Journal of
Mechanics of
Materials and Structures

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Volume 9, No. 4 July 2014

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 9, No. 4, 2014
dx.doi.org/10.2140/jomms.2014.9.427

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Effective properties of random composites are discussed in the framework of the representative volume element (RVE) theory proposed by Mityushev (2006). This theory is extended to 2D fiber composites with sections perpendicular to fibers of different radii and to 3D composites with spherical inclusions. RVE theory is applied to the mixture problem frequently met in technological processes. This problem consists in a mechanical stir process of few components to get a homogeneous material in macroscale. In particular, the AlSi9/SiC composites obtained by thermomechanical deformation are investigated.

1. Introduction

The effective properties of composites and porous media can be determined by measurement of macroscopic properties of test specimens. Analogous to the measurement, computational methods are used in theoretical investigations of the specimens, which represent the entire material. If inclusions or pores are distributed statistically homogeneously in the bulk material, the effective properties are described by constant tensors [Adler et al. 2012; Gross and Seelig 2011]. The macroscopic tensors do not depend on the size, shape of the chosen specimen or on boundary conditions [Bakhvalov and Panasenko 1989; Jikov et al. 1994]. These tensors can be determined via solution of the periodic problem when the periodicity cell represents the material under consideration. This concerns not only strictly periodic media but also statistically homogeneous media when a cell represents the macroscopic properties of the random media [Golden and Papanicolaou 1983; Jikov et al. 1994; Telega 2004]. Such media constitute a subclass of heterogeneous fields discussed in [Torquato 2002; Mityushev 1999; Mityushev and Rylko 2013] and functionally gradient materials [Jaworska et al. 2006; Rozmus et al. 2009; Gross and Seelig 2011].

Statistically homogeneous media, defined in [Golden and Papanicolaou 1983; Jikov et al. 1994; Telega 2004], can be represented by a cell, which is called the representative volume element (RVE). The homogenization theory justifies existence of the macroscopic tensors for such media. The computational problem consists in numerical and symbolic calculations of the effective tensors when the RVE is given, i.e., the microstructure is deterministically or statistically described. However, only the existence of the RVE follows from the homogenization theory. Therefore, methods to construct RVEs can be considered as the first step to determine the effective properties of composites and porous media.

Statistical methods to construct RVEs have been described in detail in [Benveniste 1987; Gusev 1997; Huet 1990; Kanit et al. 2003; Milton 2002; Ostoja-Starzewski 2008; 2011; Segurado and LLorca 2006; Torquato 2002] and works cited therein. The statistical methods are based on the overall testing process. For instance, for dispersed two-phase composites with equal inclusions, the number of particles contained

MSC2010: 74Q15.
Keywords: representative volume element, effective conductivity, nonoverlapping disks, random inclusions.

in a sample is increased and the effective constants are computed, usually by purely numerical methods [Zohdi and Wriggers 2008]. The process of increasing is stopped when the fluctuations of the effective constants become sufficiently small. The number of particles 64 per cell frequently arises in the literature [ibid.] as a sufficiently large number for the nonoverlapping uniform distribution of inclusions. Analytical and numerical results [Czapla et al. 2012a; 2012b] rigorously confirm this fact.

Purely numerical methods [Zohdi and Wriggers 2008] are restricted to special distributions of inclusions. The properties of constituents are also given numerically. These lacks were overcame in [Mityushev 2006] (hereafter referred to as [M2006]) where a rigorous and constructive theory of the RVE for plane composites with identical circular inclusions was described. The RVE theory for identical disks in that paper is based on the representation of the effective conductivity tensor [Berlyand and Mityushev 2001; Mityushev 2001] in the form of a double series depending on the concentration of inclusions and on "basic elements" which depend only on the locations of the inclusions. These basic elements are written in terms of the Eisenstein series. Coefficients in the double series depend on the physical properties of constituents. Two composites were defined as equivalent if they have the same basic elements. Therefore, the set of the composites with circular identical inclusions was divided into equivalence classes determined only by the geometrical structure of the composite. In each equivalence class, a composite having the minimal size cell is chosen. Following [M2006], this cell is called the representative cell of equivalent composite materials. A constructive algorithm to determine the representative cell for any distribution of inclusions using only pure geometrical parameters was described in [M2006]. Examples presented therein yield fast transformations from cells to the representative cell of small size. This can be used in optimal computations of the macroscopic properties by applications of numerical and analytical methods.

