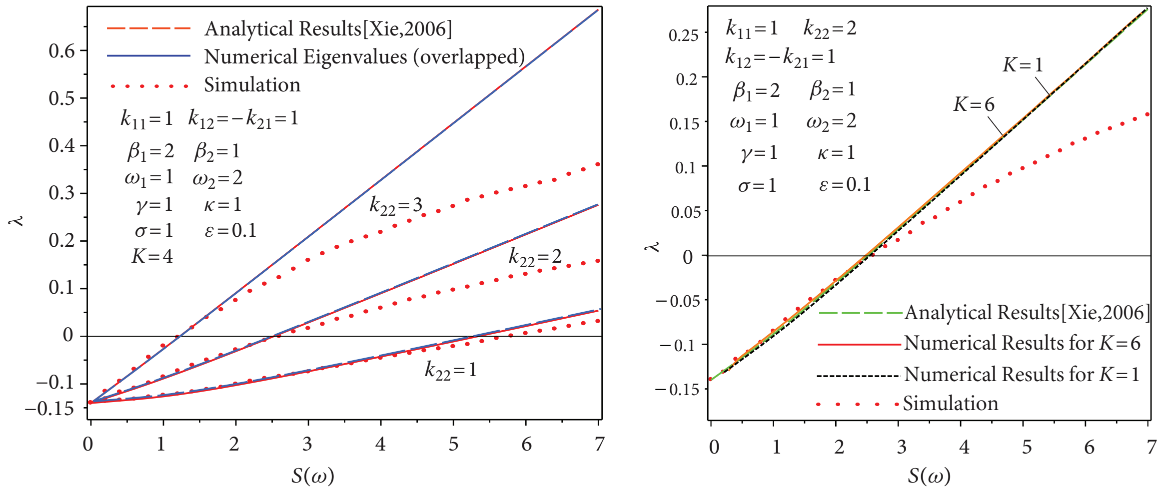

Figure 2. Comparisons of largest Lyapunov exponents for varying k_{22} (left) and varying K (right).

Expanding we get a cubic equation, $(\Lambda^{(2)})^3 + e_2^{(2)}(\Lambda^{(2)})^2 + e_1^{(2)}\Lambda^{(2)} + e_0^{(2)} = 0$. The coefficients for symmetric coupled systems with white noise ($S(\omega) = S_0$, $k_{12} = k_{21} = k$, and $k_{11} = k_{22} = 0$) are given by

$$e_2^{(2)} = -\tfrac{1}{256}\big(13p^2 + \tfrac{17}{32}p - \tfrac{27}{8}\big)S_0k^2 + \tfrac{3}{2}p(E_1 + E_2),$$

$$\begin{aligned} e_1^{(2)} = {} & S_0^2k^4\big(\tfrac{3}{2048}p^4 + \tfrac{13}{512}p^3 - \tfrac{55}{512}p^2 - \tfrac{143}{128}p + \tfrac{35}{16}\big) - S_0k^2(E_1 + E_2)\big(\tfrac{5}{64}p^3 + \tfrac{17}{32}p^2 - \tfrac{27}{8}p\big) \\ & + \big(\tfrac{9}{16}E_1^2 + \tfrac{9}{16}E_2^2 + \tfrac{15}{8}E_1E_1\big)p^2 + (E_1 - E_2)^2\big(\tfrac{1}{2} - \tfrac{3}{8}p\big), \end{aligned}$$

