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Radosław Czubacki and Tomasz Lewiński

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TOPOLOGY OPTIMIZATION OF SPATIAL CONTINUUM STRUCTURES MADE OF NONHOMOGENEOUS MATERIAL OF CUBIC SYMMETRY

RADOSŁAW CZUBACKI AND TOMASZ LEWIŃSKI

The paper deals with the minimum compliance problem of spatial structures made of a nonhomogeneous elastic material of cubic symmetry. The elastic moduli as well as the trajectories of anisotropy directions are design variables. The isoperimetric condition fixes the value of the cost of the design expressed as the integral of the unit cost assumed as a linear combination of the three elastic moduli of the cubic symmetry. The problem has been reduced to the pair of mutually dual auxiliary problems similar to those known from the theory of materials with locking and from the transshipping theory. The auxiliary minimization problem has the integrand of linear growth, which transforms the problem considered to the topology optimization problem in which simultaneously the shape of the structure and its material characteristics are constructed. In contrast to the free material design which in the single load case leads to the optimal Hooke tensor with a single nonzero eigenvalue, the optimal Hooke tensor of cubic symmetry has either three or four nonzero eigenvalues.

1. Introduction

The present paper puts forward a topology optimization method aimed at constructing a stiffest continuum structure transmitting a given load to a given boundary. The problem is specified by assuming that the structure being designed is formed of a nonhomogeneous elastic material of cubic symmetry at each point. All the fields which determine the cubic anisotropy within the whole body are design variables. The isoperimetric condition imposed is viewed as the cost of the design and is expressed by the spatial integral of a linear combination of the eigenvalues of the elastic moduli. This expression encompasses the popular definition of cost, as the integral of the trace of the Hooke tensor of cubic symmetry. Then the weight coefficients in the expression of the unit cost are equal to the multiplicities of the relevant elastic moduli. No *a priori* restrictions on the anisotropy directions are assumed; they are to be determined via the optimization process. The stiffness of the structure is defined as the inverse of the total compliance.

The problem thus formulated is a reformulation of the free material design (FMD) to the case of materials of cubic symmetry. In its original formulation the FMD involves no restrictions on the components of the Hooke tensor, apart from necessary symmetries and positive semidefiniteness conditions; see [Bendsøe et al. 1994] and [Haslinger et al. 2010]. The peculiar feature of the compliance minimizing FMD problem is possible elimination of all design variables, leading to an auxiliary problem of the form

$$\min\left\{\int_\Omega \|\boldsymbol{\tau}\|\,\mathrm{d}x \mid \boldsymbol{\tau} \in \Sigma(\Omega)\right\}, \tag{P$_1$}$$

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where $\Sigma(\Omega)$ is the set of statically admissible stress fields $\boldsymbol{\tau} = (\tau_{ij})$ on the given feasible domain Ω; the norm $\|\cdot\|$ being Euclidean one.

The problem dual to (P_1) reads

$$\max\Big\{ f(\boldsymbol{v}) \mid \boldsymbol{v} \in V(\Omega),\ \|\boldsymbol{\varepsilon}(\boldsymbol{v}(x))\|^* \le 1 \text{ a.e. in } \Omega \Big\}, \tag{P_1^*}$$

where $f(\boldsymbol{v})$ represents the virtual work of the load on the displacement field $\boldsymbol{v}$; $V(\Omega)$ being the space of kinematically admissible displacement fields; $\boldsymbol{\varepsilon}(\boldsymbol{v})$ is the strain tensor, defined as the symmetric part of the gradient of $\boldsymbol{v}$.

The problems (P_1), (P_1^*) have been derived in [Czarnecki and Lewiński 2012; 2014b; 2014a]. The mathematical structure of these mutually dual problems is similar to the Kantorovich–Rubinstein transshipping problem, while their equivalence can be proved in the manner the Theorem 3.3 in [Bouchitté et al. 2010] was proved; cf. also [Bouchitté et al. 2005]. The inequality involved in (P_1^*) can be written as the inclusion: $\boldsymbol{\varepsilon}(\boldsymbol{v}(x)) \in B$ where B is a locking locus. Here B is the unit ball with respect to the Euclidean norm. The problem (P_1^*) has much in common with the locking material problem, discussed for example in [Demengel and Suquet 1986] and in [Telega and Jemioło 1998].

Mathematically demanding questions concerning the correctness of the problems (P_1), (P_1^*), referring to the contemporary variational calculus and measure theory, are the subject of the contemporary studies, to be published soon. These subtleties will not be discussed in the present paper. Let us yet outline here the most distinguished features of these problems. Note that the minimizer $\boldsymbol{\tau} = \boldsymbol{\sigma}$ of problem (P_1) can vanish on a set Ω_0 of positive measure, being a subset of the feasible domain Ω. In the domain Ω_0 all the components of the optimal elasticity tensor vanish. In the remaining part of the feasible domain the components of Hooke's tensor $\boldsymbol{C}$ are expressed by

$$C_{ijkl}(x) = \lambda_1(x)\overset{1}{\omega}_{ij}(x)\overset{1}{\omega}_{kl}(x), \tag{1-1}$$

where $\lambda_1(x)$ is proportional to $\|\boldsymbol{\sigma}(x)\|$, while $\overset{1}{\boldsymbol{\omega}}(x) = \boldsymbol{\sigma}(x)/\|\boldsymbol{\sigma}(x)\|$.

The optimal Hooke tensor possesses only one positive eigenvalue. The form (1-1) is designed for the given loading; that is why the structure of singular properties (1-1) is capable of transmitting the given load to the given support; cf. [Czarnecki and Lewiński 2014a, §6]. Thus the solution to the problems (P_1) and (P_1^*), or only to the problem (P_1), delivers information of two kinds:

(i) Topology information which determines the shape of the domain $\Omega\backslash\Omega_0$ occupied by the structural material; this domain may be multiconnected

(ii) Information on the nonhomogeneous anisotropy: the values of the elastic moduli C_{ijkl} as well as the anisotropy directions at each point of the body.

The solutions to the problems (P_1) and (P_1^*) do not determine the underlying microstructures possibly producing given anisotropic properties.

The topological information on the shape and connectedness of the structure is crucial. This feature of the FMD method delivers the solution to the topology optimization problem implicitly comprised by the method. The regularity of the optimal shape depends on the regularity of the data.

A natural modification of the FMD is *a priori* imposing certain material symmetries. The strongest assumption is isotropy — this modification has been proposed in [Czarnecki 2015] and [Czarnecki and Wawruch 2015]; it will be called the isotropic material design (IMD). The only design variables are the bulk $k(x)$ and shear $\mu(x)$ moduli in each point x of the feasible domain. In the spatial setting the collection of the eigenvalues of the Hooke tensor is $(3k, 2\mu, 2\mu, 2\mu, 2\mu, 2\mu)$ and the trace of the Hooke tensor equals $3k + 10\mu$. Let the cost of the design be the integral of the trace of the Hooke tensor. Then the isoperimetric condition assumes the form

$$\int_{\Omega} (3k + 10\mu)\,\mathrm{d}x = \Lambda, \tag{1-2}$$

where Λ stands for the assumed cost.

Czarnecki [2015] proved that the IMD reduces to an auxiliary problem of a mathematical structure similar to (P_1) with the integrand expressed by the norm

$$|||\boldsymbol{\tau}||| = \alpha|\mathrm{tr}\,\boldsymbol{\tau}| + \beta\|\mathrm{dev}\,\boldsymbol{\tau}\|, \tag{1-3}$$

where $\mathrm{tr}\,\boldsymbol{\tau}$ is the trace of $\boldsymbol{\tau}$ and $\mathrm{dev}\,\boldsymbol{\tau}$ is the deviator of the stress:

$$\mathrm{dev}\,\boldsymbol{\tau} = \boldsymbol{\tau} - \tfrac{1}{3}(\mathrm{tr}\,\boldsymbol{\tau})\overset{2}{\boldsymbol{I}}. \tag{1-4}$$

Here $\overset{2}{\boldsymbol{I}} = (\delta_{ij})$ is a unit tensor in $\mathbb{E}^2_s$, $\mathbb{E}^2_s$ being the set of symmetric tensors of rank 2; positive parameters α, β depend on the dimension of the problem.