In this paper we discuss the mixture problem, which can be outlined as follows. Take many balls, put them in a vessel and stir. Hence, we have the original and final locations of balls in the vessel. There are various methods of stirring [Kurtyka and Rylko 2013]. The mixture problem consists in determining the time (velocity, energy, etc.) necessary to reach the required mixture. Here, we arrive at the questions of how to measure the macroscopic thermal and mechanical properties of the mixture. It can be done by expensive experiments. The RVE theory yields an effective and simple method, developed in Section 4, to resolve the mixture problem. This method essentially extends the approach of the paper [Kurtyka and Rylko 2013] based on the M-sum.

In the present paper, we extend the RVE theory of [M2006] to 2D polydisperse composites and to 3D monodisperse composites with spherical inclusions. The obtained theoretical results are applied to the porous alumina material (α-Al_2O_3) discussed in [Nowak et al. 2013] and to the mixture problem frequently met in technological processes. It is shown how to construct the RVE size following the theory of [M2006]. Advantages of using this theory are the fast reconstruction method of the RVE and applications to the mixture problem. It is worth noting that the reconstruction of the RVE in [M2006] was based on equations where unknowns were centers of inclusions. However, it is not clear how to investigate polydisperse composites by use of such equations. Our method is based on investigation of the M-sums for various fragments of the large cell. Such a modification is reduced to investigation of one fixed cell instead of the whole class, which reduces computations.

The paper is organized as follows. The 2D RVE theory for disks is developed in Section 2. The 3D RVE theory is described in Section 3 for the nonoverlapping uniformly distributed identical spherical

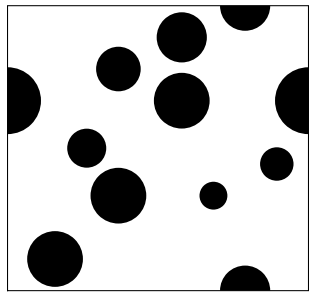

Figure 1. 2D RVE in the torus topology, a section of the 3D RVE parallel to the x_3-axis.

inclusions. Section 4 is devoted to applications of the RVE theory. Concluding remarks and relations to the RVE theory are presented in Section 5.

2. 2D RVE theory

Following [M2006], we extend the RVE theory to 2D composites with circular inclusions of different radii. Consider a 2D two-component composite made from a collection of nonoverlapping disks embedded in a matrix of a different nature (see Figure 1). It is assumed that the distribution $\mathcal{R}$ of the radii of the disks r_k is given and it does not depend on the locations of the disks. The centers of the disks satisfy a distribution $\mathcal{A}$ corresponding to a nonoverlapping-disks distribution on the plane. The distribution $\mathcal{A}$ formally does not depend on the distribution $\mathcal{R}$ of radii, but the choice of the distribution $\mathcal{A}$ is restricted by $\mathcal{R}$. This situation can be demonstrated by the following observation: for identical disks of radius r, the distance between any two centers must exceed $2r$.

It is assumed that the distribution $\mathcal{A}$ generates a random homogeneous field [Golden and Papanicolaou 1983; Jikov et al. 1994; Telega 2004] for which the macroscopic properties are correctly defined. One of the most important distributions $\mathcal{A}$ is the nonoverlapping uniform distribution $\mathcal{U}$, which corresponds to the perfect mixture of inclusions. The distribution $\mathcal{U}$ can be realized by the sequence location method or by random walks described in [Czapla et al. 2012a]. Other distributions are described in the book [Torquato 2002] in terms of the correlation functions. In the present paper, we do not discuss the question of the statistical generation of the theoretical distributions, and assume that realizations of $\mathcal{A}$ are given in the form of pairs $(\boldsymbol{a}_k, r_k)$, where $\boldsymbol{a}_k = (x_k, y_k)$ denotes the center and r_k the radius of the k-th inclusion. Further, it will be convenient to identify $\boldsymbol{a}_k$ with the complex number $a_k = x_k + iy_k$.

According to the homogenization theory [Golden and Papanicolaou 1983; Jikov et al. 1994; Telega 2004] there exists a periodicity cell with a finite number of inclusions representing the composite. First, we describe parameters of this cell. Consider a lattice $\mathfrak{L}$ on the complex plane $\mathbb{C}$ which is defined by two fundamental translation vectors ω_1 and ω_2. Without loss of generality we assume that $\omega_1 > 0$ and $\operatorname{Im}\omega_2 > 0$, where Im stands for the imaginary part. Introduce the zeroth cell

$$Q_0 := \left\{z \in \mathbb{C} : z = t_1\omega_1 + t_2\omega_2, -\tfrac{1}{2} < t_{1,2} < \tfrac{1}{2}\right\}.$$

The lattice $\mathfrak{L}$ is generated by the cells $Q_m := \{z \in \mathbb{C} : z - m_1\omega_1 - m_2\omega_2 \in Q_0\}$, where $m = m_1 + im_2$ is a complex number with m_1 and m_2 integers.