$$\begin{aligned} e_0^{(2)} = {} & -S_0^3k^6\big(\tfrac{1}{131072}p^6 + \tfrac{15}{65536}p^5 - \tfrac{1}{32768}p^4 - \tfrac{427}{16384}p^3 + \tfrac{1}{1024}p^2 + \tfrac{357}{1024}p\big) \\ & + S_0^2k^4(E_1 + E_2)\big(\tfrac{3}{4096}p^5 + \tfrac{13}{1024}p^4 - \tfrac{55}{1024}p^3 - \tfrac{143}{256}p^2 + \tfrac{35}{32}p\big) \\ & + S_0k^2\big[-\big(\tfrac{5}{512}E_1^2 + \tfrac{5}{512}E_2^2 + \tfrac{15}{256}E_1E_2\big)p^4 - \big(\tfrac{9}{128}E_1^2 + \tfrac{9}{128}E_2^2 + \tfrac{25}{64}E_1E_2\big)p^3 \\ & \qquad + \big(\tfrac{89}{128}E_1^2 + \tfrac{89}{128}E_2^2 + \tfrac{127}{64}E_1E_2\big)p^2 - \big(\tfrac{33}{32}E_1^2 + \tfrac{33}{32}E_2^2 - \tfrac{33}{16}E_1E_2\big)p\big] \\ & + \big(\tfrac{1}{32}E_1^3 + \tfrac{1}{32}E_2^3 + \tfrac{15}{32}E_1^2E_2 + \tfrac{15}{32}E_1E_2^2\big)p^3 + \big(E_1^3 + E_2^3 - E_1^2E_2 - E_1E_2^2\big)\big(\tfrac{1}{4}p - \tfrac{3}{16}p^2\big). \end{aligned} \quad (4\text{-}11)$$

The analytical expression for moment Lyapunov exponent can then be obtained by solving this cubic equation. However, for $K \geq 3$, no explicit expressions can be presented, as quartic equation is involved.

The Lyapunov exponent for $K = 2$ is given by

$$\lambda^{(2)} = -\lim_{p\to 0} \frac{e_0^{(2)}}{pe_1^{(2)}} = \frac{1}{256}\frac{\lambda_N}{\lambda_D}, \quad (4\text{-}12)$$

where

$$
\begin{aligned}
\lambda_N ={}& 22848 S_0^3 k^6 + \big(14480(k_{11}^2 + k_{22}^2) S_0 - 71680(E_1 + E_2)\big) S_0^2 k^4 \\
&+ \Big\{ (2076 k_{11}^4 + 2076 k_{22}^4 - 72 k_{11}^2 k_{22}^2) S_0^2 \\
&\quad + \big((10752 k_{11}^2 - 23040 k_{22}^2) E_2 + (10752 k_{22}^2 - 23040 k_{11}^2) E_1\big) S_0 + 67584 (E_1 - E_2)^2 \Big\} S_0 k^2 \\
&+ (k_{11}^4 k_{22}^2 + k_{11}^2 k_{22}^4 + 75 k_{11}^6 + 75 k_{22}^6) S_0^3 \\
&+ \big((512 k_{22}^4 + 256 k_{22}^2 k_{22}^2 - 1280 k_{11}^4) E_1 + (512 k_{11}^4 + 256 k_{22}^2 k_{22}^2 - 1280 k_{22}^4) E_2\big) S_0^2 \\
&+ \big((7680 k_{11}^2 - 512 k_{22}^2) E_1^2 - 7168 (k_{11}^2 + k_{22}^2) E_1 E_2 + (7680 k_{22}^2 - 512 k_{11}^2) E_2^2\big) S_0 \\
&- 16384 (E_1^3 + E_2^3 - E_1^2 E_2 - E_1 E_2^2), \\
\lambda_D ={}& 560 S_0^2 k^4 + 48 (k_{11}^2 + k_{22}^2) S_0^2 k^2 + (3 k_{11}^4 + 3 k_{22}^4 - 2 k_{22}^2 k_{22}^2) S_0^2 \\
&+ 32 \big((k_{22}^2 - k_{11}^2) E_1 + (k_{11}^2 - k_{22}^2) E_2\big) S_0 + 128 (E_1 - E_2)^2.
\end{aligned}
$$

5. Stability index

The stability index is the nontrivial zero of the moment Lyapunov exponent. Hence, it can be determined as a root-finding problem such that $\Lambda_{q(t)}(\delta_{q(t)}) = 0$. When the order K of the Fourier expansion is 0, from (4-1), the stability index is given by

$$
\delta_{q(t)} = 2\, \frac{5 k_{11}^2 S_0 + 5 k_{22}^2 S_0 + 24 k^2 S_0 - 32 (E_1 + E_2)}{3 k_{11}^2 S_0 + 3 k_{22}^2 S_0 + 8 k^2 S_0}. \tag{5-1}
$$

When $K = 1$, the stability index is the nontrivial solution of $\Lambda^{(1)}(p) = 0$ from (4-7), which is hard to express analytically. Typical results of the stability index are shown in Figure 3. It is seen that the stability index decreases from positive to negative values with the increase of the amplitude of power spectrum, which suggests that the noise destabilizes the system. The larger the pseudodamping coefficient E_1, the larger the stability index, and then the more stable the system.