The problem dual to (P_1) with the norm (1-3) assumes the form (P^*_1), in which the inequality condition has now the form

$$|||\boldsymbol{\varepsilon}(\boldsymbol{v}(x))|||^* \le 1, \tag{1-5}$$

where $\boldsymbol{\varepsilon} \in \mathbb{E}^2_s$ and the new norm in $\mathbb{E}^2_s$ is defined by

$$|||\boldsymbol{\varepsilon}|||^* = \sup_{\boldsymbol{\tau}\neq 0,\ \boldsymbol{\tau}\in\mathbb{E}^2_s} \frac{\boldsymbol{\tau}\cdot\boldsymbol{\varepsilon}}{|||\boldsymbol{\tau}|||}. \tag{1-6}$$

This is a norm dual to (1-3). In the sequel we shall show the explicit form of the norm (1-6) and the condition (1-5).

Therefore, the IMD problem, as expressed by the mutually dual problems (P_1), (P^*_1) involving the norms (1-3), (1-6), preserves the feature (i): the minimizer of (P_1) determines the domain $\Omega\backslash\Omega_0$ which is its effective domain. The process of designing of a structure made of an isotropic material is thus converted into a topology optimization algorithm admitting all possible topological changes of the initial shape of the feasible domain as far as they do keep the linear form $f(.)$ intact. Indeed, if the load is applied on a part of the boundary, then this part of the boundary cannot undergo changes during the optimization process. If the volume forces are taken into account, then also the domain of their application is kept unchanged. In fact, the method takes care of this condition, since the minimizer $\boldsymbol{\sigma}$ will not vanish in the domains where the loads are present.

The IMD method delivers as a solution: the effective domain of the minimizer, where the material is necessary as well as its isotropic properties: the layouts of the moduli $k(x)$, $\mu(x)$ optimally distributed within the feasible domain. The IMD method does not produce any information on the underlying microstructure. Note that the moduli k and μ determine the values of the Poisson ratio ν. The hitherto experiments show that the optimal Poisson ratio assumes extreme admissible values within some subdomains: in many cases ν approaches values close to -1, which is mathematically justified, while in some subdomains the optimal ν assumes values close to $\frac{1}{2}$. Thus the optimization process makes the Poisson ratio attain both bounds: $-1 < \nu < \frac{1}{2}$. In the 2D setting the bounds are even broader: $-1 < \nu < 1$. The negative values of ν make the physical interpretation difficult; only very special materials, called auxetic materials, like foams of reentrant microstructure, exhibit such properties; cf. [Friis et al. 1988]. The recent conference Auxetics'14 (Auxetics and other materials and models with "negative" characteristics) was fully devoted to this topic of materials science.

The solution of (P_1) with the norm (1-3) suffices to determine the optimal values of the moduli k and μ. It turns out that the optimal $k(x)$ is proportional to $|\operatorname{tr}\boldsymbol{\sigma}(x)|$, while the optimal $\mu(x)$ is proportional to $\|\operatorname{dev}\boldsymbol{\sigma}(x)\|$.

It is worth indicating here that similar formulae have been reported by Zohdi [2003a] in the paper on the inverse homogenization based on the minimization of the relative distance between the tensor of elastic moduli (of a composite determined by random properties of its *representative volume element* (RVE)) and a reference isotropic tensor. The optimal k occurs to be proportional to $\langle|\operatorname{tr}\boldsymbol{\sigma}|\rangle$, while the optimal μ is proportional to $\langle\|\operatorname{dev}\boldsymbol{\sigma}\|\rangle$, where $\langle\,\cdot\,\rangle$ represents averaging over RVE.

In majority of papers on composite materials the isotropy is associated with an ideal mixture of two or several constituents within an RVE. It is easy to show two-dimensional layouts of two materials within a repetitive cell resulting in isotropy of the effective Hooke tensor, see [Grigoliuk and Filshtinskii 1970]. Only recently Łukasiak [2013; 2014] has shown three-dimensional layouts resulting in isotropy of the effective Hooke tensor constructed by the homogenization method. This result has been achieved by a proper choice of RVE compatible with Kelvin's packing; see [Aste and Weaire 2008] and [Weaire 1996]. This result contradicts a remark in [Christensen 1999, p. 95]: "cubic symmetry is the highest order symmetry that can be obtained by a space filling periodic repeating cell pattern".

The composite materials and crystals are usually nonisotropic. Thus it is thought useful to extend the FMD method to the class of designs of lower symmetry. According to Xia [1997] and Ting [2003] there are 8 symmetry classes: triclinic, monoclinic, orthotropic, tetragonal, trigonal, transversely isotropic, cubic and isotropic. In the present paper the material design will be confined to the cubic symmetry case. The aim is to put forward a method to construct — within a given feasible domain — the stiffest structure capable of transmitting a given load to a given supporting surface by appropriate choice of the material characteristics of the cubic symmetry class. In each point of the structure the six parameters: the three elastic moduli and a triplet $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{p})$ of mutually orthogonal unit vectors satisfying

$$\begin{gathered}\|\boldsymbol{n}\| = \|\boldsymbol{m}\| = \|\boldsymbol{p}\| = 1,\\ \boldsymbol{n}\cdot\boldsymbol{m} = 0, \quad \boldsymbol{n}\cdot\boldsymbol{p} = 0, \quad \boldsymbol{m}\cdot\boldsymbol{p} = 0\end{gathered} \tag{1-7}$$

are to be determined. The Hooke tensor of a material of cubic symmetry is represented by the celebrated formula by Walpole [1984]:

$$\boldsymbol{C} = a\boldsymbol{J} + b\boldsymbol{L} + c\boldsymbol{M}, \tag{1-8}$$

where a, b, c are elastic moduli while the fourth-rank tensors $\boldsymbol{J}$, $\boldsymbol{L}$, $\boldsymbol{M}$ are expressed as

$$\boldsymbol{J} = \tfrac{1}{3}\overset{2}{\boldsymbol{I}}\otimes\overset{2}{\boldsymbol{I}}, \quad \boldsymbol{L} = \overset{4}{\boldsymbol{I}} - \boldsymbol{S}, \quad \boldsymbol{M} = \boldsymbol{S} - \boldsymbol{J}, \quad \text{and} \tag{1-9}$$

$$\boldsymbol{S} = \boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{m}\otimes\boldsymbol{m}\otimes\boldsymbol{m}\otimes\boldsymbol{m} + \boldsymbol{p}\otimes\boldsymbol{p}\otimes\boldsymbol{p}\otimes\boldsymbol{p}, \tag{1-10}$$

$$\overset{4}{I}_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}), \tag{1-11}$$

the latter being a unit tensor in $\mathbb{E}_s^4$ or in the space of Hooke tensors obeying usual symmetry rules.

The formula (1-8) is also a spectral representation, since the tensors $\boldsymbol{J}$, $\boldsymbol{L}$, $\boldsymbol{M}$ are projection operators (see [Walpole 1984]):

$$\begin{gathered} \boldsymbol{J}^2 = \boldsymbol{J}, \quad \boldsymbol{L}^2 = \boldsymbol{L}, \quad \boldsymbol{M}^2 = \boldsymbol{M}, \\ \boldsymbol{L}\boldsymbol{J} = \boldsymbol{J}\boldsymbol{L} = \boldsymbol{0}, \quad \boldsymbol{M}\boldsymbol{J} = \boldsymbol{J}\boldsymbol{M} = \boldsymbol{0}, \quad \boldsymbol{M}\boldsymbol{L} = \boldsymbol{L}\boldsymbol{M} = \boldsymbol{0}. \end{gathered} \tag{1-12}$$

The eigenvalues of tensor (1-8) are (a, b, b, b, c, c). Moreover,

$$\operatorname{tr} \boldsymbol{J} = 1, \quad \operatorname{tr} \overset{4}{\boldsymbol{I}} = 6, \quad \operatorname{tr} \boldsymbol{S} = 3 \tag{1-13}$$

or

$$\operatorname{tr} \boldsymbol{C} = a + 3b + 2c. \tag{1-14}$$

In the optimization problem considered the design variables are the scalar fields $a(x)$, $b(x)$, $c(x)$ and the vector fields $\boldsymbol{n}(x)$, $\boldsymbol{m}(x)$, $\boldsymbol{p}(x)$ satisfying the conditions (1-7).

The spectral representation of the inverse of $\boldsymbol{C}$ reads

$$\boldsymbol{C}^{-1} = \frac{1}{a}\boldsymbol{J} + \frac{1}{b}\boldsymbol{L} + \frac{1}{c}\boldsymbol{M}, \tag{1-15}$$

provided that all the moduli a, b, c are positive. If, for instance, $b = 0$, the tensor $\boldsymbol{C}^{-1}$ will be assumed in the form

$$\boldsymbol{C}^{-1} = \frac{1}{a}\boldsymbol{J} + \frac{1}{c}\boldsymbol{M}, \tag{1-16}$$

remembering that then the formula $\boldsymbol{C}^{-1}\boldsymbol{C} = \overset{4}{\boldsymbol{I}}$ is broken.