Let $\mathscr{C}_N$ denote the set of the elements (a_k, r_k), $k = 1, 2, \ldots, N$, where the radii r_k satisfy the distribution $\mathscr{R}$ and the centers a_k correspond to nonoverlapping uniformly distributed disks in the cell Q_0. Let $\mathscr{C}_{N+1} = \mathscr{C}_N \cup \{(a_{N+1}, r_{N+1})\}$. Introduce the set $\mathscr{C} = \bigcup_{N=N_0}^{\infty} \mathscr{C}_N$ with sufficiently large N_0. The number N_0 gives the size of the minimal representative set $\mathscr{C}_{N_0}$. The set $\mathscr{C}$ consists of all the configurations of mutually disjoint disks uniformly distributed on the plane whose radii satisfy the distribution $\mathscr{R}$. It is worth noting that $\mathscr{C}$ describes random locations of disks on the plane. In practical measurements, we observe finite fragments of $\mathscr{C}$. If these fragments represent the considered material, it is possible to statistically recover the distributions $\mathscr{R}$ and $\mathscr{A}$.

The radii distribution $\mathscr{R}$ can be easily constructed since it describes a 1D random variable. The 2D distribution $\mathscr{A}$ is theoretically described by correlation functions [Torquato 2002]. But we do not follow that reference, and consider $\mathscr{A}$ as a set of the given center coordinates a_k (measured and statistically presented). In particular, the 2D concentration of inclusions ϕ_2 can be measured. Theoretically, the 2D concentration ϕ_2 can be considered as the mean value

$$\phi_2 = \frac{1}{|Q|} \sum_{k=1}^{N} \pi r_k^2, \tag{2-1}$$

where $|Q|$ stands for the area of the domain Q.

According to the theory [M2006], we have to compare two different representative elements of $\mathscr{A}$. Consider a large fundamental region Q' constructed from the fundamental translation vectors ω_1' and ω_2'. Let Q' contain N' nonoverlapping circular disks D_k' of radius r_k' with centers $a_k' \in Q'$ $(k = 1, 2, \ldots, N')$ representing the distributions $\mathscr{R}$ and $\mathscr{A}$. Let $\widehat{\Lambda}'$ be the effective tensor of the composite represented by the region Q' with inclusions D_k'. Let the cell Q' correspond to another small cell Q which contains inclusions $D_k = \{z \in \mathbb{C} : |z - a_k| < r_k\}$ $(k = 1, \ldots, N)$ also representing the distributions $\mathscr{R}$ and $\mathscr{A}$, such that the effective tensor $\widehat{\Lambda}$ is close to $\widehat{\Lambda}'$. Closeness is defined by the concentration accuracy $O(\phi_2^{L+1})$ for the difference $\Delta\widehat{\Lambda} = \widehat{\Lambda} - \widehat{\Lambda}'$, with prescribed L. According to [M2006], the cell Q is a representative cell for the region Q' with the accuracy $O(\phi_2^{L+1})$ if

$$\Delta\widehat{\Lambda} = O(\phi_2^{L+1}). \tag{2-2}$$

Let a representative cell Q have the minimal possible area from all the representative cells equivalent to Q'. This cell is called the RVE. The existence of the RVE is evident since in the worst case one can take $Q = Q'$. The numerical statistical methods [Kanit et al. 2003] are also based on the relation (2-2). Since ϕ_2 is fixed in numerical computations, (2-2) becomes $\Delta\widehat{\Lambda} \approx 0$.

Instead of (2-2), Mityushev proposed to compare the basic elements of the expansion of the effective tensor. These basic elements are introduced as follows. Let a cell Q contain N inclusions with centers a_k. Let $E_m(z)$ denote the Eisenstein function of order m (see for instance Appendix A of [Czapla et al. 2012a]). Let $\mathbf{C}$ denote the operator of complex conjugation, which satisfies $\mathbf{C}^q z = z$ for even q and $\mathbf{C}^q z = \mathbf{C}z = \bar{z}$ for odd q.

