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INTERACTION BETWEEN A CIRCULAR INCLUSION AND A CIRCULAR VOID UNDER PLANE STRAIN CONDITIONS

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The interaction force between a circular inclusion characterized by uniform eigenstrain and a nearby circular void is determined by evaluating the J-integral around the void. The Kienzler–Zhuping formula was used to determine the hoop stress along the boundary of the void in terms of the infinite-medium solution to the inclusion problem. Specific results are given for the inclusion with dilatational eigenstrain. The M-integrals around the void and inclusion are then evaluated, the former being proportional to the energy release rates associated with a self-similar expansion of the void. The energy rate associated with an isotropic expansion of the inclusion differs from the M-integral around the inclusion. The relationship between the two is derived. It is shown that the greater the distance from the void, the greater the energy associated with the presence of the inclusion and the greater the energy rate associated with its growth, which suggests that the presence of nearby free surfaces facilitates the eigenstrain transformations. The attraction exerted on a circular inclusion with a uniform shear eigenstrain by the free surface of a half-space is also evaluated. Peculiar variation of this configurational force with the distance between the inclusion and the free surface is noted and discussed.

1. Introduction

Kienzler and Zhuping [1987] derived an appealing formula for the hoop stress along the boundary of a circular void of radius a in an infinite medium under plane stress or plane strain conditions, due to remote loading or an internal source of stress, which reads

$$\sigma_\theta(a,\theta) = 2[\sigma_\theta^0(a,\theta) - \sigma_r^0(a,\theta)] + \tfrac{1}{2}\,[\sigma_\theta^0(0,\theta) + \sigma_r^0(0,\theta)]. \tag{1}$$

The stress field in an infinite medium without a void due to the same source of stress is denoted by the superscript ^{0}, and (r,θ) are the polar coordinates with the origin at the center of the void. This formula received relatively little attention in the literature, although it did stimulate research efforts that led to the formulation of the so-called heterogenization procedure [Honein and Herrmann 1988; 1990], according to which the solution to the problem of two or more inhomogeneities under remote or other types of loading is expressed in terms of the solution to the corresponding homogeneous problem. An important part of the latter analysis is that the elastic field produced by the prescribed tractions or displacements over the boundary of a circular hole in an infinite medium can be expressed in terms of the elastic field produced by the same quantities acting on the boundary of a circular disk, and vice versa. We use the Kienzler–Zhuping formula in this paper to evaluate the J-integral along the boundary of a circular void, or

Keywords: configurational force, conservation integrals, dilatation, eigenstrain, half-space, inclusion, plane strain, shear, void.

along the straight edge of a half-space, without using or deriving the solution to the entire boundary value problem at hand. Based on this, we determine the configurational force exerted by the free surface on a nearby source of internal stress. The latter is taken to be a circular inclusion with a uniform dilatational eigenstrain. The M-integrals around the void and the inclusion are evaluated. The M_O-integral around the void is proportional to the energy release rate associated with a self-similar expansion of the void, if the surface of the void is traction-free. The energy rate associated with an isotropic expansion of the inclusion differs from the ratio M_C/b, where the M_C-integral is evaluated around the boundary of the inclusion of radius b. The relationship between the two quantities is derived. The attraction exerted on a circular inclusion with a uniform shear eigenstrain by the free surface of a half-space is also evaluated. Peculiar variation of this configurational force with the distance between the inclusion and the free surface is noted and discussed.

The determination of physical quantities, such as the dislocation interaction force or the stress-intensity factor in various dislocation and fracture mechanics problems, without solving the corresponding boundary value problems, was earlier considered by Eshelby [1975], Freund [1978], Rice [1985], Kienzler and Kordisch [1990], and Lubarda and Markenscoff [2007], among others. More recently, Lubarda [2015] employed the antiplane-strain version of the Kienzler–Zhuping formula, first recognized by Lin et al. [1990], to determine the configurational force between a circular void and inclusion characterized by uniform eigenshear. The study of inclusions and interactions among them plays an important role in materials science problems concerned with displacive transformations and transformation toughening mechanisms in crystalline metallic materials and ceramics. A comprehensive review of recent works on inclusions can be found in [Zhou et al. 2013].

2. Circular inclusion near a circular void

Suppose that a circular cylinder of radius b is taken out from an infinite medium and given a uniform plane eigenstrain $(\epsilon^\bullet_x, \epsilon^\bullet_y, \epsilon^\bullet_{xy})$. When the cylinder is inserted back into the medium, with their interface perfectly bonded, the stress fields expressed in polar coordinates (ρ, φ) with respect to the center C of the inclusion (Figure 1), are

$$\begin{aligned}\sigma_\rho^{0,\mathrm{in}} &= -2k\big[(\epsilon^\bullet_\rho + \epsilon^\bullet_\varphi) + \tfrac{1}{2}(\epsilon^\bullet_\rho - \epsilon^\bullet_\varphi)\big],\\ \sigma_\varphi^{0,\mathrm{in}} &= -2k\big[(\epsilon^\bullet_\rho + \epsilon^\bullet_\varphi) - \tfrac{1}{2}(\epsilon^\bullet_\rho - \epsilon^\bullet_\varphi)\big],\\ \sigma_{\rho\varphi}^{0,\mathrm{in}} &= -2k\epsilon^\bullet_{\rho\varphi},\end{aligned} \tag{2}$$

where $\epsilon^\bullet_\rho + \epsilon^\bullet_\varphi = \epsilon^\bullet_x + \epsilon^\bullet_y$, $k = \mu/[4(1-\nu)]$, and

$$\begin{aligned}\epsilon^\bullet_\rho - \epsilon^\bullet_\varphi &= (\epsilon^\bullet_x - \epsilon^\bullet_y)\cos 2\varphi + 2\epsilon^\bullet_{xy}\sin 2\varphi,\\ 2\epsilon^\bullet_{\rho\varphi} &= -(\epsilon^\bullet_x - \epsilon^\bullet_y)\sin 2\varphi + 2\epsilon^\bullet_{xy}\cos 2\varphi.\end{aligned} \tag{3}$$