The cost of the design will be taken as the integral of the linear combination of the elastic moduli:

$$\int_\Omega (\alpha_1 a + \alpha_2 b + \alpha_3 c)\, \mathrm{d}x = \Lambda, \tag{1-17}$$

where $\alpha_i > 0$ are fixed. If $\alpha_1 = 1$, $\alpha_2 = 3$, $\alpha_3 = 2$ then the unit cost is equal to $\operatorname{tr} \boldsymbol{C}$; see (1-14). We consider the following problem of optimum design:

> *Find the layout of the elastic moduli a, b, c and the orthogonal trajectories of the vector fields* $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p})$ *at each point of the feasible domain* Ω*, satisfying the isoperimetric condition* (1-17)*, such that the structure made of this nonhomogeneous material (of cubic symmetry at each point) is characterized by the smallest total compliance among all structures designed in the same feasible domain, obeying the same isoperimetric condition and capable of transmitting the same load to the same boundary.*

We shall show that the problem above can be reduced to the two auxiliary problems (P_1), (P_1^*) with norms $|||\cdot|||$, $|||\cdot|||^*$ but with different coefficients α and β than those involved in the auxiliary problems of the IMD method. This result is surprising, since the auxiliary optimization problems for the isotropic and cubic symmetries differ very slightly and preserve their main property: the integrand of (P_1) is still expressed in terms of the two invariants of the stress field. Upon finding the minimizer $\boldsymbol{\tau} = \boldsymbol{\sigma}$ of problem (P_1) one can determine the design variables: the scalars $a(x)$, $b(x)$, $c(x)$ and the vector fields $\boldsymbol{n}(x)$, $\boldsymbol{m}(x)$, $\boldsymbol{p}(x)$ at each point of the domain $\Omega \backslash \Omega_0$ where $\boldsymbol{\sigma}$ does not vanish.

The mentioned similarity between optimized cubic symmetry and isotropy has already been noted previously in a different context of constructing a stationary form of energy density of a material of cubic symmetry; Norris [2006] noted that "the extreme values of the energy for cubic materials have the same form of the energy for an isotropic solid"; see Equations (4.59, 46) therein.

Considered in the present paper the optimal materials of cubic symmetry can be manufactured as cellular foams of small density (see [Christensen 1999]), or the lotus-type porous copper (see [Xie et al. 2004]).

Moreover, the metal matrix composites belong to the class of materials whose cubic symmetry can be tailored by appropriate choice of microstructural parameters determining the physical, topological as well as geometrical properties of second phase particles. Towards this end one should state an inverse problem by appropriate extension of Zohdi's [2001; 2003b] normalizing objective functionals. Although constructed for the isotropic design, they can serve as well for the cubic design provided that the set of design variables is augmented by Euler's angles of particles, following the lines of smooth change of the triplet fields $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p})$. Appropriate liquid state processing makes it possible to align the particles along prescribed directions; see comments in [Zohdi 2001, §2].

Reduction of the optimum design problem to the problem of type (P$_1$) with the norm $|||\cdot|||$ means that *the method put forward implies simultaneously the topology and the material optimization*. The effective domain $\Omega\backslash\Omega_0$ of the minimizer $\boldsymbol{\tau}=\boldsymbol{\sigma}$ determines the shape of the structure and admits its multiconnectedness. Depending on the shape of the feasible domain Ω and on the type of the surface load applied the method forms the domain $\Omega\backslash\Omega_0$ occupied by the material. We shall prove in the sequel that depending on the sign of $(\alpha_2-\alpha_3)$ the nonzero optimal moduli are either (a, b) or (a, c). *This means that in all cases exactly one modulus of three* (a, b, c) *vanishes to make the whole structure as stiff as possible*. The optimal cubic material turns out to be degenerated in all cases. Yet the optimal material properties are perfectly suited for the given load and support, thus making the optimal structure fulfill all conditions of equilibrium as well as the boundary conditions. The displacement and strain fields in the optimum structure are not uniquely determined, the constitutive equations being noninvertible, yet the corresponding stress field transmitting the load to the support is unique.

The following conventions are adopted. The feasible domain Ω in $\mathbb{R}^3$ is parametrized by the Cartesian system (x_1, x_2, x_3) of vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ satisfying $\boldsymbol{e}_i\boldsymbol{e}_j=\delta_{ij}$; the Latin indices $i, j, \ldots$ run over 1, 2, 3. The summation convention over repeated indices is adopted. An arbitrary point x of Ω is identified with its coordinates (x_1, x_2, x_3). The symmetric tensors of rank two form the set $\mathbb{E}^2_s$, while rank-four Hooke tensors of usual symmetries form the set $\mathbb{E}^4_s$. Comma implies partial differentiation: $(\cdot)_{,i}=\partial(\cdot)/\partial x_i$. A symmetric part of the gradient of a vector field $\boldsymbol{v}=(v_1, v_2, v_3)$ is denoted by $\varepsilon_{ij}(\boldsymbol{v})=\frac{1}{2}(v_{i,j}+v_{j,i})$. The Euclidean norms are defined for vectors by $\|\boldsymbol{v}\|=(v_iv_i)^{1/2}$ for $\boldsymbol{v}\in\mathbb{R}^3$, and for tensors by $\|\boldsymbol{\tau}\|=(\tau_{ij}\tau_{ij})^{1/2}$ for $\boldsymbol{\tau}\in\mathbb{E}^2_s$. The scalar products are defined as

$$\boldsymbol{v}\cdot\boldsymbol{w}=v_iw_i, \quad \boldsymbol{\tau}\cdot\boldsymbol{\varepsilon}=\tau_{ij}\varepsilon_{ij} \quad \text{for } \boldsymbol{v}, \boldsymbol{w}\in\mathbb{R}^3,\ \boldsymbol{\tau}, \boldsymbol{\varepsilon}\in\mathbb{E}^2_s.$$

2. Optimum design problem

The main data is the spatial feasible domain Ω in which the designed structure is to be placed. On a boundary Γ_1 the tractions $\boldsymbol{T}=(T_i)$ are applied. This surface is not subject to optimization. Under the equilibrium problem we understand construction of the stress fields $\boldsymbol{\tau}$ within Ω, transmitting the given load $\boldsymbol{T}$ to the given part Γ_2 of the boundary. Not all points of Γ_2 need to be supporting points — along some part of the boundary the material can disappear. The vector fields $\boldsymbol{v} = (v_1, v_2, v_3)$ within Ω vanishing on Γ_2 form the space $V(\Omega)$ of kinematically admissible displacements. In this paper we do not formulate the regularity conditions for the fields involved; these conditions can be assumed by analogy with those assumed in [Bouchitté et al. 2005; 2010] concerning the transshipping problem in the scalar version.

The stress field $\boldsymbol{\tau}$ is said to be statically admissible if it satisfies the variational equation

$$\int_\Omega \boldsymbol{\tau} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x = f(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V(\Omega), \tag{2-1}$$

where the linear form is defined by

$$f(\boldsymbol{v}) = \int_{\Gamma_1} \boldsymbol{T} \cdot \boldsymbol{v}\,\mathrm{d}s. \tag{2-2}$$

The value $f(\boldsymbol{v})$ is the virtual work of the tractions on the field $\boldsymbol{v}$. The set of all stress fields $\boldsymbol{\tau}$ satisfying (2-1), (2-2) forms the linear affine set $\Sigma(\Omega)$.

The body occupying the domain Ω, supported on Γ_2, loaded on Γ_1 of given anisotropy determined by the field of the Hooke tensor $\boldsymbol{C}(x)$ will be called a structure if it is capable of transmitting the given load to the support Γ_2. Its total compliance is expressed by the Castigliano formula:

$$Y = \min_{\boldsymbol{\tau} \in \Sigma(\Omega)} \int_\Omega \boldsymbol{\tau} \cdot (\boldsymbol{C}^{-1}\boldsymbol{\tau})\,\mathrm{d}x. \tag{2-3}$$

Assume that the tensor field $\boldsymbol{C}(x)$ exhibits a cubic symmetry at each point x of the feasible domain. Let the moduli a, b, c and the triplet $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p})$ be design variables. The moduli a, b, c must satisfy the isoperimetric condition (1-17) while the triplet $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p})$ must satisfy the conditions (1-7). The problem of optimum design expressing the compliance minimization is formulated as

$$J = \min_{\substack{(\boldsymbol{m},\boldsymbol{n},\boldsymbol{p})\\ \text{satisfying (1-7)}}} \; \min_{\substack{(a,b,c)\\ \text{satisfying (1-17)}}} Y \tag{2-4}$$

The fields (a, b, c) and $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p})$ minimizing Y determine the moduli and the trajectories of cubic anisotropy directions. It will be shown in the sequel that minimization over the design variables can be performed analytically, which reduces the problem (2-4) to an auxiliary problem (P_1) with a certain norm of type (1-3).