6. Simulation

Monte Carlo simulation is applied to determine the p-th moment Lyapunov exponents and to check the accuracy of the approximate results from the stochastic averaging.

Suppose the excitation is approximated by a Gaussian white noise with spectral density $S(\omega) = \vartheta^2 =$ constant for all ω, and then $\xi(t)\,\mathrm{d}t = \vartheta\,\mathrm{d}W(t)$. Let

$$
\begin{aligned}
&x_1(t) = q_1(t), \quad x_2(t) = \dot{q}_1(t), \quad x_3(t) = q_2(t), \quad x_4(t) = \dot{q}_2(t), \\
&x_5(t) = \int_0^t \gamma\, \mathrm{e}^{-\zeta(t-s)} q_1(s)\,\mathrm{d}s, \quad x_6(t) = \int_0^t \gamma\, \mathrm{e}^{-\zeta(t-s)} q_2(s)\,\mathrm{d}s.
\end{aligned} \tag{6-1}
$$

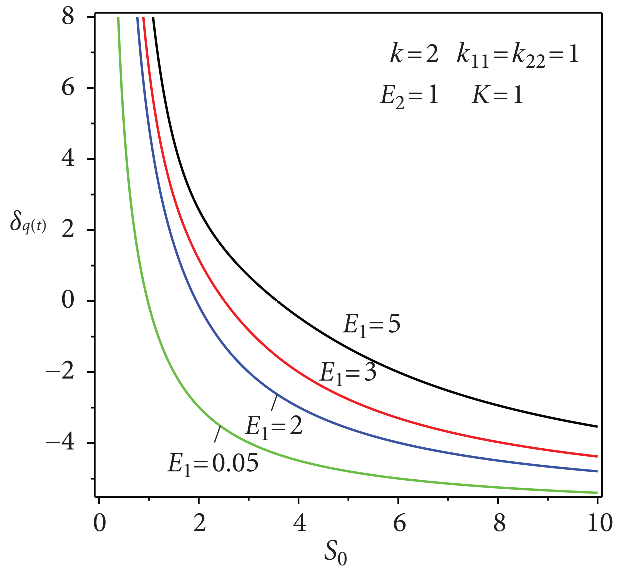

Figure 3. Stability index for system under white noise.

Equation (2-1) can be written as a six-dimensional system of Itô differential equations

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{A}\,\boldsymbol{x}\,\mathrm{d}t + \boldsymbol{B}\,\boldsymbol{x}\vartheta\,\mathrm{d}W, \tag{6-2}$$

where $\boldsymbol{x} = \{x_1, x_2, x_3, x_4, x_5, x_6\}^T$, and

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_1^2 & -2\varepsilon\beta_1 & 0 & 0 & -\varepsilon\omega_1^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_2^2 & -2\varepsilon\beta_2 & 0 & -\varepsilon\omega_2^2 \\ \gamma & 0 & 0 & 0 & -\zeta & 0 \\ 0 & 0 & \gamma & 0 & 0 & -\zeta \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\varepsilon^{1/2}\omega_1 k_{11} & 0 & -\varepsilon^{1/2}\omega_1 k_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\varepsilon^{1/2}\omega_2 k_{21} & 0 & -\varepsilon^{1/2}\omega_2 k_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6-3}$$

Equation (6-2) is linear homogeneous. We apply the algorithm proposed in [Xie 2006; Wolf et al. 1985] to simulate the moment Lyapunov exponents and Lyapunov exponents. The norm for simulations is