The expressions (2) are obtained from the results for circular inclusions presented in [Lubarda 1998] and [Lubarda and Markenscoff 1999], which were derived by using the Papkovich–Neuber potentials. They can also be obtained from the general ellipsoidal inclusion analysis in [Eshelby 1957; 1959]. The

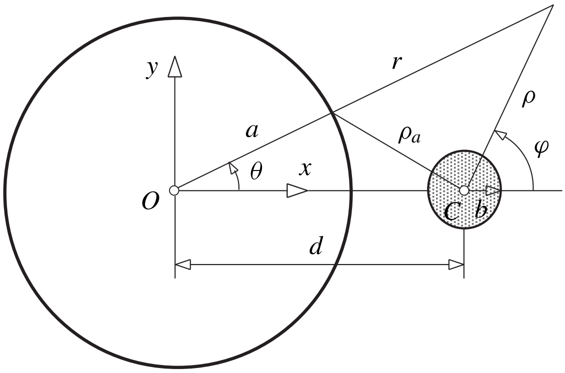


Figure 1. A circular inclusion of radius b in an infinite medium near the void of radius a. The eigenstrain of the inclusion is $(\epsilon_x^\bullet, \epsilon_y^\bullet, \epsilon_{xy}^\bullet)$. The centers of the inclusion and void are separated by the distance d. The distance from the center C of the inclusion to an arbitrary point on the boundary of the void is ρ_a. The polar coordinates (ρ, φ) are with respect to the point C, and (r, θ) with respect to the point O.

in-plane stress components outside of the inclusion are

$$\begin{aligned}\sigma_\rho^{0,\mathrm{out}} &= -2k\frac{b^2}{\rho^2}\left[(\epsilon_\rho^\bullet+\epsilon_\varphi^\bullet)+\frac{1}{2}\left(4-3\frac{b^2}{\rho^2}\right)(\epsilon_\rho^\bullet-\epsilon_\varphi^\bullet)\right],\\ \sigma_\varphi^{0,\mathrm{out}} &= 2k\frac{b^2}{\rho^2}\left[(\epsilon_\rho^\bullet+\epsilon_\varphi^\bullet)-\frac{3}{2}\frac{b^2}{\rho^2}(\epsilon_\rho^\bullet-\epsilon_\varphi^\bullet)\right],\\ \sigma_{\rho\varphi}^{0,\mathrm{out}} &= 2k\frac{b^2}{\rho^2}\left(2-3\frac{b^2}{\rho^2}\right)\epsilon_{\rho\varphi}^\bullet.\end{aligned} \tag{4}$$

When rewritten in terms of the polar coordinates (r, θ) with respect to the point O, the outside stress field (omitting the superscript "out") is

$$\begin{aligned}\sigma_r^0+\sigma_\theta^0 &= \sigma_\rho^0+\sigma_\varphi^0,\\ \sigma_r^0-\sigma_\theta^0 &= (\sigma_\rho^0-\sigma_\varphi^0)\cos 2(\varphi-\theta)-2\sigma_{\rho\varphi}^0\sin 2(\varphi-\theta),\\ 2\sigma_{r\theta}^0 &= (\sigma_\rho^0-\sigma_\varphi^0)\sin 2(\varphi-\theta)+2\sigma_{\rho\varphi}^0\cos 2(\varphi-\theta).\end{aligned} \tag{5}$$

In these expressions,

$$\begin{gathered}\sin\varphi=\frac{r\sin\theta}{\rho},\quad \cos\varphi=\frac{r\cos\theta-d}{\rho},\\ \sin(\varphi-\theta)=\frac{d\sin\theta}{\rho},\quad \cos(\varphi-\theta)=\frac{r-d\cos\theta}{\rho},\\ \rho^2=r^2+d^2-2rd\cos\theta.\end{gathered} \tag{6}$$

The total strain energy in the entire medium (inside and outside of the inclusion, per unit thickness) can be determined from the equation [Lubarda and Markenscoff 1999]

$$E_T^0=-\frac{1}{2}\int_{A^{\mathrm{in}}}\sigma_{ij}^{0,\mathrm{in}}\epsilon_{ij}^\bullet\,\mathrm{d}A=-\tfrac{1}{2}\big(\sigma_\rho^{0,\mathrm{in}}\epsilon_\rho^\bullet+\sigma_\varphi^{0,\mathrm{in}}\epsilon_\varphi^\bullet+2\sigma_{\rho\varphi}^{0,\mathrm{in}}\epsilon_{\rho\varphi}^\bullet\big)b^2\pi, \tag{7}$$

which gives

$$E_T^0 = \pi k b^2 \big\{ (\epsilon_x^\bullet + \epsilon_y^\bullet)^2 + \tfrac{1}{2}\big[(\epsilon_x^\bullet - \epsilon_y^\bullet)^2 + 4\epsilon_{xy}^{\bullet 2} \big] \big\}. \tag{8}$$

The rate of energy change associated with an infinitesimal increase of the inclusion radius δb is

$$\frac{\delta E_T^0}{\delta b} = 2\pi k b \big\{ (\epsilon_x^\bullet + \epsilon_y^\bullet)^2 + \tfrac{1}{2}\big[(\epsilon_x^\bullet - \epsilon_y^\bullet)^2 + 4\epsilon_{xy}^{\bullet 2} \big] \big\}. \tag{9}$$

This can also be deduced from the expression for the specific configurational force (f^0) (per unit length of the circumference of the inclusion, at each point orthogonal to the circumference), defined such that [Gavazza 1977]

$$\delta E_T^0 = -(2\pi b) f^0 \delta b, \qquad f^0 = \tfrac{1}{2}(\sigma_{ij}^{0,\mathrm{in}} + \sigma_{ij}^{0,\mathrm{out}})_{\rho=b} \epsilon_{ij}^\bullet. \tag{10}$$

This formula has recently been extended to problems of dynamically expanding inclusions in a comprehensive sequence of papers by Markenscoff and Ni [2010; 2011].