3. Elimination of design variables

Let us change the sequence of minimization operators in (2-4) and (2-3). We rewrite (2-4) in the form

$$J = \min_{\boldsymbol{\tau} \in \Sigma(\Omega)} Y(\boldsymbol{\tau}), \tag{3-1}$$

where

$$Y(\boldsymbol{\tau}) = \min_{\substack{(\boldsymbol{m},\boldsymbol{n},\boldsymbol{p})\\ \text{satisfying (1-7)}}} \; \min_{\substack{(a,b,c)\\ \text{satisfying (1-17)}}} \int_\Omega \boldsymbol{\tau} \cdot (\boldsymbol{C}^{-1}\boldsymbol{\tau})\,\mathrm{d}x. \tag{3-2}$$

Let us compute the integrand in (3-2) by using (1-15) and (1-9):

$$\boldsymbol{\tau} \cdot (\boldsymbol{C}^{-1}\boldsymbol{\tau}) = \frac{1}{a}\boldsymbol{\tau} \cdot (\boldsymbol{J}\boldsymbol{\tau}) + \frac{1}{b}(\|\boldsymbol{\tau}\|^2 - \boldsymbol{\tau} \cdot (\boldsymbol{S}\boldsymbol{\tau})) + \frac{1}{c}(\boldsymbol{\tau} \cdot (\boldsymbol{S}\boldsymbol{\tau}) - \boldsymbol{\tau} \cdot (\boldsymbol{J}\boldsymbol{\tau})). \tag{3-3}$$

Let us write

$$\boldsymbol{\tau}\cdot(\boldsymbol{C}^{-1}\boldsymbol{\tau})=\frac{1}{a}Y_1(\boldsymbol{\tau})+\frac{1}{b}Y_2(\boldsymbol{\tau})+\frac{1}{c}Y_3(\boldsymbol{\tau}) \tag{3-4}$$

with

$$\begin{aligned} Y_1(\boldsymbol{\tau})&=\tfrac{1}{3}(\operatorname{tr}\boldsymbol{\tau})^2,\\ Y_2(\boldsymbol{\tau})&=\|\boldsymbol{\tau}\|^2-\boldsymbol{\tau}\cdot(\boldsymbol{S}\boldsymbol{\tau}),\\ Y_3(\boldsymbol{\tau})&=\boldsymbol{\tau}\cdot(\boldsymbol{S}\boldsymbol{\tau})-\tfrac{1}{3}(\operatorname{tr}\boldsymbol{\tau})^2. \end{aligned} \tag{3-5}$$

One can prove that for each choice of the triplet $(\boldsymbol{m},\boldsymbol{n},\boldsymbol{p})$ satisfying (1-7) the following estimates hold:

$$\tfrac{1}{3}(\operatorname{tr}\boldsymbol{\tau})^2\le\boldsymbol{\tau}\cdot(\boldsymbol{S}\boldsymbol{\tau})\le\|\boldsymbol{\tau}\|^2. \tag{3-6}$$

The right inequality becomes sharp if the triplet $(\boldsymbol{m},\boldsymbol{n},\boldsymbol{p})$ coincides with principal directions of the stress tensor $\boldsymbol{\tau}$. The left inequality is also attainable, which is much more difficult to prove. It can be inferred from the stationarity criterion of Norris [2006, (4.32)]. The minimizer $(\boldsymbol{m}^*,\boldsymbol{n}^*,\boldsymbol{p}^*)$ of the quadratic form $\boldsymbol{\tau}\cdot(\boldsymbol{S}\boldsymbol{\tau})$, for fixed $\boldsymbol{\tau}$, satisfies the condition

$$\boldsymbol{S}(\boldsymbol{m}^*,\boldsymbol{n}^*,\boldsymbol{p}^*)\boldsymbol{\tau}=\tfrac{1}{3}(\operatorname{tr}\boldsymbol{\tau})\overset{2}{\boldsymbol{I}}, \tag{3-7}$$

which implies

$$\min_{\substack{(\boldsymbol{m},\boldsymbol{n},\boldsymbol{p})\\ \text{satisfying (1-7)}}}(\boldsymbol{\tau}\cdot(\boldsymbol{S}\boldsymbol{\tau}))=\tfrac{1}{3}(\operatorname{tr}\boldsymbol{\tau})^2. \tag{3-8}$$

We conclude that both $Y_2(\boldsymbol{\tau})$ and $Y_3(\boldsymbol{\tau})$ are nonnegative.

Consider the auxiliary problem

$$W=\min_{\substack{a>0,\,b>0,\,c>0\\ \text{satisfying (1-17)}}}\int_\Omega\Big(\frac{1}{a}Y_1+\frac{1}{b}Y_2+\frac{1}{c}Y_3\Big)\,\mathrm{d}x, \tag{3-9}$$

in which $\boldsymbol{\tau}$ is fixed while the quantities Y_i are positive. Such a problem has been solved in [Czarnecki and Lewiński 2014b, §3.1]. Its solution has the form

$$W=\frac{1}{\Lambda}\bigg[\int_\Omega\big(\sqrt{\alpha_1}\sqrt{Y_1}+\sqrt{\alpha_2}\sqrt{Y_2}+\sqrt{\alpha_3}\sqrt{Y_3}\big)\,\mathrm{d}x\bigg]^2. \tag{3-10}$$

Let us proceed to perform minimization over the triplets $(\boldsymbol{m},\boldsymbol{n},\boldsymbol{p})$. Let us introduce the notation

$$z=\boldsymbol{\tau}\cdot(\boldsymbol{S}\boldsymbol{\tau}),\qquad z_0=\tfrac{1}{3}(\operatorname{tr}\boldsymbol{\tau})^2,\qquad z_1=\|\boldsymbol{\tau}\|^2. \tag{3-11}$$

Let us consider the auxiliary problem

$$\min_{z\in[z_0,z_1]}f(z), \tag{3-12}$$

where

$$f(z)=\sqrt{\alpha_2}\sqrt{z_1-z}+\sqrt{\alpha_3}\sqrt{z-z_0}. \tag{3-13}$$

It is easy to check that $f''(z)<0$ if $z\in[z_0,z_1]$. Thus the minima of $f(z)$ can only be attained at the ends of the interval $[z_0,z_1]$:

$$\min_{z\in[z_0,z_1]}f(z)=\min(f(z_0),f(z_1)). \tag{3-14}$$

Let us compute

$$f(z_0) = \sqrt{\alpha_2}\sqrt{z_1 - z_0}, \qquad f(z_1) = \sqrt{\alpha_3}\sqrt{z_1 - z_0}. \tag{3-15}$$

Thus

$$\min_{z \in [z_0, z_1]} f(z) = \sqrt{z_1 - z_0}\,\min(\sqrt{\alpha_2}, \sqrt{\alpha_3}). \tag{3-16}$$

According to the Norris result (3-7) both the minima are attainable by a certain triplet $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p})$.

The results (3-10), (3-16) solve the problem (3-2):

$$Y(\boldsymbol{\tau}) = \frac{1}{\Lambda}\left(\int_\Omega \left(\sqrt{\alpha_1}\sqrt{Y_1(\boldsymbol{\tau})} + \min(\sqrt{\alpha_2}, \sqrt{\alpha_3})\sqrt{\|\boldsymbol{\tau}\|^2 - Y_1(\boldsymbol{\tau})}\right) \mathrm{d}x\right)^2.$$

Let us note that

$$\sqrt{\|\boldsymbol{\tau}\|^2 - Y_1(\boldsymbol{\tau})} = \|\operatorname{dev} \boldsymbol{\tau}\|. \tag{3-17}$$

The problem (3-7) is thus reduced to the form

$$J = \frac{1}{\Lambda} Z^2, \tag{3-18}$$

where

$$Z = \min\left\{\textstyle\int_\Omega |\!|\!|\boldsymbol{\tau}|\!|\!| \,\mathrm{d}x \mid \boldsymbol{\tau} \in \Sigma(\Omega)\right\} \tag{3-19}$$

and

$$|\!|\!|\boldsymbol{\tau}|\!|\!| = \sqrt{\alpha_1/3}\,|\operatorname{tr} \boldsymbol{\tau}| + \min(\sqrt{\alpha_2}, \sqrt{\alpha_3})\|\operatorname{dev} \boldsymbol{\tau}\| \tag{3-20}$$

is the norm of type (1-3). Thus the problem (3-20) has almost the same form as that occurring in the similar problem concerning nonhomogeneous isotropy; see [Czarnecki 2015].