$$\|\boldsymbol{x}(t)\| = \sqrt{\textstyle\sum_{i=1}^{6} x_i^2}.$$

The iteration equations are given by, using the explicit Euler scheme,

$$x_1^{k+1} = x_1^k + x_2^k \cdot \Delta t,$$
$$x_2^{k+1} = x_2^k + \left(-\omega_1^2 x_1^k - 2\varepsilon\beta_1 x_2^k + \varepsilon\omega_1^2 x_5^k\right)\Delta t - \varepsilon^{1/2}\omega_1(k_{11}x_1^k + k_{12}x_3^k)\vartheta \cdot \Delta^k,$$
$$x_3^{k+1} = x_3^k + x_4^k \cdot \Delta t,$$
$$x_4^{k+1} = x_4^k + \left(-\omega_2^2 x_3^k - 2\varepsilon\beta_2 x_4^k + \varepsilon\omega_2^2 x_6^k\right)\Delta t - \varepsilon^{1/2}\omega_2(k_{21}x_1^k + k_{22}x_3^k)\vartheta \cdot \Delta^k,$$

$$x_5^{k+1} = x_5^k + (\gamma x_1^k - \zeta x_5^k)\Delta t, \quad \text{and} \quad x_6^{k+1} = x_6^k + (\gamma x_3^k - \zeta x_6^k)\Delta t,$$

where Δt is the time step and k denotes the k-th iteration.

The analytical and numerical Lyapunov exponents in Figure 2 are tangent lines of results from Monte Carlo simulation, which confirms the method of stochastic averaging is a valid first-order approximation method. In Monte Carlo simulation, the sample size for estimating the expected value is $N = 5000$, time step is $\Delta t = 0.0005$, and the number of iteration is 10^8.

7. Application: flexural-torsional stability of a rectangular beam

As an application, the flexural-torsional stability of a simply supported, uniform, narrow, rectangular, viscoelastic beam of length L subjected to a stochastically varying concentrated load $P(t)$ acting at the center of the beam cross-section as shown in Figure 4 is considered. Both nonfollower and follower loading cases are studied.

For the elastic beam under dynamic loading, the flexural and torsional equations of motion are given by (see [Xie 2006])

$$\begin{aligned} &EI_y \frac{\partial^4 u}{\partial z^4} + \frac{\partial^2 (M_x \psi)}{\partial z^2} - \frac{\partial^2 M_y}{\partial z^2} + m\frac{\partial^2 u}{\partial t^2} + D_u \frac{\partial u}{\partial t} = 0, \\ &-GJ\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial M_x}{\partial z}\frac{\partial u}{\partial z} + M_x \frac{\partial^2 u}{\partial z^2} + \frac{\partial M_z}{\partial z} + mr^2 \frac{\partial^2 \psi}{\partial t^2} + D_\psi \frac{\partial \psi}{\partial t} = 0, \end{aligned} \tag{7-1}$$

with boundary conditions

$$u(0,t) = u(L,t) = \frac{\partial^2 u(0,t)}{\partial z^2} = \frac{\partial^2 u(L,t)}{\partial z^2} = 0, \quad \psi(0,t) = \psi(L,t) = 0, \tag{7-2}$$

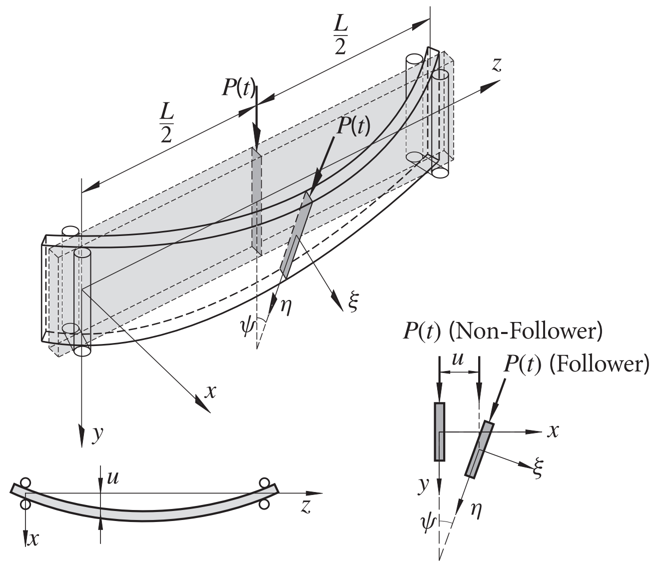

Figure 4. Flexural-torsional vibration of a rectangular beam.

where M_x, M_y, and M_z are the bending moments in the x, y, and z direction, respectively. $u(t)$ is the lateral deflection and $\psi(t)$ is the angle of twist. E is Young's modulus of elasticity and G is the shear modulus. I_y is the moment of inertia in the y direction. J is Saint-Venant's torsional constant. EI_y is the flexural rigidity, GJ is the torsional rigidity, m is the mass per unit length of the beam, r is the polar radius of gyration of the cross section, D_ψ and D_u are the viscous damping coefficients in the ψ and u directions,respectively, t is time and z is the axial coordinate.