2.1. ***Dilatational eigenstrain.*** If the inclusion is given a purely dilatational eigenstrain $\epsilon_x^\bullet = \epsilon_y^\bullet = \epsilon^\bullet$ and $\epsilon_{xy}^\bullet = 0$, the outside stress field in polar coordinates (ρ, φ), relative to the center of the inclusion, is

$$\sigma_\varphi^{0,\mathrm{out}} = -\sigma_\rho^{0,\mathrm{out}} = 4k\epsilon^\bullet \frac{b^2}{\rho^2}, \qquad \sigma_{\rho\varphi}^{0,\mathrm{out}} = 0. \tag{11}$$

When this is substituted into (5), the outside stress field in polar coordinates (r, θ), relative to the center of the void, becomes

$$\begin{aligned} \sigma_\theta^0(r,\theta) &= -\sigma_r^0(r,\theta) = 4k\epsilon^\bullet \frac{b^2}{\rho^2}\left(1 - \frac{2d^2\sin^2\theta}{\rho^2}\right), \\ \sigma_{r\theta}^0(r,\theta) &= -8k\epsilon^\bullet \frac{b^2 d}{\rho^3}(r - d\cos\theta)\sin\theta. \end{aligned} \tag{12}$$

The outside stress field is deviatoric ($\sigma_r^0 + \sigma_\theta^0 = 0$). Thus, from the Kienzler–Zhuping formula (1), it follows that along the boundary of the void $r = a$, the hoop stress due to inserted inclusion is

$$\sigma_\theta(a,\theta) = 2[\sigma_\theta^0(a,\theta) - \sigma_r^0(a,\theta)] = 4\sigma_\theta^0(a,\theta); \tag{13}$$

that is, in view of (12),

$$\sigma_\theta(a,\theta) = \frac{4\mu\epsilon^\bullet}{1-\nu}\frac{b^2}{\rho_a^2}\left(1 - \frac{2d^2\sin^2\theta}{\rho_a^2}\right), \qquad \rho_a^2 = a^2 + d^2 - 2ad\cos\theta. \tag{14}$$

The total strain energy (per unit thickness) due to inserted inclusion in an infinite medium, and its rate, are

$$E_T^0 = \frac{\pi\mu b^2 \epsilon^{\bullet 2}}{1-\nu}, \qquad \frac{\delta E_T^0}{\delta b} = \frac{2\pi\mu b \epsilon^{\bullet 2}}{1-\nu}, \tag{15}$$

which follows from (8) and (9). Thus, from (10), the configurational force per unit length of the circumference of the inclusion is $f^0 = -p^0\epsilon^\bullet$, and $p^0 = \mu\epsilon^\bullet/(1-\nu)$ is the interface pressure.

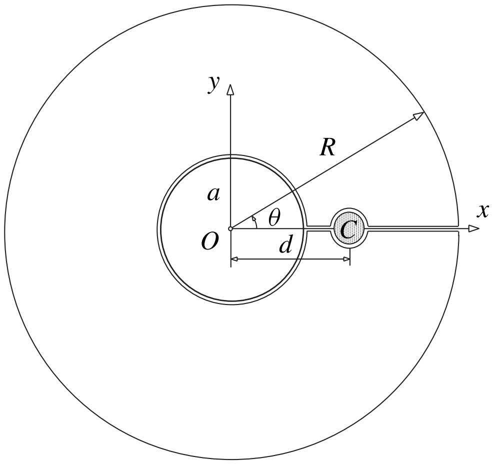


Figure 2. The closed contour around the void of radius a and its center at O, and the inclusion of radius b and its center at C, used to evaluate the J- and M-integrals. The radius R of the remote circle tends to infinity.

3. Configurational force between the inclusion and void

The J_β-integral for the plane strain elasticity problems can be expressed in terms of the Eshelby's energy momentum tensor $P_{\alpha\beta}$ [Eshelby 1956; 1957] as

$$J_\beta = \oint P_{\alpha\beta} n_\alpha \, \mathrm{d}l, \qquad P_{\alpha\beta} = W\delta_{\alpha\beta} - \sigma_{\alpha\gamma} u_{\gamma,\beta}, \qquad (\alpha, \beta = x, y). \tag{16}$$

In particular,

$$J_x = \oint (W n_x - t_x u_{x,x} - t_y u_{y,x}) \, \mathrm{d}l, \tag{17}$$

where $t_x = \sigma_x n_x + \sigma_{xy} n_y$ and $t_y = \sigma_{yx} n_x + \sigma_y n_y$ are the traction components over the contour whose outward normal has the components (n_x, n_y). An analysis based on dual conservation integrals, originally introduced by Bui [1973; 1974], could also be used, but is not pursued in this paper; see [Lubarda and Markenscoff 2007] and [Lubarda 2012] for its application to other problems. When evaluated over a closed contour which does not embrace a singularity or a defect, the integral in (17) vanishes. Such a contour, going around the void and the inclusion, along the positive x-axis, and around a remote circle of large radius $R \gg (a, b, d)$, is shown in Figure 2. The contributions to J_x along the lines just above and below the x-axis cancel each other, and the contribution from the remote circle vanishes because the stresses fall off as $1/R^2$ in the limit $R \to \infty$, as in an unvoided infinite medium, because for large R the stress field becomes increasingly unaware of the presence of the void near the inclusion. Thus, $J_x = J_x^{\text{void}} + J_x^{\text{incl}} = 0$; i.e.,

$$J_x^{\text{incl}} = -J_x^{\text{void}}. \tag{18}$$

The J_x-integral along the boundary of the void ($r = a$, $n_x = \cos\theta$, $n_y = \sin\theta$) is