Remark 1. The problem (3-19) is the tensorial counterpart of the following problem with a vectorial unknown:

$$Z_s = \min\left\{\textstyle\int_\Omega \|\boldsymbol{p}\| \,\mathrm{d}x \mid \boldsymbol{p} \in \Sigma_s(\Omega)\right\}, \tag{3-21}$$

where $\|\boldsymbol{p}\|$ is a certain norm of the vector $\boldsymbol{p} \in \mathbb{R}^3$; $\Sigma_s(\Omega)$ is the set of vector fields $\boldsymbol{p} = (p_1, p_2, p_3)$ on Ω such that

$$\int_\Omega \boldsymbol{p} \cdot \nabla v \,\mathrm{d}x = \int_{\Gamma_1} T \cdot v \,\mathrm{d}s \quad \text{for all } v \in V_s(\Omega), \tag{3-22}$$

where T is given on $\Gamma_1 \subset \partial\Omega$ while $V_s(\Omega)$ is the set of scalar fields v vanishing on the segment Γ_2 of the boundary. The problem (3-21) appears in the theory of transshipping; cf. [Bouchitté et al. 2010; 2005]. This problem can be rearranged to a well-posed form by completing it with appropriate assumptions concerning the data.

The subtle problems concerning possible well-posedness of the problem (3-19) will be the subject of an independent work. Assume that $\boldsymbol{\tau} = \boldsymbol{\sigma}$ is the minimizer of (3-19). Let $\Omega \backslash \Omega_0$ be the effective domain of $\boldsymbol{\sigma}$. The values of the optimal moduli $a^*(x)$, $b^*(x)$, $c^*(x)$ can be computed by the formulae which follow from minimization of (3-9):

$$a^*(x) = \frac{1}{\sqrt{\lambda\alpha_1}}\sqrt{Y_1(\boldsymbol{\sigma}(x))}, \quad b^*(x) = \frac{1}{\sqrt{\lambda\alpha_2}}\sqrt{Y_2(\boldsymbol{\sigma}(x))}, \quad c^*(x) = \frac{1}{\sqrt{\lambda\alpha_3}}\sqrt{Y_3(\boldsymbol{\sigma}(x))}, \tag{3-23}$$

where the Lagrange multiplier λ is positive and given by the formula

$$\sqrt{\lambda}=\frac{1}{\Lambda}\int_{\Omega}\left(\sqrt{\alpha_1 Y_1(\boldsymbol{\sigma})}+\sqrt{\alpha_2 Y_2(\boldsymbol{\sigma})}+\sqrt{\alpha_3 Y_3(\boldsymbol{\sigma})}\right)\mathrm{d}x \tag{3-24}$$

and $Y_i(\boldsymbol{\sigma})$ are determined by (3-5), where the triplet $(\boldsymbol{m}^*, \boldsymbol{n}^*, \boldsymbol{p}^*)$ corresponds either to the lower or to the upper bound in (3-6).

In case of $\alpha_2<\alpha_3$ the minimum in (3-14) is attained if $z=z_0$ then the triplet $(\boldsymbol{m}^*, \boldsymbol{n}^*, \boldsymbol{p}^*)$ makes sharp the left inequality in (3-6). The triplet should be found from the Norris condition (3-7). In case of $\alpha_2>\alpha_3$ the minimum in (3-14) is attained if $z=z_1$. Then the triplet $(\boldsymbol{m}^*, \boldsymbol{n}^*, \boldsymbol{p}^*)$ makes sharp the right inequality in (3-6). The triplet coincides with eigenvectors of the tensor $\boldsymbol{\tau}=\boldsymbol{\sigma}$.

Let us discuss these two cases in more detail.

Case 1: $\alpha_2<\alpha_3$. The choice of $(\boldsymbol{m}^*, \boldsymbol{n}^*, \boldsymbol{p}^*)$ by (3-7) implies that $\boldsymbol{\sigma}\cdot(\boldsymbol{S}\boldsymbol{\sigma})=\frac{1}{3}(\operatorname{tr}\boldsymbol{\sigma})^2$. Thus

$$Y_2(\boldsymbol{\sigma})=\|\operatorname{dev}\boldsymbol{\sigma}\|^2,\qquad Y_3(\boldsymbol{\sigma})=0. \tag{3-25}$$

The optimal moduli are expressed as

$$a^*(x)=\frac{1}{\sqrt{\lambda\alpha_1}}\frac{1}{\sqrt{3}}|\operatorname{tr}\boldsymbol{\sigma}(x)|,\qquad b^*(x)=\frac{1}{\sqrt{\lambda\alpha_2}}\|\operatorname{dev}\boldsymbol{\sigma}(x)\|,\qquad c^*(x)=0. \tag{3-26}$$

The optimal Hooke tensor $\boldsymbol{C}^*(x)$ is characterized by the eigenvalues $(a^*(x), b^*(x), b^*(x), b^*(x), 0, 0)$. Let us write down its spectral representation:

$$\boldsymbol{C}^*(x)=a^*(x)\boldsymbol{J}+b^*(x)\boldsymbol{L}(x), \tag{3-27}$$

where $\boldsymbol{L}(x)=\overset{4}{\boldsymbol{I}}-\boldsymbol{S}(x)$ and the tensor $\boldsymbol{S}(x)$ is determined by the triplet $(\boldsymbol{m}^*, \boldsymbol{n}^*, \boldsymbol{p}^*)$ satisfying the condition (3-7). Let us write down the formula for the multiplier λ:

$$\sqrt{\lambda}=\frac{1}{\Lambda}\int_{\Omega}\left(\sqrt{\alpha_1/3}\,|\operatorname{tr}\boldsymbol{\sigma}|+\sqrt{\alpha_2}\,\|\operatorname{dev}\boldsymbol{\sigma}\|\right)\mathrm{d}x, \tag{3-28}$$

which completes the formulae (3-26) for the effective moduli.

Let us write down the constitutive equations of the optimal body taking at the given point x the axes x_i as directed along the triplet $(\boldsymbol{m}^*, \boldsymbol{n}^*, \boldsymbol{p}^*)$ determined by the condition (3-7). The components of the tensor $\boldsymbol{C}$ read

$$C_{ijkl}=\tfrac{1}{3}(a-c)\delta_{ij}\delta_{kl}+\tfrac{1}{2}b(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+(c-b)(\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}+\delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2}+\delta_{i3}\delta_{j3}\delta_{k3}\delta_{l3}). \tag{3-29}$$

This formula can be inferred from (1-8)–(1-11). The nonzero components C_{ijkl} are

$$\begin{aligned}C_{1111}&=C_{2222}=C_{3333}=\tfrac{1}{3}a+\tfrac{2}{3}c,\\ C_{1122}&=C_{1133}=C_{2233}=\tfrac{1}{3}a-\tfrac{1}{3}c,\\ C_{1212}&=C_{1313}=C_{2323}=\tfrac{1}{2}b.\end{aligned} \tag{3-30}$$

Hence the constitutive equations assume the form

$$\begin{aligned}\tilde{\sigma}_{11} &= \left(\tfrac{1}{3}a + \tfrac{2}{3}c\right)\tilde{\varepsilon}_{11} + \tfrac{1}{3}(a-c)(\tilde{\varepsilon}_{22}+\tilde{\varepsilon}_{33}),\\ \tilde{\sigma}_{22} &= \left(\tfrac{1}{3}a + \tfrac{2}{3}c\right)\tilde{\varepsilon}_{22} + \tfrac{1}{3}(a-c)(\tilde{\varepsilon}_{11}+\tilde{\varepsilon}_{33}),\\ \tilde{\sigma}_{33} &= \left(\tfrac{1}{3}a + \tfrac{2}{3}c\right)\tilde{\varepsilon}_{33} + \tfrac{1}{3}(a-c)(\tilde{\varepsilon}_{11}+\tilde{\varepsilon}_{22}),\\ \tilde{\sigma}_{12} &= b\tilde{\varepsilon}_{12}, \quad \tilde{\sigma}_{13} = b\tilde{\varepsilon}_{13}, \quad \tilde{\sigma}_{23} = b\tilde{\varepsilon}_{23}\end{aligned} \tag{3-31}$$

The stresses $\tilde{\sigma}_{ij}$ and strains $\tilde{\varepsilon}_{ij}$ refer to the optimal body. These tensors should not be misled with the solution to the auxiliary problems: (3-19) and the dual to the latter.