For a three-dimensional loading, the stress and strain tensors σ and ϵ are given by

$$\sigma = \sigma_I + s, \quad \epsilon = \epsilon_I + e, \tag{7-3}$$

where σ_I and ϵ_I are the stress and strain first invariants, which produce dilatation and no distortion, also called hydrostatic or volumetric tensor, and s and e are the stress and strain deviatoric components, which tend to distort the body. The deviatoric stress tensor can be obtained by subtracting the hydrostatic stress tensor from the stress tensor.

According to Boltzmann's superposition principle, a constitutive relation for a linear viscoelastic material under three-dimensional loading can be obtained by replacing the bulk modulus B and the shear modulus G by appropriate Volterra operators [Drozdov 1998]. Most natural and polymer materials exhibit a elastic dilatation whereas the shear deformation is viscoelastic. The bulk modulus is assumed to time-independent and then the viscoelastic constitutive relation is given by

$$\sigma_I = 3B\epsilon_I, \quad s = 2G(1-\mathscr{H})e, \tag{7-4}$$

where κ is the bulk modulus, G is the shear modulus, and $\mathscr{H}$ is the relaxation operator defined in (2-3) for Maxwell viscoelastic materials. Therefore, for problems with viscoelastic materials, one may simply replace the elastic moduli E and G in elastic problems by the Voltera operators $E(1-\mathscr{H})$ and $G(1-\mathscr{H})$, respectively. The governing equation for a viscoelastic deep beam can be obtained from (7-1) as

$$\begin{aligned} &EI_y(1-\mathscr{H})\frac{\partial^4 u}{\partial z^4} + \frac{\partial^2(M_x\psi)}{\partial z^2} - \frac{\partial^2 M_y}{\partial z^2} + m\frac{\partial^2 u}{\partial t^2} + D_u\frac{\partial u}{\partial t} = 0, \\ &-GJ(1-\mathscr{H})\frac{\partial^2\psi}{\partial z^2} + \frac{\partial M_x}{\partial z}\frac{\partial u}{\partial z} + M_x\frac{\partial^2 u}{\partial z^2} + \frac{\partial M_z}{\partial z} + mr^2\frac{\partial^2\psi}{\partial t^2} + D_\psi\frac{\partial\psi}{\partial t} = 0. \end{aligned} \tag{7-5}$$

These equations are partial stochastic differential equations and are difficult to solve. An approximate solution can be obtained by using Galerkin method. Seek solutions of the form

$$u(z,t) = q_1 \sin\frac{\pi z}{L}, \quad \psi(z,t) = q_2 \sin\frac{\pi z}{L}, \tag{7-6}$$

in which $q_1 = u_m = u(\frac{1}{2}L, t)$, $q_2 = \psi_m = \psi(\frac{1}{2}L, t)$. Substituting into (7-5), multiplying by $\sin(\pi z/L)$, and integrating with respect to z from 0 to L yields

$$\begin{aligned} &\ddot{Q}_1 + 2\beta_1\dot{Q}_1 + \omega_1^2(1-\mathscr{H})Q_1 + \omega_1 k_{12}\xi(t)Q_2 = 0, \\ &\ddot{Q}_2 + 2\beta_2\dot{Q}_2 + \omega_2^2(1-\mathscr{H})Q_2 + \omega_2 k_{21}\xi(t)Q_1 = 0, \end{aligned} \tag{7-7}$$

where

$$K = \sqrt{\frac{\omega_2(12-\pi^2)}{\omega_1(4+\pi^2)}}, \quad \beta_1 = \frac{D_u}{2m}, \quad \beta_2 = \frac{D_\psi}{2mr^2}, \quad \omega_1^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI_y}{m}, \quad \omega_2^2 = \left(\frac{\pi}{L}\right)^2 \frac{GJ}{mr^2},$$