$$J_x^{\text{void}} = a \int_0^{2\pi} W(a, \theta) \cos\theta \, \mathrm{d}\theta = \frac{1-\nu}{4\mu} a \int_0^{2\pi} \sigma_\theta^2(a, \theta) \cos\theta \, \mathrm{d}\theta, \tag{19}$$

because $\sigma_x n_x + \sigma_{xy} n_y = 0$, $\sigma_{yx} n_x + \sigma_y n_y = 0$, and $W = \sigma_\theta \epsilon_\theta / 2 = (1-\nu)\sigma_\theta^2/(4\mu)$ along the traction-free boundary of the void. By incorporating the stress expression (14) into (19), it follows that

$$J_x^{\text{void}} = \frac{8\mu a \epsilon^{\bullet 2}}{1-\nu}\left(\frac{b}{a}\right)^4 \left[I_1 - 4\left(\frac{d}{a}\right)^2 I_2 + 4\left(\frac{d}{a}\right)^4 I_3 \right], \tag{20}$$

where

$$\begin{aligned} I_1 &= \int_0^\pi \frac{\cos\theta \, \mathrm{d}\theta}{\rho_0^4} = \frac{2\pi(d/a)}{[(d/a)^2-1]^3}, \\ I_2 &= \int_0^\pi \frac{\sin^2\theta \cos\theta \, \mathrm{d}\theta}{\rho_0^6} = \frac{\pi[3(d/a)^2-1]}{4(d/a)^3[(d/a)^2-1]^3}, \\ I_3 &= \int_0^\pi \frac{\sin^4\theta \cos\theta \, \mathrm{d}\theta}{\rho_0^8} = \frac{\pi[2(d/a)^2-1]}{4(d/a)^5[(d/a)^2-1]^3}, \end{aligned} \tag{21}$$

and $\rho_0^2 = 1 - 2(d/a)\cos\theta + (d/a)^2$. These integral expressions were derived by using the general formula [Gradshteyn and Ryzhik 1965]

$$\int_0^\pi \frac{\cos n\theta \, \mathrm{d}\theta}{(1-2\alpha\cos\theta+\alpha^2)^m} = \frac{\pi}{\alpha^n(\alpha^2-1)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k} \binom{2m-k-2}{m-1} (\alpha^2-1)^k, \tag{22}$$

valid for $\alpha^2 > 1$, in conjunction with the appropriate integration by parts. The trigonometric identity $8\sin^4\theta = 3 - 4\cos 2\theta + \cos 4\theta$ was also conveniently utilized. The expressions were also verified by the Matlab evaluation of integrals. Consequently, by substituting (21) into (20), the J-integral around the void becomes

$$J_x^{\text{void}} = \bar{J}_x \left(\frac{b}{a}\right)^4 \frac{d/a}{[(d/a)^2-1]^3}, \qquad \bar{J}_x = \frac{8\pi\mu a \epsilon^{\bullet 2}}{1-\nu}. \tag{23}$$

This represents the material or configurational force exerted on the void by a nearby inclusion (per unit length in the z-direction). It represents the energy release rate associated with an imagined void translation within the material toward the inclusion (by diffusion or otherwise), keeping the position of the inclusion fixed. The opposite force of the same magnitude is exerted on the inclusion by the surface of the void. The maximum value of the force is reached at the minimum distance between the centers of the inclusion and the void for which the presented analysis applies ($d_{\min} = a + b$), and is equal to

$$J_{x,\max}^{\text{void}} = \bar{J}_x \frac{(b/a)(1+b/a)}{(2+b/a)^3}. \tag{24}$$

The variation of J_x^{void} (normalized by $\bar{J}_x$) versus d/a for several values of b/a is shown in Figure 3.

In the limit of infinitely large radius of the void, the inclusion behaves as if it is near the free surface of a half-space. If the distance from the center of the inclusion to that surface is denoted by c, the force on the inclusion is found to be

$$J_x^{\text{incl}} = -\frac{\pi\mu\epsilon^{\bullet 2}}{1-\nu}\frac{b^4}{c^3}. \tag{25}$$

The force on an ellipsoidal inclusion with uniform dilatational eigenstrain near the free surface of a half-space was evaluated and discussed by Mura [1987]. More recently, Zhou et al. [2013] provided

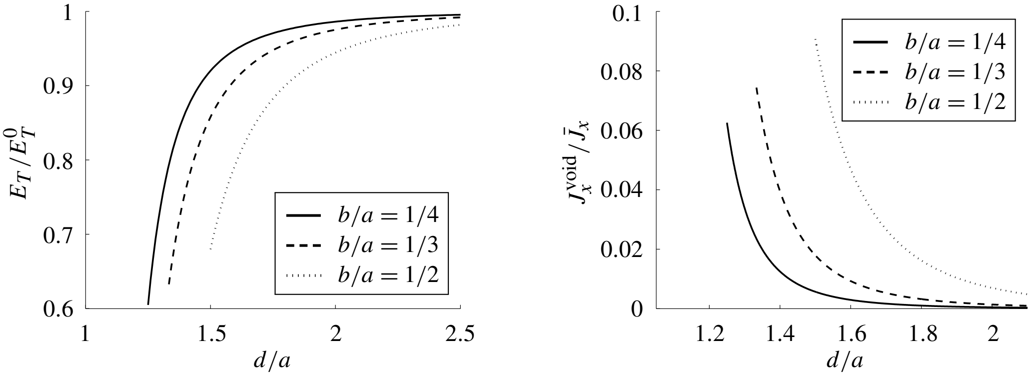


Figure 3. Left: the variation of the strain energy E_T/E_T^0 with d/a, according to (29), for the indicated values of the ratio b/a. Right: the variation of the configurational forces $J_x^{\rm void}$ with d/a, according to (23), for the same values of the ratio b/a. The scaling force parameter is $\bar{J}_x = 8\pi\mu a\epsilon^{\bullet 2}/(1-\nu)$. The left endpoints of the curves (at $d/a = 1 + b/a$, when the inclusion is tangent to the void) specify the minimum strain energy and maximum configurational force corresponding to a selected value of the ratio b/a.

a comprehensive survey of recent works on inclusions and inhomogeneities in an infinite space and a half-space, addressing the problems of a single inclusion, two inclusions, and multiple inclusions, as well as dislocations and cracks.