In the case discussed $a = a^*$, $b = b^*$, $c = c^* = 0$ the equations (3-31) assume the form

$$\begin{aligned}\tilde{\sigma}_{ii} &= \tfrac{1}{3}a^*(\tilde{\varepsilon}_{11}+\tilde{\varepsilon}_{22}+\tilde{\varepsilon}_{33}), && \text{(do not sum over } i),\\ \tilde{\sigma}_{ij} &= b^*\tilde{\varepsilon}_{ij}, && i \neq j.\end{aligned} \tag{3-32}$$

The principal directions of tensors $\tilde{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$ do not coincide. Consequently, these fields do not coincide.

Case 2: $\alpha_2 > \alpha_3$. The triplet $(\boldsymbol{m}^*, \boldsymbol{n}^*, \boldsymbol{p}^*)$ coincides with eigenvectors of tensor $\boldsymbol{\sigma}$. Thus we have $\boldsymbol{\sigma}\cdot(\boldsymbol{S}\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}\|^2$. Then

$$Y_2(\boldsymbol{\sigma}) = 0, \quad Y_3(\boldsymbol{\sigma}) = \|\operatorname{dev}\boldsymbol{\sigma}\|^2. \tag{3-33}$$

The optimal moduli assume the form

$$a^*(x) = \frac{1}{\sqrt{\lambda\alpha_1}}\frac{1}{\sqrt{3}}|\operatorname{tr}\boldsymbol{\sigma}(x)|, \quad b^*(x) = 0, \quad c^*(x) = \frac{1}{\sqrt{\lambda\alpha_3}}\|\operatorname{dev}\boldsymbol{\sigma}(x)\|. \tag{3-34}$$

The optimal Hooke tensor has the eigenvalues $(a^*(x), 0, 0, 0, c^*(x), c^*(x))$. The spectral representation of the Hooke tensor reads

$$\boldsymbol{C}^*(x) = a^*(x)\boldsymbol{J} + c^*(x)\boldsymbol{M}(x), \tag{3-35}$$

with $\boldsymbol{M}(x) = \boldsymbol{S}(x) - \boldsymbol{J}$ and tensor $\boldsymbol{S}(x)$ is determined by the triplet of eigenvectors of tensor $\boldsymbol{\sigma}$ at point x. Let us write down the expression for the Lagrange multiplier

$$\sqrt{\lambda} = \frac{1}{\Lambda}\int_\Omega \left(\sqrt{\alpha_1/3}\,|\operatorname{tr}\boldsymbol{\sigma}| + \sqrt{\alpha_3}\,\|\operatorname{dev}\boldsymbol{\sigma}\|\right)\mathrm{d}x. \tag{3-36}$$

Thus the formulae for the optimal moduli have been put in their final form.

According to (3-30) the moduli C^*_{1212}, C^*_{1313}, C^*_{2323} vanish. Thus, in the assumed coordinate system the constitutive equations assume the form

$$\tilde{\sigma}_{11} = \left(\tfrac{1}{3}a^* + \tfrac{2}{3}c^*\right)\tilde{\varepsilon}_{11} + \tfrac{1}{3}\left(a^*-c^*\right)\left(\tilde{\varepsilon}_{22}+\tilde{\varepsilon}_{33}\right), \quad \tilde{\sigma}_{12} = 0, \quad \tilde{\sigma}_{13} = 0, \quad \tilde{\sigma}_{23} = 0. \tag{3-37}$$

The remaining equations follow from change of indices. We conclude that the eigendirections of the tensor $\boldsymbol{\sigma}$ (i.e. the minimizer of (3-19)) and of the tensor $\tilde{\boldsymbol{\sigma}}$ (stress field in the optimal structure) coincide. Thus the trajectories of both the fields coincide.

Both the fields $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are statically admissible and have the same trajectories. Thus, except very specific situations in which the stress field has multiple principal values and undefined eigenvectors, both the stress fields should coincide. Indeed, if writing the equilibrium equations along the trajectories (or — in the curvilinear coordinate system formed by the trajectories) these three equations involve three unknown fields (principal stresses). This suggests that there is only one solution to this system satisfying the same boundary conditions on the boundary Γ_1. Consequently, the fields $\tilde{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$ must, in general, coincide, except for very specific cases.

The case of $\alpha_2 > \alpha_3$ is particularly important as encompassing the isoperimetric condition expressed in terms of the trace of the Hooke tensor; see (1-14). Then $\alpha_1 = 1$, $\alpha_2 = 3$, $\alpha_3 = 2$ and the norm (3-20) assumes the form

$$|||\boldsymbol{\tau}||| = \frac{\sqrt{3}}{3}|\operatorname{tr}\boldsymbol{\tau}| + \sqrt{2}\|\operatorname{dev}\boldsymbol{\tau}\|, \tag{3-38}$$

the parameters α and β being equal to $\frac{\sqrt{3}}{3}$ and $\sqrt{2}$, respectively.

The algorithm for solving the problem (2-4) with the isoperimetric condition concerning the trace of $\boldsymbol{C}$ can be summarized as below:

(1) Solve the problem (3-19) or construct its minimizer $\boldsymbol{\tau} = \boldsymbol{\sigma}$.

(2) Determine the domain $\Omega\backslash\Omega_0$ where $\boldsymbol{\sigma}$ does not vanish.

(3) Compute by (3-34), (3-36) the moduli a^*, c^* within the domain $\Omega\backslash\Omega_0$.

(4) Find the triplets $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p})$ as eigenvectors of the tensor $\boldsymbol{\sigma}$.

(5) Compute components of $\boldsymbol{C}^*$ by (3-35).

Remark 2. The FMD method in its original version in which no additional conditions are imposed on the structure of the Hooke tensor $\boldsymbol{C}$ leads to the singular optimal solution $\boldsymbol{C}^*$ with only one nonzero eigenvalue. The method proposed in the present paper in which the cubic symmetry is imposed leads to the singular representations (3-27) or (3-35) with two or three vanishing eigenvalues. The method IMD in which the material symmetry is imposed as isotropic leads to the nondegenerate optimal Hooke tensors. This distinguishes the method IMD from other versions of FMD-type, since nonsingular results are obtained even if only one load condition is taken into account.

To make the presented results (3-27) and (3-35) nonsingular one should, e.g., consider multiple load optimization, with the simplest scalarization concept, as used recently in [Czarnecki and Lewiński 2014b] for the classical FMD. This is an open problem to be discussed elsewhere.

4. Formulation dual to the auxiliary minimization problem

As has been stressed in Section 1, the problem dual to the stress-based auxiliary problem of the FMD has the form (P_1^*), which can be expressed as

$$\max\{f(\boldsymbol{v}) \mid \boldsymbol{v} \in V(\Omega),\ \boldsymbol{\varepsilon}(\boldsymbol{v}(x)) \in B\}, \tag{4-1}$$

where B is the unit ball in $\mathbb{E}_s^2$ with respect to the Euclidean norm. It is seen that the problem (4-1) has a very specific mathematical structure, since it involves the conditions to be satisfied pointwise. This is the consequence of the integrand in (P_1) being of linear growth. Since the integrand in (3-19) has also a linear growth, the problem dual to (3-19) will have again the mathematical form similar to (4-1); the modification will touch the shape of the ball B, called the locking locus. As mentioned in Section 1, the ball B is now defined by the norm dual to the norm $|||\cdot|||$ involved in (3-19), hence

$$B = \{\boldsymbol{\varepsilon} \in \mathbb{E}_s^2 \mid |||\boldsymbol{\varepsilon}|||^* \le 1\}, \tag{4-2}$$

while the norm $|||\cdot|||^*$ is defined by (1-6). A passage from (3-19) to (4-1)–(4-2) can be done by using the arguments invoked in [Czarnecki and Lewiński 2014a], along the lines of the derivation shown in [Strang and Kohn 1983], where, however, other norms are involved.