The following sums, of multi-order $(m_1, \ldots, m_q)$, were introduced in [M2006]:

$$e_{m_1 \ldots m_q} := \left(\frac{\pi}{\phi_2}\right)^{1+\frac{1}{2}(m_1+\cdots+m_q)} \sum_{k_0, k_1, \ldots, k_q} r_{k_0}^2 r_{k_1}^{2t_1} \ldots r_{k_q}^{2t_q} E_{m_1}(a_{k_0} - a_{k_1}) \overline{E_{m_2}(a_{k_1} - a_{k_2})} \cdots \times \mathbf{C}^q E_{m_q}(a_{k_{q-1}} - a_{k_q}), \tag{2-3}$$

where $k_s = 1, 2, \dots, N$ for $0 \le s \le q$, $t_0 = 1$ and $t_s = m_s - t_{s-1}$. We call (2-3) the M-sum of order $(m_1, \dots, m_q)$.[1] For instance, the M-sum of order $(2, 2)$ has the form

$$e_{22} := \left(\frac{\pi}{\phi_2}\right)^3 \sum_{k_0,k_1,k_2} r_{k_0}^2 r_{k_1}^2 r_{k_2}^2 E_2(a_{k_0} - a_{k_1}) \overline{E_2(a_{k_1} - a_{k_2})}. \tag{2-4}$$

It is justified in [M2006] that the effective conductivity tensor for 2D composites can be presented in the form of a power series in the total concentration ϕ_2 with coefficients linearly depending on $e_{m_1 \dots m_q}$. An explicit iterative scheme for the coefficients of this series was explicitly presented in [Berlyand and Mityushev 2001; 2005; Mityushev 2001]. Plane elastic stress and strain fields are described by biharmonic functions [Grigolyuk and Filshtinsky 1991]. Any harmonic function is biharmonic. Therefore, harmonic functions describe not only conductivity problems but also special elastic stress and strain fields. This implies that the M-sums also describe (perhaps partially) macroscopic properties of elastic composites. We conjecture that this description is complete for the following reason. An iterative scheme analogous to [Berlyand and Mityushev 2001; 2005; Mityushev 2001] for conductivity problems can be extended to 2D elastic problems. Then, a formula for the effective elastic tensor could be deduced. Such a scheme was actually constructed in particular cases for regular structures [Grigolyuk and Filshtinsky 1991]. Preliminary results in [ibid.] and [Mityushev 2000] demonstrate that the effective elastic constants linearly depend on the M-sums and, let us say, M^*-sums having the same structure as (2-3), but with the quasi-elliptic functions $E_m^*(z)$ introduced in Appendix 2 of the book [Grigolyuk and Filshtinsky 1991] instead of the Eisenstein functions $E_m(z)$. Hence, the effective elastic tensor can be expressed through a linear combinations of the M- and M^*- sums (the form of this linear combination is not known, but it exists). The quasi-elliptic functions $E_m^*(z)$ are expressed via the Eisenstein functions by algebraic equations [ibid.]. Therefore, it is sufficient to consider only the M-sums (2-3) for elastic media.

Not all the M-sums participate in the effective tensor. For instance, the effective conductivity up to $O(\phi_2^5)$ contains eight M-sums: e_2, e_{22}, e_{33}, e_{222}, e_{44}, e_{322}, e_{223}, e_{2222}. For macroscopically isotropic composites, $e_2 = \pi$, and many other M-sums are dependent [Mityushev and Rylko 2012]. This reduces the number of basic elements to achieve the accuracy $O(\phi_2^4)$ to the following four M-sums:

$$e_{22},\ e_{33},\ e_{2222},\ e_{44}. \tag{2-5}$$

The M-sum $(2, 2)$ can be calculated by (2-4). Explicit forms of other M-sums (2-5) are given by the following formulae:

$$e_{33} = \left(\frac{\pi}{\phi_2}\right)^4 \sum_{k_0,k_1,k_2} r_{k_0}^2 r_{k_1}^4 r_{k_2}^2 E_3(a_{k_0} - a_{k_1}) \overline{E_3(a_{k_1} - a_{k_2})}, \tag{2-6}$$

$$e_{2222} = \left(\frac{\pi}{\phi_2}\right)^5 \sum_{k_0,k_1,k_2,k_3,k_4} r_{k_0}^2 r_{k_1}^2 r_{k_2}^2 r_{k_3}^2 r_{k_4}^2 E_2(a_{k_0} - a_{k_1}) \overline{E_2(a_{k_1} - a_{k_2})} E_2(a_{k_2} - a_{k_3}) \overline{E_2(a_{k_3} - a_{k_4})}, \tag{2-7}$$

$$e_{44} = \left(\frac{\pi}{\phi_2}\right)^5 \sum_{k_0,k_1,k_2} r_{k_0}^2 r_{k_1}^6 r_{k_2}^2 E_4(a_{k_0} - a_{k_1}) \overline{E_4(a_{k_1} - a_{k_2})}. \tag{2-8}$$

[1] M-sum is short for Mityushev's sum.