4. Total strain energy

The total strain energy in the medium with inserted inclusion near a traction-free void is the sum of the strain energy term corresponding to the inclusion inserted in an infinite medium far away from the void, thus given by the first expression in (15), and the term dependent on d which accounts for the interaction between the void and inclusion. The latter includes the strain energy of the auxiliary (image) field and the cross term due to energy of the auxiliary $\hat{\sigma}_{ij}$ field on the infinite-medium strain field, as well as the energy of the infinite-medium stress field on the auxiliary $\hat{\epsilon}_{ij}$ strain field. Thus,

$$E_T = \frac{\pi\mu b^2\epsilon^{\bullet 2}}{1-\nu} + \hat{E}_T(d, b). \tag{26}$$

The energy $\hat{E}_T$ can be conveniently determined by noting that its negative gradient with respect to the distance between the inclusion and the void must be equal to the configurational force on the void, as given by the expression (23):

$$-\frac{\partial\hat{E}_T}{\partial d} = \frac{8\pi\mu\epsilon^{\bullet 2}}{1-\nu}\,\frac{adb^4}{(d^2-a^2)^3}. \tag{27}$$

Upon integration, this gives

$$\hat{E}_T = -\frac{2\pi\mu\epsilon^{\bullet 2}}{1-\nu}\,\frac{a^2b^4}{(d^2-a^2)^2}, \tag{28}$$

up to an immaterial constant term. Consequently, by substituting (28) into (26), the total strain energy becomes

$$E_T = E_T^0\left[1 - \frac{2a^2b^2}{(d^2-a^2)^2}\right], \quad E_T^0 = \frac{\pi\mu b^2\epsilon^{\bullet 2}}{1-\nu}. \tag{29}$$

Therefore, the introduction of the void in an infinite medium with the inserted inclusion decreases the strain energy, the decrease being greater for larger voids that are closer to the inclusion. This is sketched in Figure 3 (left image).

For later use, we also evaluate from (29) the following quantities:

$$-a\frac{\partial E_T}{\partial a} = \frac{4\pi\mu a^2\epsilon^{\bullet 2}}{1-\nu}\left(\frac{b}{a}\right)^4 \frac{(d/a)^2+1}{[(d/a)^2-1]^3}, \tag{30}$$

$$-b\frac{\partial E_T}{\partial b} = \frac{2\pi\mu a^2\epsilon^{\bullet 2}}{1-\nu}\left[-\left(\frac{b}{a}\right)^2 + \left(\frac{b}{a}\right)^4 \frac{4}{[(d/a)^2-1]^2}\right], \tag{31}$$

which will be discussed in relation to M-integrals in the next two sections.

5. M-integral around the void

The M-integral of the plane strain elasticity with respect to the coordinate origin at the point O can be expressed as [Knowles and Sternberg 1972; Budiansky and Rice 1973]

$$M_O = \oint P_{\alpha\beta}x_\beta n_\alpha \,\mathrm{d}l = \oint \left[x(P_{xx}n_x + P_{yx}n_y) + y(P_{xy}n_x + P_{yy}n_y)\right]\mathrm{d}l; \tag{32}$$

that is,

$$M_O = \oint \left[W(xn_x + yn_y) - t_x(xu_{x,x} + yu_{x,y}) - t_y(xu_{y,x} + yu_{y,y})\right]\mathrm{d}l. \tag{33}$$

By using the same closed contour from Figure 2, as in the evaluation of the J_x-integral, the contribution from the remote circle M_O^R tends to 0 as R tends to ∞, because the stresses decay as $1/R^2$ far away from the inclusion. Thus, $M_O^{\text{void}} + M_O^{\text{incl}} = 0$, where

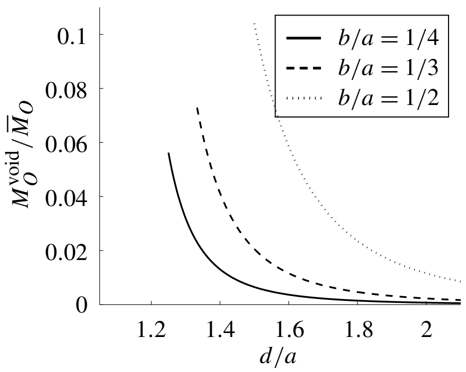
$$M_O^{\text{void}} = a^2\int_0^{2\pi} W(a,\theta)\,\mathrm{d}\theta = \frac{(1-\nu)a^2}{4\mu}\int_0^{2\pi}\sigma_\theta^2(a,\theta)\,\mathrm{d}\theta. \tag{34}$$

Upon using the circumferential stress expression (14), the integration in (34) gives

$$M_O^{\text{void}} = \frac{8\mu a^2\epsilon^{\bullet 2}}{1-\nu}\left(\frac{b}{a}\right)^4\left[K_1 - 4\left(\frac{d}{a}\right)^2 K_2 + 4\left(\frac{d}{a}\right)^4 K_3\right], \tag{35}$$

where

$$K_1 = \int_0^\pi \frac{\mathrm{d}\theta}{\rho_0^4} = \frac{\pi[(d/a)^2+1]}{[(d/a)^2-1]^3},$$
$$K_2 = \int_0^\pi \frac{\sin^2\theta\,\mathrm{d}\theta}{\rho_0^6} = \frac{\pi}{2[(d/a)^2-1]^3}, \tag{36}$$
$$K_3 = \int_0^\pi \frac{\sin^4\theta\,\mathrm{d}\theta}{\rho_0^8} = \frac{\pi[3(d/a)^2-1]}{8(d/a)^4[(d/a)^2-1]^3}.$$