Our aim now is now to find the explicit form of the norm $|||\cdot|||^*$ given by (1-6). The scalar product $\boldsymbol{\tau}\cdot\boldsymbol{\varepsilon}$ can be decomposed into the hydrostatic and deviatoric parts:

$$\boldsymbol{\tau}\cdot\boldsymbol{\varepsilon}=\tfrac{1}{3}(\operatorname{tr}\boldsymbol{\tau})(\operatorname{tr}\boldsymbol{\varepsilon})+\operatorname{dev}\boldsymbol{\tau}\cdot\operatorname{dev}\boldsymbol{\varepsilon}.$$

Let $\eta=\operatorname{tr}\boldsymbol{\tau}$, $\boldsymbol{s}=\operatorname{dev}\boldsymbol{\tau}$. Let us rewrite (1-6) using definition (1-3) of the norm $|||\cdot|||$:

$$|||\boldsymbol{\varepsilon}|||^*=\frac{1}{\alpha}\max_{\substack{\eta\in\mathbb{R},\,\boldsymbol{s}\in\mathbb{E}^2_s\\ \operatorname{tr}\boldsymbol{s}=0,\,\boldsymbol{s}\neq\boldsymbol{0}}}\frac{\frac{1}{3}\eta(\operatorname{tr}\boldsymbol{\varepsilon})+\boldsymbol{s}\cdot\operatorname{dev}\boldsymbol{\varepsilon}}{|\eta|+\left(\frac{\beta}{\alpha}\right)\|\boldsymbol{s}\|}. \tag{4-3}$$

Now we shall make use of the following elementary result, valid for positive a and b:

$$\max_{x\in\mathbb{R}}\frac{cx+a}{|x|+b}=\max\left(|c|,\,\frac{a}{b}\right). \tag{4-4}$$

This result will be used by interpreting

$$x=\eta,\quad c=\tfrac{1}{3}\operatorname{tr}\boldsymbol{\varepsilon},\quad a=\boldsymbol{s}\cdot\operatorname{dev}\boldsymbol{\varepsilon},\quad b=\frac{\beta}{\alpha}\|\boldsymbol{s}\|. \tag{4-5}$$

Tensor $\boldsymbol{s}$ can always be chosen such that $a>0$. Thus the norm (4-3) assumes the form

$$|||\boldsymbol{\varepsilon}|||^*=\frac{1}{\alpha}\max\left\{\tfrac{1}{3}|\operatorname{tr}\boldsymbol{\varepsilon}|,\ \frac{\alpha}{\beta}\max_{\substack{\boldsymbol{s}\in\mathbb{E}^2_s\\ \operatorname{tr}\boldsymbol{s}=0,\,\boldsymbol{s}\neq\boldsymbol{0}}}\frac{\boldsymbol{s}\cdot\operatorname{dev}\boldsymbol{\varepsilon}}{\|\boldsymbol{s}\|}\right\}, \tag{4-6}$$

hence

$$|||\boldsymbol{\varepsilon}|||^*=\max\left\{\frac{1}{3\alpha}|\operatorname{tr}\boldsymbol{\varepsilon}|,\ \frac{1}{\beta}\|\operatorname{dev}\boldsymbol{\varepsilon}\|\right\} \tag{4-7}$$

since

$$\max_{\substack{\boldsymbol{s}\in\mathbb{E}^2_s\\ \operatorname{tr}\boldsymbol{s}=0,\,\boldsymbol{s}\neq\boldsymbol{0}}}\frac{\boldsymbol{s}\cdot\operatorname{dev}\boldsymbol{\varepsilon}}{\|\boldsymbol{s}\|}=\max_{\substack{\boldsymbol{s}\in\mathbb{E}^2_s\\ \boldsymbol{s}\neq\boldsymbol{0}}}\frac{\boldsymbol{s}\cdot\operatorname{dev}\boldsymbol{\varepsilon}}{\|\boldsymbol{s}\|}=\|\operatorname{dev}\boldsymbol{\varepsilon}\|.$$

We omit the proof that $|||\cdot|||^*$ given by (4-7) is a norm in $\mathbb{E}^2_s$.

If the cost of the design is expressed by the trace of $\boldsymbol{C}$ then one should put $\alpha=\frac{\sqrt{3}}{3}$ and $\beta=\sqrt{2}$ into (4-7) to find the locking locus in the form

$$\max\left\{\tfrac{\sqrt{3}}{3}|\operatorname{tr}\boldsymbol{\varepsilon}|,\ \tfrac{\sqrt{2}}{2}\|\operatorname{dev}\boldsymbol{\varepsilon}\|\right\}\le 1. \tag{4-8}$$

The above set can be expressed in terms of principal strains upon using the formulae

$$|\operatorname{tr}\boldsymbol{\varepsilon}|=|\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{II}}+\varepsilon_{\mathrm{III}}|,\qquad \|\operatorname{dev}\boldsymbol{\varepsilon}\|=\tfrac{\sqrt{3}}{3}\sqrt{(\varepsilon_{\mathrm{I}}-\varepsilon_{\mathrm{II}})^2+(\varepsilon_{\mathrm{II}}-\varepsilon_{\mathrm{III}})^2+(\varepsilon_{\mathrm{I}}-\varepsilon_{\mathrm{III}})^2}. \tag{4-9}$$

In the space of principal strains the locking locus assumes the shape of a cylindrical domain of the axis along the vector of $\boldsymbol{e}=(1,1,1)$; see Figure 1. The length of the cylinder equals 2, while its radius equals $\frac{2\sqrt{3}}{3}$.

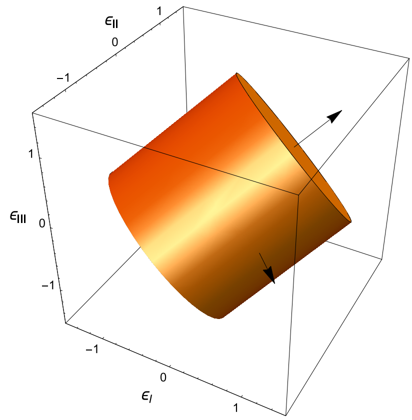

Figure 1. The locking locus for the isoperimetric condition expressed by $\operatorname{tr}\boldsymbol{C}$.

5. The question of correctness of the equilibrium problem of the optimal structure

The equilibrium of the optimal structure is governed by the following variational problem: find $\boldsymbol{u} \in V(\Omega)$ such that

$$\mathsf{a}^*(\boldsymbol{u}, \boldsymbol{v}) = f(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V(\Omega), \tag{5-1}$$

the bilinear form being associated with the optimal Hooke tensor $\boldsymbol{C}^*$

$$\mathsf{a}^*(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot (\boldsymbol{C}^* \boldsymbol{\varepsilon}(\boldsymbol{u}))\, \mathrm{d}x \tag{5-2}$$

given by the formula (3-27) in case $\alpha_2 < \alpha_3$ and by the formula (3-35) if $\alpha_2 > \alpha_3$. Consider the case of $\alpha_2 > \alpha_3$ encompassing the case of the isoperimetric condition expressed by $\operatorname{tr}\boldsymbol{C}$. The bilinear form $\mathsf{a}^*(\,\cdot\,,\,\cdot\,)$ is nonnegative, since

$$\begin{aligned}\boldsymbol{\varepsilon} \cdot (\boldsymbol{C}^* \boldsymbol{\varepsilon}) &= a^*(\boldsymbol{\varepsilon} \cdot (\boldsymbol{J}\boldsymbol{\varepsilon})) + c^*\big[\boldsymbol{\varepsilon} \cdot (\boldsymbol{S}\boldsymbol{\varepsilon}) - \boldsymbol{\varepsilon} \cdot (\boldsymbol{J}\boldsymbol{\varepsilon})\big] \\ &= \tfrac{1}{3} a^* (\operatorname{tr}\boldsymbol{\varepsilon})^2 + c^*\big(\boldsymbol{\varepsilon} \cdot (\boldsymbol{S}\boldsymbol{\varepsilon}) - \tfrac{1}{3}(\operatorname{tr}\boldsymbol{\varepsilon})^2\big) \ge 0.\end{aligned} \tag{5-3}$$

Let $\mathcal{R}$ be the kernel of the bilinear form $\mathsf{a}^*(\,\cdot\,,\,\cdot\,)$:

$$\mathcal{R} = \big\{\boldsymbol{v} \in H^1(\Omega, \mathbb{R}^3) \mid \mathsf{a}^*(\boldsymbol{v}, \boldsymbol{v}) = 0\big\}. \tag{5-4}$$

Any field v of the class $\mathcal{R}$ is such that there exists a field $\boldsymbol{w} \in H^1(\Omega, \mathbb{R}^3)$ such that

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \boldsymbol{L}\boldsymbol{\varepsilon}(\boldsymbol{w}), \tag{5-5}$$

where $\boldsymbol{L} = \overset{4}{\boldsymbol{I}} - \boldsymbol{S}$; cf. (1-9).