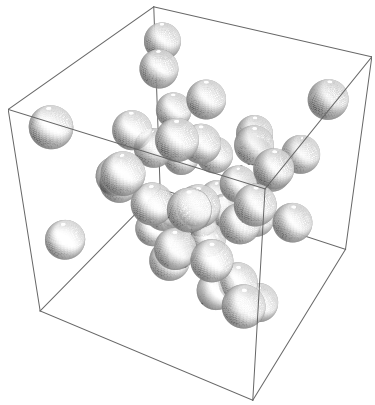

Figure 2. 3D RVE, which can be triply periodically continued to the whole space.

Remark. The M-sums (2-3), in particular (2-5), can be considered as the moments of the correlation functions [Torquato 2002]. Hence, the RVE theory [M2006] implicitly uses the correlation functions and does not require their explicit computations.

3. 3D RVE theory

The 2D M-sums (2-3) can be applied to the 3D RVE by use of the sections method. Consider a 3D cell P with fundamental translation vectors $\boldsymbol{\omega}_j$ $(j = 1, 2, 3)$. For simplicity, consider the unit cubic cell P, where $\boldsymbol{\omega}_1 = (1, 0, 0)$, $\boldsymbol{\omega}_2 = (0, 1, 0)$ and $\boldsymbol{\omega}_3 = (0, 0, 1)$. Suppose that sufficiently many identical balls (n, say) are uniformly located in P without overlapping (see Figure 2). This distribution corresponds to perfectly mixed balls of the same radius R in a uniform host with prescribed volume concentration

$$\phi = \frac{4\pi n}{3} R^3. \tag{3-1}$$

Consider a typical section of P parallel to the plane generated by $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ (see Figure 1). This section can be considered as a plane cell Q_0 with the complex coordinates introduced in Section 2. Let Q_0 contain N disks of radii r_k with centers located at a_k. The concentration of disks in Q_0 has the form

$$\phi_2 = \sum_{k=1}^{N} \pi r_k^2 \tag{3-2}$$

since Q_0 is the unit square. The uniform nonoverlapping distribution of the identical balls in P yields a uniform nonoverlapping distribution of disks in Q_0 with radii distributed in a special way described by a distribution denoted for definiteness by $\mathcal{R}_s$. Let r be the random value distributed in accordance with $\mathcal{R}_s$. It can be described by the uniformly distributed 1D random variable on the segment $(0, R)$. Then the mathematical expectation of the area of a disk is calculated by formula

$$\langle \pi r^2 \rangle = \frac{1}{R} \int_0^R \pi (R^2 - x^2)\, dx = \frac{2\pi}{3} R^2. \tag{3-3}$$

Let $\langle N \rangle$ denote the expected number of disks in the cell Q_0. Then (3-2) and (3-3) imply that

$$\frac{\phi_2}{\langle N\rangle}=\left\langle\frac{1}{N}\sum_{k=1}^{N}\pi r_k^2\right\rangle=\frac{2\pi}{3}R^2. \tag{3-4}$$

The volume concentration (3-1) is equal to the probability that a point from P belongs to one of the balls. The concentration (3-2) is equal to the probability that a point from Q_0 belongs to one of the disks. The balls are uniformly distributed in P. Moreover, Q_0 is a typical section of P, hence the 2D and 3D concentrations coincide:

$$\phi_2=\phi. \tag{3-5}$$

Then (3-4) and (3-1) yield

$$\langle N\rangle=2Rn. \tag{3-6}$$

This formula relates the expected number $\langle N\rangle$ of disks in the cell Q_0 with the given number n of balls per cell.

The expected values of (2-4), (2-6)–(2-8) are calculated by use of (3-3)–(3-4) as

$$\langle e_{m_1\dots m_q}\rangle=q_{m_1\dots m_q}, \tag{3-7}$$

where

$$q_{m_1\dots m_q}:=\frac{1}{\langle N\rangle^{1+\frac{1}{2}(m_1+\cdots+m_q)}}\sum_{k_0k_1\dots k_q}E_{m_1}(a_{k_0}-a_{k_1})\overline{E_{m_2}(a_{k_1}-a_{k_2})}\cdots\mathbf{C}^qE_{m_q}(a_{k_{q-1}}-a_{k_q}). \tag{3-8}$$