$$\xi(t) = \frac{P(t)}{P_{\text{cr}}}, \quad P_{\text{cr}} = \frac{4mrL\,|\omega_1^2-\omega_2^2|}{[(12-\pi^2)(4+\pi^2)]^{1/2}}, \quad k_{12} = -k_{21} = \frac{|\omega_1^2-\omega_2^2|}{2\sqrt{\omega_1\omega_2}} = k_F, \tag{7-8}$$

where β_1 and β_2 are reduced viscous damping coefficients, P_{cr} is the critical force for the simply supported narrow rectangular beam.

For nonfollower force case, the only difference is that $M_y = 0$ in (7-5). The equations of motion are of the same form of (7-7), but the parameters are different,

$$K = -\sqrt{\frac{\omega_2}{\omega_1}}, \quad 2\beta_1 = \frac{D_u}{m}, \quad 2\beta_2 = \frac{D_\psi}{mr^2}, \quad \omega_1^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI_y}{m}, \quad \omega_2^2 = \left(\frac{\pi}{L}\right)^2 \frac{GJ}{mr^2},$$

$$\xi(t) = \frac{P(t)}{P_{\text{cr}}}, \quad P_{\text{cr}} = \frac{8mrL\omega_1\omega_2}{4+\pi^2}, \quad k_{12} = k_{21} = \sqrt{\omega_1\omega_2} = k_N. \tag{7-9}$$

It is seen that (7-7) has the same form of (2-1), except that $k_{11} = k_{22} = 0$. By introducing the polar transformation and using the method of stochastic averaging, Equation (7-7) can be approximated in amplitude by the Itô stochastic differential equations in (2-8), where the drift and diffusion terms are given by

$$m_i^a = a_i\big[-\beta_i - \tau_\varepsilon \underset{t}{\mathscr{M}}(I_i^{sc})\big] + \frac{1}{8}\frac{a_j^2}{a_i}k_{ij}^2 S_0, \quad b_{ii}^a = \tfrac{1}{4}k_{ij}^2 a_j^2 S_0, \quad b_{ij}^a = 0. \tag{7-10}$$

Substituting (7-10) and $a_1 = r\cos\varphi$, $a_2 = r\sin\varphi$ into (3-20) yields the eigenvalue problem, from which moment Lyapunov exponents can be determined by solving (3-26).

Some analytical stability boundaries are discussed here. For nonfollower symmetric coupled systems under white noise excitation, the moment stability boundaries can be obtained from (4-1) for $K = 0$ and (4-7) for $K = 1$, and the almost-sure stability boundaries is from (4-3) for $K = 0$ and (4-9) for $K = 1$, which is shown in Figure 5, left. The moment stability boundaries are more conservative than the almost-sure boundary. With the moment order p increase, these moment boundaries become more and more conservative.

One can also obtain the critical amplitude of power spectral density of white noise excitation for the case $K = 1$ from (4-7) and (4-9), which is illustrated in Figure 5, right. The critical excitation increases with the pseudodamping coefficient, which confirms that damping and viscoelasticity would consume some energy of motion during vibration. Smaller amplitude of excitation (S_0) would destabilize the system in terms of higher-order moment stability, which suggests that stability region with higher-order moment is more conservative than that with lower-order moment stability. Almost-sure stability is the least conservative. Some numerical values for discrete points already shown in Figure 5, right, are listed in Table 1 for reference.