Let us compute the integrand of $\mathsf{a}^*(\boldsymbol{v}, \boldsymbol{v})$:

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot (\boldsymbol{C}^* \boldsymbol{\varepsilon}(\boldsymbol{v})) = (\boldsymbol{L}\boldsymbol{\varepsilon}(\boldsymbol{w})) \cdot (\boldsymbol{C}^* \boldsymbol{L}\boldsymbol{\varepsilon}(\boldsymbol{w})).$$

According to (3-35) and using (1-12) we find

$$\boldsymbol{C}^*\boldsymbol{L} = (a^*\boldsymbol{J} + c^*\boldsymbol{M})\boldsymbol{L} = a^*(\boldsymbol{J}\boldsymbol{L}) + c^*(\boldsymbol{M}\boldsymbol{L}) = \boldsymbol{0},$$

which proves that $\boldsymbol{v} \in \mathcal{R}$.

Let us proceed now to prove

$$f(\boldsymbol{v}) = 0 \quad \text{for all } \boldsymbol{v} \in \mathcal{R} \cap V(\Omega), \tag{5-6}$$

which is the main necessary condition of well-posedness of problem (5-1). Let us note that the minimizer $\boldsymbol{\tau} = \boldsymbol{\sigma}$ of (3-19) satisfies (2-1). Thus the linear form $f(\cdot)$ may be represented by

$$f(\boldsymbol{v}) = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x \quad \text{for all } \boldsymbol{v} \in V(\Omega). \tag{5-7}$$

For showing (5-6) it is sufficient to prove that

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) = 0 \tag{5-8}$$

holds if the field $\boldsymbol{v}$ is chosen such that $\boldsymbol{\varepsilon}(\boldsymbol{v}) = \boldsymbol{L}\boldsymbol{\varepsilon}(\boldsymbol{w})$ and $\boldsymbol{w} \in H^1(\Omega, \mathbb{R}^3)$. We compute

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) = \boldsymbol{\sigma} \cdot (\boldsymbol{L}\boldsymbol{\varepsilon}(\boldsymbol{w})) = \boldsymbol{\sigma} \cdot ((\overset{4}{\boldsymbol{I}} - \boldsymbol{S})\boldsymbol{\varepsilon}(\boldsymbol{w})) = \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\sigma} \cdot (\boldsymbol{S}\boldsymbol{\varepsilon}(\boldsymbol{w})). \tag{5-9}$$

Let $\overset{m}{\sigma}, \overset{m}{\boldsymbol{n}}$ be the eigenvalues and eigenvectors of the tensor $\boldsymbol{\sigma}$, $m = 1, 2, 3$. In the case when $\alpha_2 > \alpha_3$, the following representations hold:

$$\begin{aligned} S_{ijkl} &= \overset{1}{n}_i \overset{1}{n}_j \overset{1}{n}_k \overset{1}{n}_l + \overset{2}{n}_i \overset{2}{n}_j \overset{2}{n}_k \overset{2}{n}_l + \overset{3}{n}_i \overset{3}{n}_j \overset{3}{n}_k \overset{3}{n}_l, \\ \sigma_{ij} &= \overset{1}{\sigma} \overset{1}{n}_i \overset{1}{n}_j + \overset{2}{\sigma} \overset{2}{n}_i \overset{2}{n}_j + \overset{3}{\sigma} \overset{3}{n}_i \overset{3}{n}_j, \end{aligned} \tag{5-10}$$

since the triplet $(\boldsymbol{m}^*, \boldsymbol{n}^*, \boldsymbol{p}^*)$ coincides with the eigenvectors of $\boldsymbol{\sigma}$. Let us write $\eta_{ij} = \varepsilon_{ij}(\boldsymbol{w})$ for brevity and compute

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}) = \sum_{m=1}^{3} \overset{m}{\sigma} (\overset{m}{n}_i \eta_{ij} \overset{m}{n}_j) \tag{5-11}$$

as well as

$$\boldsymbol{\sigma} \cdot (\boldsymbol{S}\boldsymbol{\varepsilon}(\boldsymbol{w})) = \boldsymbol{\sigma} \cdot (\boldsymbol{S}\boldsymbol{\eta}) = \sigma_{ij} S_{ijkl} \eta_{kl} = \sum_{m=1}^{3} \sigma_{ij} \overset{m}{n}_i \overset{m}{n}_j \overset{m}{n}_k \overset{m}{n}_l \eta_{kl}. \tag{5-12}$$

Taking into account that

$$\sigma_{ij} \overset{m}{n}_i \overset{m}{n}_j = \overset{m}{\sigma}, \tag{5-13}$$

we find

$$\boldsymbol{\sigma} \cdot (\boldsymbol{S}\boldsymbol{\varepsilon}(\boldsymbol{w})) = \sum_{m=1}^{3} \overset{m}{\sigma} (\overset{m}{n}_k \eta_{kl} \overset{m}{n}_l), \tag{5-14}$$

which proves that $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) = 0$ if $\boldsymbol{v}$ satisfies the condition (5-5). Consequently, the condition (5-6) is fulfilled.

Assume that the problem (5-1) possesses two solutions $\overset{1}{\boldsymbol{u}}, \overset{2}{\boldsymbol{u}}$. Then

$$\mathsf{a}^*(\overset{1}{\boldsymbol{u}} - \overset{2}{\boldsymbol{u}}, \boldsymbol{v}) = 0 \quad \text{for all } \boldsymbol{v} \in V(\Omega). \tag{5-15}$$

Let us choose $\boldsymbol{v} = \overset{1}{\boldsymbol{u}} - \overset{2}{\boldsymbol{u}}$ to conclude that $\overset{1}{\boldsymbol{u}} - \overset{2}{\boldsymbol{u}} \in \mathcal{R}$. By (5-5) a field $\boldsymbol{w}$ exists such that

$$\boldsymbol{\varepsilon}(\overset{1}{\boldsymbol{u}} - \overset{2}{\boldsymbol{u}}) = \boldsymbol{L}\boldsymbol{\varepsilon}(\boldsymbol{w}). \tag{5-16}$$

Then

$$\boldsymbol{C}^*\boldsymbol{\varepsilon}(\overset{1}{\boldsymbol{u}}) - \boldsymbol{C}^*\boldsymbol{\varepsilon}(\overset{2}{\boldsymbol{u}}) = 0 \tag{5-17}$$

since

$$\boldsymbol{C}^*\boldsymbol{L} = \boldsymbol{0}.$$

Hence to both the fields $\overset{1}{\boldsymbol{u}}$ and $\overset{2}{\boldsymbol{u}}$ the same stress tensor is assigned:

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{C}^*\boldsymbol{\varepsilon}(\overset{1}{\boldsymbol{u}}) = \boldsymbol{C}^*\boldsymbol{\varepsilon}(\overset{2}{\boldsymbol{u}}). \tag{5-18}$$

We conclude that the stress field in the optimal structure is uniquely determined, provided it exists. We have previously noted that this stress field is simultaneously the minimizer of the problem (3-19), except for some specific cases when the eigenvectors of the stress field are not uniquely determined.

Let us stress that the displacement field and strain field are not uniquely determined, but this property does not contradict uniqueness of the stress field. Moreover, the compliance $f(\boldsymbol{u})$ is uniquely determined since $\boldsymbol{u}$, a solution to (5-1), is determined modulo fields from $\mathcal{R}$ on which the linear form $f(\cdot)$ vanishes.

6. Final remarks

The minimum compliance problem (2-4) has been reduced to the auxiliary, stress-based problem (3-19) with the integrand of linear growth. A numerical method for this problem is now available; it has been successfully developed by Czarnecki [2015] to construct optimum isotropic bodies. Case studies concerning cubic symmetry will be published in separate papers.

The optimal designs depend drastically on the sign of $(\alpha_2 - \alpha_3)$, α_i being the weight coefficients in the isoperimetric condition (1-17). In case of $\alpha_2 < \alpha_3$ the eigenvalues of the optimal Hooke tensor $\boldsymbol{C}^*$ are $(a, b, b, b, 0, 0)$. In case of $\alpha_2 > \alpha_3$ these eigenvalues are $(a, 0, 0, 0, c, c)$. Moreover, in the latter case the stress field in the optimal body coincides (except for very specific cases) with the stress field being the minimizer of the auxiliary problem (3-19). In particular, the trajectories of both the stress fields are the same.

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RADOSŁAW CZUBACKI: r.czubacki@il.pw.edu.pl
Department of Structural Mechanics and Computer Aided Engineering, Faculty of Civil Engineering, Warsaw University of Technology, ul. Armii Ludowej 16, 00-637 Warsaw, Poland

TOMASZ LEWIŃSKI: t.lewinski@il.pw.edu.pl
Department of Structural Mechanics and Computer Aided Engineering, Faculty of Civil Engineering, Warsaw University of Technology, ul. Armii Ludowej 16, 00-637 Warsaw, Poland

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