The M-sums (3-8) correspond to the distribution of the identical disks when the centers a_k obey the uniform nonoverlapping distribution $\mathcal{U}_s$ consistent with the special radii distribution $\mathcal{R}_s$. The relation (3-7) contains the mathematical expectation $\langle e_{m_1\dots m_q}\rangle$ over the radii distribution $\mathcal{R}_s$. We now write the mathematical expectation of (3-7) over the distribution $\mathcal{A}_s$ of centers in the form

$$\langle\!\langle e_{m_1\dots m_q}\rangle\!\rangle=\langle\!\langle q_{m_1\dots m_q}f\rangle\!\rangle. \tag{3-9}$$

It is difficult to calculate analytically the operator $\langle\!\langle\cdot\rangle\!\rangle$ because the distribution $\mathcal{A}_s$ has a complicated support domain for the centers of nonoverlapping disks. However, numerical statistical estimations of $\langle\!\langle q_{m_1\dots m_q}\rangle\!\rangle$ can be performed by algorithms developed in [Czapla et al. 2012a; 2012b]. Consider the M-sums (2-5) to estimate the RVE with the accuracy $O(\phi^4)$. It was proved in [Mityushev and Rylko 2012] that

$$q_{pp}=\frac{(-1)^p}{\langle N\rangle^{p+1}}\sum_{m=1}^{\langle N\rangle}\left|\sum_{k=1}^{\langle N\rangle}E_p(a_m-a_k)\right|^2,\quad p=2,3,4, \tag{3-10}$$

and

$$q_{2222}=\frac{1}{\langle N\rangle}\sum_{k=1}^{\langle N\rangle}\left|\frac{1}{\langle N\rangle^2}\sum_{k_1,k_2}E_2(a_k-a_{k_1})\overline{E_2(a_{k_1}-a_{k_2})}\right|^2. \tag{3-11}$$

This implies that $q^{(1)}=q_{22}$, $q^{(2)}=-q_{33}$, $q^{(3)}=q_{44}$ and $q^{(4)}=q_{2222}$ are nonnegative for any location of a_k. It was justified in [Berlyand and Mityushev 2005; Berlyand et al. 2013] that, in the framework of the shaking model, the $q^{(j)}$ $(j=1,2,3,4)$ decrease when order (regularity) of the center locations increases. It is conjectured that the decrease principle for ordered structures takes place in the general case. Consider the degenerate plane radii distribution $\mathcal{R}_R$ in which all the radii are equal to R. This distribution

yields the plane uniform nonoverlapping distribution $\mathfrak{U}_R$ of the identical disks. The probabilistic space of events described by $\mathfrak{U}_s$ is wider than that described by $\mathfrak{U}_R$. Hence, $\mathfrak{U}_R$ is "more regular" than $\mathfrak{U}_s$. This yields the inequality $q^{(j)}(\mathfrak{U}_R) \le q^{(j)}(\mathfrak{U}_s)$. For moderate concentrations, one can expect that

$$q^{(j)}(\mathfrak{U}_R) \approx q^{(j)}(\mathfrak{U}_s), \quad j = 1, 2, 3, 4. \tag{3-12}$$

4. Applications

In this section, various applications of the RVE theory are demonstrated. Section 4 is devoted to the size problem of the RVE. It is a traditional problem of the RVE which can be easily solved by simple and fast computations based on the M-sums. Sections 4–4 concern investigations of the structure of the special composites.

Example 1: size of the RVE. We consider a polydisperse random "shaking" composite [Berlyand and Mityushev 2001; 2005]. Let the periodic square array be perturbed by random deviations of the disks. Consider the model as displayed in Figure 3. The doubly periodic unit cell $Q_0 = (-0.5, 0.5) \times (-0.5, 0.5)$ contains 900 disks of radii between 0.0045 and 0.0135, displaced from their position in a regular array by the distance d, taken for each disk as a random variable uniformly distributed on the interval $(0, 0.006)$. Following the RVE theory [M2006], we have to restrict the infinite set of M-sums $\{e_2, e_{22}, e_{33}, e_{2222}, \ldots\}$. For definiteness, take the set $\{e_2, e_{22}\}$. Now, let the unit cell Q_0 be cut to the cells $Q_c = (-0.5c, 0.5c) \times (-0.5c, 0.5c)$, where $0 \le c \le 1$. New M-sums $\{e_2(c), e_{22}(c)\}$ with reduced numbers of disks $N(c)$ are calculated (after necessary normalization to the unit cell) and presented in Figures 4 and 5. Here, the parameter c is omitted and dependencies of the M-sums on the number of inclusions per cell are shown.

Theoretically, $e_2 = \pi \approx 3.14159$. Hence, the imaginary part $\frac{1}{\pi} \operatorname{Im} e_2$ can be considered as a measure of accuracy. For instance, if the precision 1% is chosen, it is sufficient to take 100 inclusions per cell. The presented computations were performed in about 40 seconds on a standard computer. The results of the computations are given in Table 1.