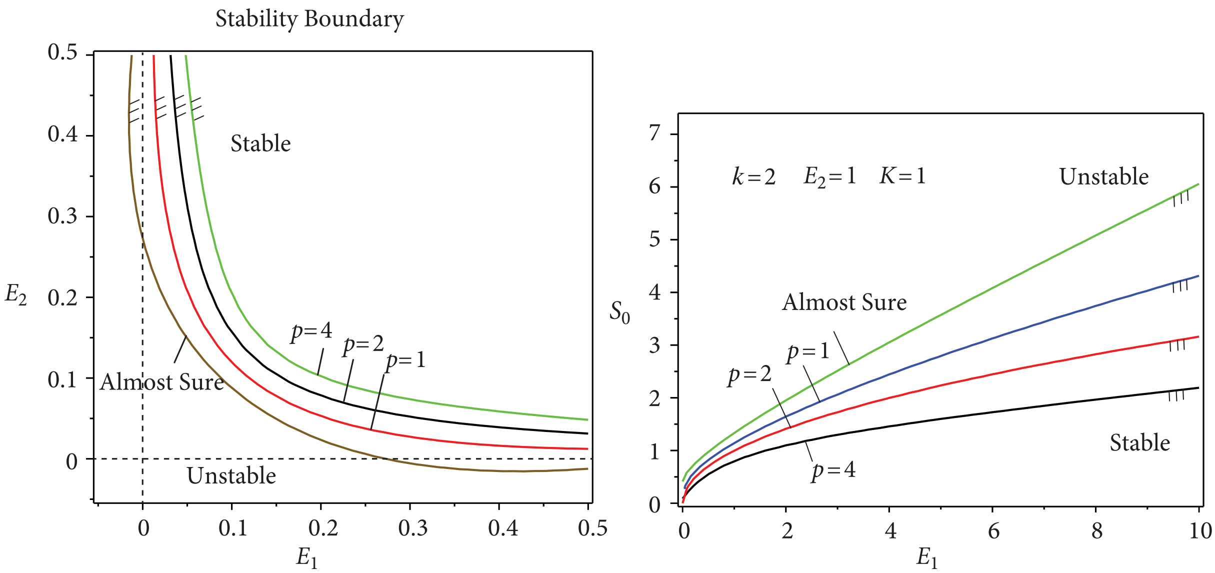

Figure 5. Left: Almost-sure and p-th moment stability boundaries. Right: Critical excitation and pseudodamping.

E_1	$p=0$	$p=1$	$p=2$	$p=4$
0	0.6667	0.2857	0.2500	0.2000
2	2.0000	0.8571	0.7500	0.6000
4	3.3333	1.4286	1.2500	1.0000
6	4.6667	2.0000	1.7500	1.4000
8	6.0000	2.5714	2.2500	1.8000
10	7.3333	3.1429	2.7500	2.2000

E_1	$p=0$	$p=1$	$p=2$	$p=4$
0	0.4593	0.2500	-	0.1000
2	1.9512	1.6443	1.4142	1.1000
4	3.0526	2.4459	2.0000	1.4600
6	4.0836	3.1290	2.4495	1.7286
8	5.0816	3.7463	2.8284	1.9667
10	6.0605	4.3184	3.1623	2.1909

Table 1. Critical excitations S_0 for the cases $K=0$ (left) and $K=1$ (right).

8. Discussion

Figure 6, top, shows that with the increase of β, the slope of the moment curves at the original point change from negative to positive, which suggests the beam's status from instability to stability and so damping plays a stabilizing role in flexural-torsional analysis of the beam under white noise excitation.

Figure 6, bottom, shows that the viscoelastic intensity γ has a stabilizing effect (left panel) but η plays a destabilizing effect on systems under white noise (right panel). With an increase of γ, the stability index also increases, which means the stability region for $p>0$ becomes wider, so the viscoelasticity help stabilization. However, the decrease of η means the relaxation time increase, the stability region for $p>0$ also becomes wider, which shows smaller η or larger relaxation time helps to stabilize the system. The effect of stabilization is prominent only when η is small. As η exceeds 10, large increase of η produces very small effect on stabilization.