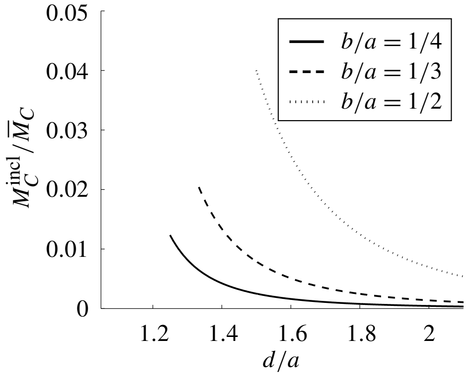


Figure 4. The variation of M_O^{void} (left) and M_C^{void} (right) with d/a for the indicated values of the ratio b/a. The scaling factor for both plots is $\overline{M}_O = \overline{M}_C$, as defined in (37).

The substitution of (36) into (35) yields

$$M_O^{\text{void}} = \overline{M}_O \left(\frac{b}{a}\right)^4 \frac{(d/a)^2 + 1}{[(d/a)^2 - 1]^3}, \qquad \overline{M}_0 = \frac{1}{2}\,\overline{J}_x a = \frac{4\pi\mu a^2 \epsilon^{\bullet 2}}{1-\nu}. \tag{37}$$

Physically, the ratio M_O^{void}/a represents the energy release rate associated with isotropic void growth (by material absorption over the surface of the void), keeping the position of the inclusion fixed relative to the center of the void. Indeed, the comparison of (37) with (30) shows that $M_O^{\text{void}} = -a(\partial E_T/\partial a)$.

6. M integral around the inclusion

By using the well-known relationship between the M-integrals relative to the coordinate origins at O and C [Rice 1985],

$$M_O^{\text{incl}} = M_C^{\text{incl}} + d \cdot J_x^{\text{incl}}, \tag{38}$$

and in view of the relationship $M_O^{\text{void}} + M_O^{\text{incl}} = 0$, it follows that

$$M_C^{\text{incl}} = -(M_O^{\text{void}} + d \cdot J_x^{\text{incl}}). \tag{39}$$

Consequently, by substituting into (39) the expression for M_O^{void} from (37) and the expression for $J_x^{\text{incl}} = -J_x^{\text{void}}$ from (23), the M_C integral around the inclusion is found to be

$$M_C^{\text{incl}} = \overline{M}_C \left(\frac{b}{a}\right)^4 \frac{1}{[(d/a)^2 - 1]^2}, \qquad \overline{M}_C = \overline{M}_O. \tag{40}$$

If $d \gg a$, then M_C^{incl} approaches zero, as if the inclusion was in an infinite medium without the void ($M_C^{\text{incl},\infty} = 0$). The variation of M_O^{void} and M_C^{incl} with d/a for several values of the ratio b/a is shown in Figure 4. In the limit of infinitely large radius of the void, the M-integral around the inclusion is

$$M_C^{\text{incl}} = \frac{\pi\mu\epsilon^{\bullet 2}}{1-\nu}\,\frac{b^4}{c^2}, \tag{41}$$

in duality with (25) through the relation $M_C^{\text{incl}} = -cJ_x^{\text{incl}}$.

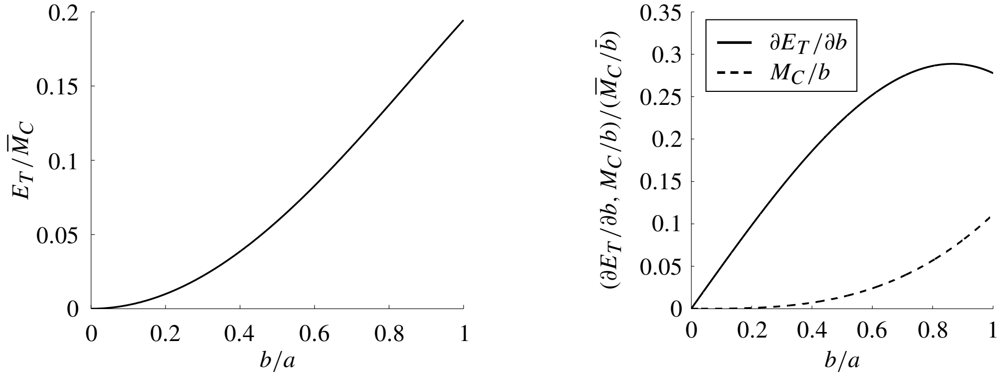


Figure 5. Left: the variation of the strain energy E_T, given by (29) and normalized by $\overline{M}_C$, with b/a in the case $d = 2a$. Right: the corresponding variation of $\delta E_T/\delta b$ and M_C^{incl}/b, given by (42) and normalized by $\overline{M}_C/\bar{b}$, where $\bar{b} = a$.

6.1. ***Expansion of the inclusion.*** The ratio M_C^{incl}/b does not represent the energy rate associated with the isotropic growth (self-similar expansion) of the inclusion ($\delta E_T/\delta b$), which is given by (31). In fact, the two are related by

$$\frac{\delta E_T}{\delta b} = \frac{2\pi\mu b \epsilon^{\bullet 2}}{1-\nu} - \frac{2M_C^{\text{incl}}}{b}. \tag{42}$$

Figure 5 (right image) shows the variation of $\delta E_T/\delta b$ and M_C^{incl}/b (both normalized by $\overline{M}_C/b$) with b/a, in the case $d = 2a$. The energy (E_T) plot itself is shown in Figure 5 (left image). The maximum value of the rate $\delta E_T/\delta b$ is 0.2887 $\overline{M}_C/a$, taking place for $b = 0.866a$. The energy rate in the case when the inclusion is tangent to the void ($b = a$) is 0.2778 $\overline{M}_C/a$. Figure 6 shows the variation of the energy E_T and the energy rate $\delta E_T/\delta b$ with the distance d/a in the case of a circular inclusion of radii $b = a$, $b = 3a/4$, and $b = a/2$. In the limit as $d \to \infty$, the energy E_T approaches the value E_T^0 according to (29), while the energy rate $\delta E_T/\delta b$ approaches the value $\frac{1}{2}(b/a)^2$ times $\overline{M}_C/b$, according to (42). Therefore, the greater the distance from the void, the greater the energy E_T associated with the presence of the inclusion, and the greater the rate of energy $\delta E_T/\delta b$ associated with the increase of the inclusion. This suggests that the presence of a nearby free surface facilitates the eigenstrain transformation, which may be of importance for the study of displacive transformations and transformation toughening mechanisms in crystalline metallic materials and ceramics.