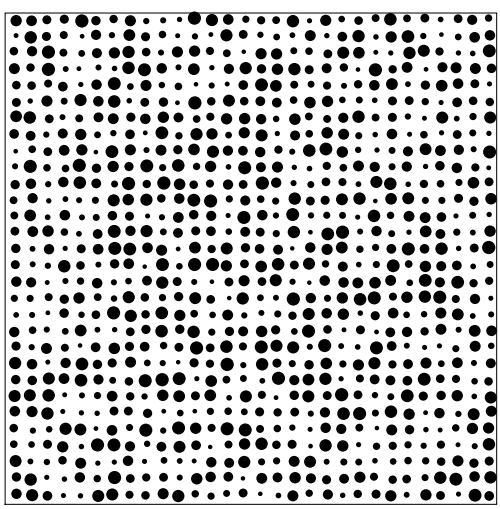

Figure 3. Shaking array of disks.

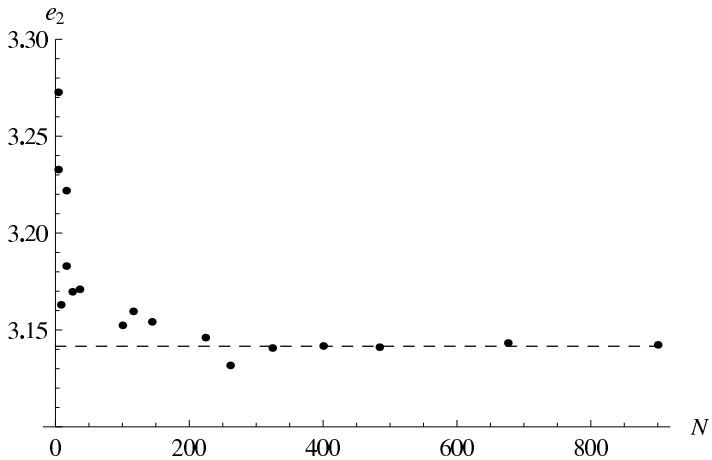

Figure 4. Plot of e_2 against N, the number of disks in the corresponding cut cell.

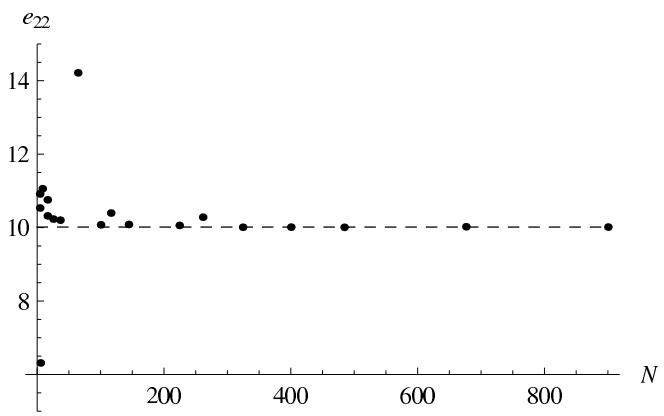

Figure 5. Plot of e_{22} against N.

It is worth noting that representation of the shaking geometries (and any other random geometry) by one inclusion per cell gives the worst possible approximation of the effective properties, since the effective conductivity of the periodic regular structure attains extremal values in locally disturbed composites with a fixed concentration [Berlyand and Mityushev 2001; 2005].

N	4	5	8	16	25	36	64	100
$\operatorname{Re} e_2$	3.27	1.59	3.16	3.61	3.17	3.171	3.40	3.15
$\operatorname{Im} e_2$	0.122	0.461	−0.00165	0.294	0.0172	0.01	0.061	−0.001
e_{22}	10.915	6.314	11.059	10.319	10.232	10.202	14.214	10.075
N	224	261	324	400	484	676	900	
$\operatorname{Re} e_2$	3.1461	3.1317	3.1407	3.1417	3.1411	3.1433	3.1423	
$\operatorname{Im} e_2$	0.0066	0.006	0.00264	−0.0013	0.000755	0.00204	0.00165	
e_{22}	10.0587	10.2828	10.0092	10.0123	10.0075	10.023	10.014	

Table 1. M-sums computed for the shaking geometry; N is the number of inclusions per cell.

concentration	0.05	0.1	0.2	0.45	0.7
q_{22}	54.462	**30.334**	18.195	11.525	10.056
q_{33}	−382.333	**−94.686**	−23.367	−3.506	−0.408
q_{44}	4190.05	**554.766**	77.573	7.988	1.094
q_{2222}	9643.33	**2135.82**	575.241	161.135	103.932

Table 2. Theoretical M-sums for the distribution $\mathcal{U}_R$.

q_2^*	$3.195 + 1.473i$
q_{22}^*	**39.1971**
q_{33}^*	**−282.112**
q_{44}^*	**2127.11**
q_{2222}^*	**3527.98**

Table 3. M-sums computed for data from [Kurtyka and Rylko 2013].