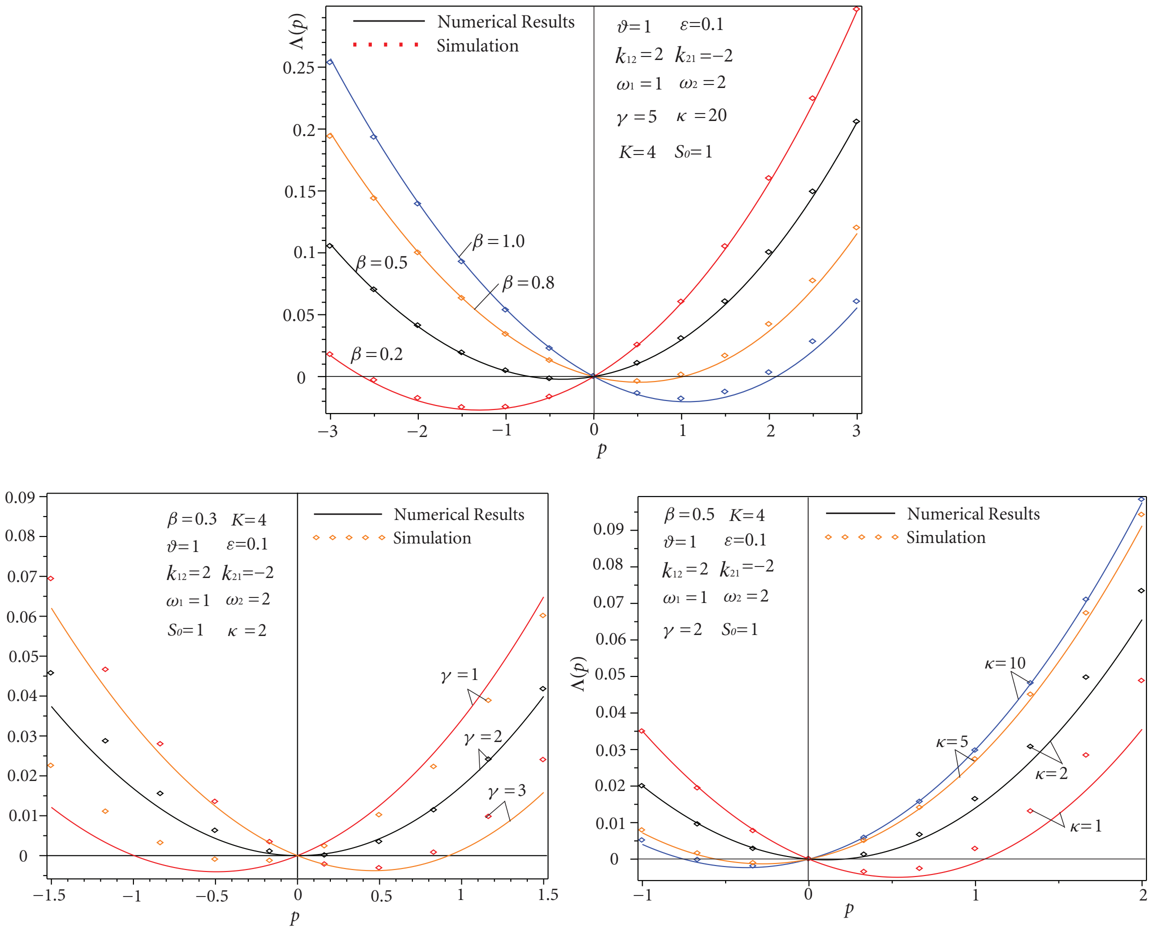

Figure 6. Effect of damping (top), viscosity γ (bottom left), and viscosity η (bottom right on moment Lyapunov exponents.

9. Conclusions

The stochastic stability of coupled viscoelastic systems described by Stratonovich stochastic integro-differential equations of 2DOF was investigated. The system was parametrically excited by white noise of small intensity with small damping. The Stratonovich equations of motion were first decoupled into two-dimensional Itô stochastic differential equations, by making use of the method of stochastic averaging for the nonviscoelastic terms and the method of Larionov for viscoelastic terms. An elegant scheme for determining the moment Lyapunov exponents was presented by only using Khasminskii and Wedig's mathematical transformations from the decoupled Itô equations. The moment Lyapunov exponents and Lyapunov exponents are compared well to the Monte Carlo simulation results and other analytical expressions from the literature.

As an application, the flexural-torsional stability of a simply supported rectangular viscoelastic beam subjected to a stochastically varying concentrated load acting at the center of the beam cross-section is considered. The moment and almost-sure stability boundaries and the critical amplitude of power spectral density are obtained. It is found that, under white noise excitation, the parameters of damping β and the viscoelastic intensity γ have stabilizing effects on the moment and almost-sure stability. However, viscosity parameter η plays a destabilizing role. The stability index decreases from positive to negative values with the increase of the amplitude of power spectrum, which suggests that the noise destabilize the system. These results are useful in engineering applications.

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