If the specific configurational force f is introduced, the rate of total strain energy associated with a uniform expansion of inclusion can be expressed as

$$\frac{\delta E_T}{\delta b} = -\int_0^{2\pi} f(b,\theta) b \, \mathrm{d}\theta, \quad f = \left[\tfrac{1}{2}(\sigma_{ij}^{0,\text{in}} + \sigma_{ij}^{0,\text{out}})_{\rho=b} + \hat{\sigma}_{ij}(b,\theta)\right]\epsilon_{ij}^{\bullet}, \tag{43}$$

where $\hat{\sigma}_{ij}(b,\theta)$ are the stress components of the auxiliary problem around the inclusion. This type of expression for f was originally derived by Gavazza [1977]. For the dilatational eigenstrain, (43) reduces to

$$\frac{\delta E_T}{\delta b} = \frac{2\pi\mu b \epsilon^{\bullet 2}}{1-\nu} - b\epsilon^{\bullet} \int_0^{2\pi} \hat{\sigma}_{kk}(b,\theta) \, \mathrm{d}\theta. \tag{44}$$

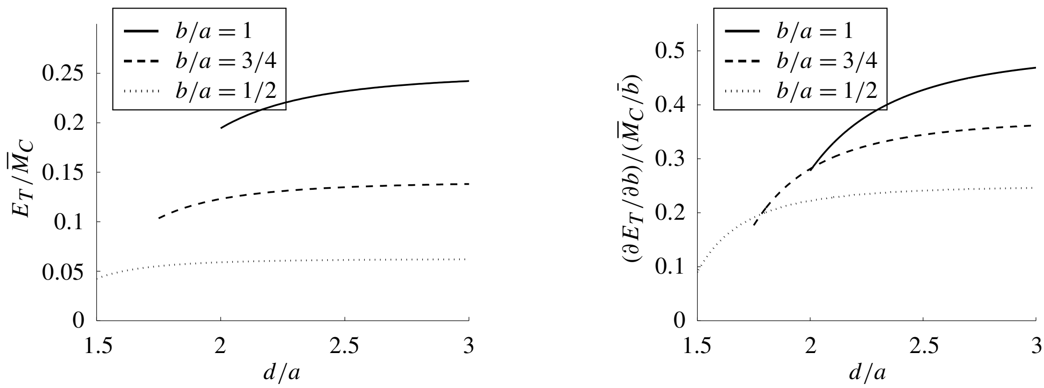


Figure 6. Left: the variation of the strain energy E_T, given by (29) and normalized by $\overline{M}_C$, with the normalized distance d/a, for the three indicated radii of the inclusion. Right: the corresponding variation of $\delta E_T/\delta b$, given by (42) and normalized by $\overline{M}_C/\bar{b}$, where $\bar{b} = a$.

To evaluate $\hat{\sigma}_{kk}(b, \theta)$, one would have to solve the auxiliary boundary-value problem, which was circumvented in the earlier derivation of the rate $\delta E_T/\delta b$. The average normal stress around the inclusion can, however, be determined immediately by comparing (44) with (31):

$$\frac{1}{2\pi}\int_0^{2\pi} \hat{\sigma}_{kk}(b, \theta)\,\mathrm{d}\theta = \frac{4\mu\epsilon^{\bullet 2}}{1-\nu}\,\frac{a^2b^2}{(d^2-a^2)^2}. \tag{45}$$

7. Circular inclusion in a half-space

Figure 7 (left image) shows a circular inclusion of radius b whose center is at distance c from the free surface ($x = 0$) of a half-space. If the inclusion was given a uniform shear eigenstrain $\epsilon^{\bullet}_{xy}$, the longitudinal stress along the free surface can be calculated from

$$\sigma_y(0, y) = 2[\sigma_y^0(0, y) - \sigma_x^0(0, y)], \tag{46}$$

where the infinite medium stress field outside the inclusion is specified in Section 2. Upon using the stress-transformation formulas between the (ρ, φ) and (x, y) coordinate systems, it follows that

$$\sigma_y(0, y) = -\bar{\sigma}_y c b^2\left(2 - \frac{3b^2}{c^2+y^2}\right)\frac{y(c^2-y^2)}{(c^2+y^2)^3}, \quad \bar{\sigma}_y = \frac{8\mu\epsilon^{\bullet}_{xy}}{1-\nu}. \tag{47}$$

The plot of this stress along the y-axis for several values of the ratio c/b is shown in Figure 8.