Example 2. The uniform distribution $\mathcal{U}_R$ has been studied extensively, in particular, the expectations $q^{(j)}$ were computed [Czapla et al. 2012a; 2012b]. The results are gathered in Table 2. The composite material F3K.10S reinforced with SiC particles with concentration 0.1 was discussed in [Kurtyka and Rylko 2013]. This composite was obtained by a process of thermomechanical deformation (FSP), which can be considered as a stir method. The results of measurement and computations are presented in Table 3.

First of all, we have to note that the theoretical value $q_2 = \pi \approx 3.14159$ [Czapla et al. 2012a] differs from the measured data $q_2^* = 3.195 + 1.473i$, which demonstrates anisotropy of the experimental sample. This means that SiC particles were not well stirred in the host. The coefficient of anisotropy introduced in [Mityushev 2001] can be used to measure anisotropy of composites. Let us compare now the theoretical $q^{(j)}$ and experimental results $q^{(j)*}$ for high-order M-sums with the fixed concentration 0.1 (bold numbers in the tables). One can see that the $|q^{(j)*}|$ exceed the corresponding values $|q^{(j)}|$. This follows from the initial compact location of the inclusions, because smaller distances $|a_k - a_m|$ between the centers yield larger values of the Eisenstein functions $|E_p(a_k - a_m)|$, hence larger values of the M-sums. Therefore, comparison of the values from Tables 2 and 3 demonstrates that SiC particles are not sufficiently well stirred during the FSP process. Besides the macroscopic anisotropy confirmed by $q_2^* = 3.195 + 1.473i$, other values $q^{(j)*}$ of the M-sums also confirm that the final mixture is not sufficiently well stirred.

Example 3. In the present section, the structure of the Al_2O_3 composite from [Nowak et al. 2013] (see Figure 1 in that paper) is investigated. The 2D concentration of inclusions is equal to 0.52 (3D porosity 0.84 in [ibid.] gives another volume characteristic of the considered composite).

The comparison of the results in Tables 2 and 4 demonstrates the inverse situation to Example 2. First, the value q_2^* is approximately equal to π. The data from Table 4 are similar to the data from the column of Table 2 corresponding to the concentration 0.45. This implies that the "gelcasting of foams" manufacturing method [Nowak et al. 2013] yields isotropic and homogeneous structures similar to the theoretical uniform nonoverlapping distribution.

q_2^*	$3.126 + 0.06118i$
q_{22}^*	11.7983
q_{33}^*	−5.0835
q_{44}^*	9.61203
q_{2222}^*	163.672

Table 4. M-sums computed for digitized data obtained from Figure 1 of [Nowak et al. 2013].

5. Conclusion

In the present paper, we extend the RVE theory of [M2006] to 2D polydisperse composites and to 3D monodisperse composites with spherical inclusions and apply it to the mixture problem. The considered example refers to Al_2O_3 composites obtained by gelcasting of foams and to F3K.10S samples reinforced with SiC particles obtained by FSP.

The developed method gives a simple procedure to construct a typical RVE whose existence is predicted by the homogenization theory. Further, the effective constants can be computed by numerical methods.

The presented methodology can be applied to arbitrary dispersed composites. In the present paper, we consider balls uniformly distributed in host material. This follows from the restriction that we have at our disposal only those theoretical M-sums computed in [Czapla et al. 2012a; 2012b]. Other distributions can be investigated by simulations described in these papers.

Acknowledgement

I am grateful to Professor Barbara Gambin (Warsaw) for useful references.

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Received 17 Jan 2014. Revised 17 May 2014. Accepted 13 Jun 2014.

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JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

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JoMMS (ISSN 1559-3959) at Mathematical Sciences Publishers, 798 Evans Hall #6840, c/o University of California, Berkeley, CA 94720-3840, is published in 10 issues a year. The subscription price for 2014 is US $555/year for the electronic version, and $710/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to MSP.

JoMMS peer-review and production is managed by EditFLOW® from Mathematical Sciences Publishers.

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Journal of Mechanics of Materials and Structures

Volume 9, No. 4 July 2014