The configurational force between the inclusion and the free surface of a half-space is determined from

$$J_x^{\mathrm{incl}} = -\int_{-\infty}^{\infty} W(0, y)\,\mathrm{d}y = -\frac{1-\nu}{4\mu}\int_{-\infty}^{\infty} \sigma_y^2(0, y)\,\mathrm{d}y. \tag{48}$$

Upon the substitution of (47) and integration, (48) becomes

$$J_x^{\mathrm{incl}} = \bar{J}_x\,\frac{32(c/b)^4 - 72(c/b)^2 + 45}{5(c/b)^7}, \quad \bar{J}_x = -\frac{5\pi\mu}{32(1-\nu)}\,b\epsilon^{\bullet 2}_{xy}, \tag{49}$$

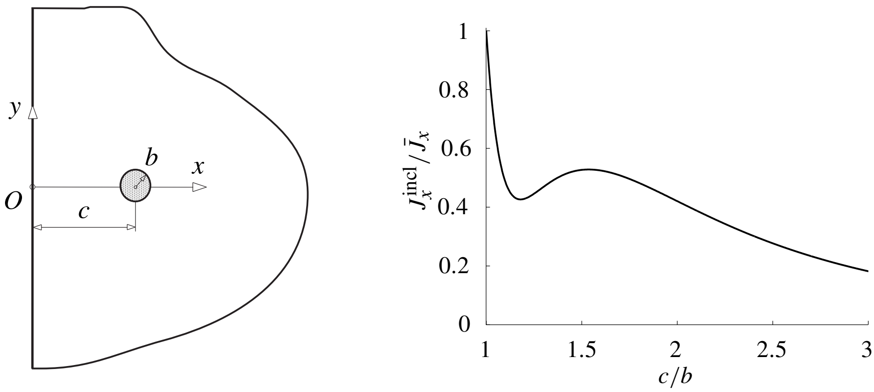


Figure 7. Left: a circular inclusion of radius b with its center at the distance c from the free surface of the half-space. Right: the variation of J_x with c/b. The scaling factor is $\bar{J}_x = -5\pi\mu\epsilon^{\bullet}_{xy}/[32(1-\nu)]$.

where $\bar{J}_x$ is the value of J_x when $c = b$. In the evaluation of integrals, the following recursive relations were used [Gradshteyn and Ryzhik 1965]:

$$\begin{aligned} \int_0^\infty \frac{y^m\,\mathrm{d}y}{(1+y^2)^n} &= \frac{m-1}{2n-m-1}\int_0^\infty \frac{y^{m-2}\,\mathrm{d}y}{(1+y^2)^n}, \quad n > m/2, \\ \int_0^\infty \frac{\mathrm{d}y}{(1+y^2)^n} &= \frac{2n-3}{2n-2}\int_0^\infty \frac{\mathrm{d}y}{(1+y^2)^{n-1}}, \quad n > 1. \end{aligned} \tag{50}$$

The variation of J_x^{incl} with c/b is shown in Figure 7 (right image). For the values of the ratio c/b between 1.1794 and 1.5359 there is a peculiar increase of the force with the increase of c/b, after which the configurational force resumes its expected decay with the increase of c/b. The brief interruption of the force decay in this region with the increase of the ratio c/b is a consequence of the markedly different nature of the stress variation $\sigma_y(0, y)$, corresponding to different values of the ratio c/b, as shown in Figure 8.

8. Conclusion

The J- and M-integrals along the boundary of a circular void, or along the straight edge of a half-space, due to a nearby circular inclusion with uniform dilatational or shear eigenstrain, are evaluated by using the Kienzler–Zhuping formula, without solving the entire boundary value problem at hand. Three types of interactions between inclusion and void are considered. First, the configurational force exerted on the void by the inclusion is determined by means of the J-integral evaluation. This represents the energy release rate associated with an imagined void translation within the material toward the inclusion (by diffusion or otherwise), keeping the position of the inclusion fixed. Second, the energy release rate is evaluated associated with a self-similar expansion of the void (by material absorption over the surface of the void), keeping the position of the inclusion fixed relative to the center of the void. This energy rate is related to the M-integral around the void. Third, the energy rate associated with an isotropic expansion of the inclusion is evaluated and related to the M-integral around the inclusion. The former differs from the

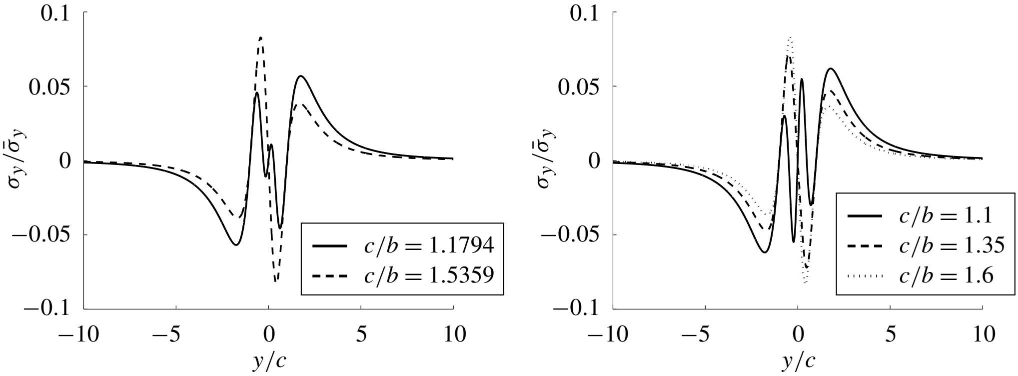


Figure 8. The variation of the stress component $\sigma(0, y)$ along the y-axis for several values of the ratio c/b. The left image is for the values of c/b corresponding to the local minimum and maximum of J_x, and the right image is for three other values of c/b. The utilized scaling factor is $8\mu\epsilon^{\bullet}_{xy}/(1-\nu)$. The values of the ratio c/b at which the configurational force has a local maximum and minimum are the roots of $(c/b)^2 = (15 \pm \sqrt{15})/8$. The corresponding force values are $0.5280\bar{J}_x$ and $0.4262\bar{J}_x$.

latter. The relationship between the two quantities is derived and discussed. It is shown that the greater the distance from the void, the greater the energy associated with the presence of the inclusion and the energy rate associated with its growth. This suggests that the presence of a nearby free surface facilitates the eigenstrain transformation, which may be of importance for the study of displacive transformations and transformation toughening mechanisms in crystalline metallic materials and ceramics. The attraction exerted on a circular inclusion with a uniform shear eigenstrain by the free surface of a half-space is also evaluated. Peculiar variation of this configurational force with the distance between the inclusion and the free surface is noted and discussed.

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