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THE EFFECT OF SMALL SCALE ON THE FREE VIBRATION OF FUNCTIONALLY GRADED TRUNCATED CONICAL SHELLS

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In this paper, the FG thin truncated conical shell formulation is developed using the modified couple stress theory. The material distributions in FG conical shell are assumed to vary continuously along shell thickness according to volume fraction of constituents based on power law distribution. The governing equations and boundary conditions are derived using Hamilton's principle, and, in the special case, the free vibration of the simply supported FG conical nanoshell is investigated using Galerkin method. Finally, the effects of parameters such as dimensionless length scale parameter, apex angle, gradient index and length on the natural frequency are examined. According to the studies conducted, the modified couple stress theory predicts the stiffness of conical nanoshell with higher accuracy than the classical continuum theory. Besides, the increasing effect of the length scale parameter on increase in natural frequency caused by decrease in length and increase in circumferential and axial wave numbers is investigated as well.

1. Introduction

Compared to homogeneous materials, laminated composite materials are considered as structural elements in different industries because of their high stiffness-weight and strength-weight ratios and the ability of changing structural properties to meet specific needs. However, due to discontinuity existing in material properties in the interfaces between two different materials, the stress concentration resulting in cracks and delamination phenomenon has rendered the use of these materials difficult [Sahoo and Singh 2014; Xie et al. 2014; Li et al. 2014; Furlotti et al. 2014]. Today, FGMs which are made up two isotropic materials — e.g., metal and ceramics — and enjoy the benefits of both materials are in demand by many industries [Tajalli et al. 2013; Khalili et al. 2010; Shahba and Rajasekaran 2012]. The material properties of FGMs are variable continuously and smoothly along a definite direction from one surface to the next, solving the delamination problem in composite structures [Jomehzadeh et al. 2009]. The material's high toughness, low density and thermal resistance are properties that appeal to industries as varied as the biomedical, defense and aerospace industries [Kahrobaiyan et al. 2012]. Hence, it is necessary to study the behavior of FGMs in order to make a correct prediction of their static and dynamic behavior, as various studies have to date been carried out on them [Tadi Beni et al. 2015a; 2015b; Tornabene and Viola 2013; Sankar 2001; Aydogdu and Taskin 2007; Tornabene et al. 2015; Ying et al. 2008; Viola et al. 2012]. For example, using Navier's solution for the rectangular plate and finite element model based on the third order shear deformation plate theory, Reddy investigated the deformations and stresses under the influence of material distribution based on the linear third-order theory and the nonlinear first order

Keywords: free vibration, modified couple stress theory, thin shell model, functionally graded material, truncated conical shell, length scale parameter.

theory by using von Karman's geometric nonlinearity [Reddy 2000]. Li studied the dynamic and static behaviors of the functionally graded beam by considering rotational inertia and shear deformation [Li 2008]. Using higher order shear and normal deformation theories, Kant et al. [2010] investigated the displacements and stresses of the functionally graded beams and plates and examined the analytical formulation of static behavior of simply supported functionally graded beams and plates based on higher order theories.

Today, FGMs have extensive applications in nano/micro structures such as micro/nano-electromechanical systems (MEMS/NEMS), thin films in the shape of memory alloys and atomic force microscopes (AFMs) [Craciunescu and Wuttig 2003; Fu et al. 2004; Witvrouw and Mehta 2005]. The classical continuum theory is used by most of the aforementioned studies to study micro-nanoscale devices [Yoon et al. 2005; Zhang et al. 2005; Ansari et al. 2010]. However, the results obtained in the nanoscale reveal a difference between the predictions of the results based on the classical continuum theory and the experimental results in the nanoscale. For instance, McFarland et al. observed a considerable difference between the values of stiffness obtained by the classical continuum theory and those obtained by the bending test of a polypropylene micro cantilever [McFarland and Colton 2005]. Therefore, the appropriate theory is required to correct investigation of the FGMs behavior in micro-nanodimensions. On the other hand, obtained experimental results demonstrate that mechanical properties of materials in the micro-nanoscale are size-dependent; hence, the theories should be able to correctly predict and investigate size-dependent behavior in micro-nanostructures. It should be noted that some conventional methods used in the study of micro-nanostructures are: atomistic simulations, molecular dynamics, and higher order continuum theories such as the modified couple stress theory, the strain gradient theory, and the nonlocal theory. Using these theories and by considering the size effect, researchers have to date conducted many studies showing the capabilities of these theories [Tadi Beni 2012; Tadi Beni and Abadyan 2013; Tadi Beni et al. 2012; Ke et al. 2012; Zeverdejani and Tadi Beni 2013; Abadyan et al. 2011; Sahmani and Ansari 2013; Wang et al. 2013].

The couple stress theory, as a nonclassical continuum theory containing higher order stresses is initially introduced by Toupin [1962], Mindlin and Tiersten [1962], Mindlin [1964] and Koiter [1964a; 1964b]. This theory which contains four material parameters, two lame constants and two higher order material constants is employed by researchers such as Anthoine [2000] in an attempt to investigate beam bending. Using moments of couples equilibrium equation as well as forces and moments of forces equilibrium equations, the modified couple stress theory is introduced by Yang et al. [2002]. Many studies are carried out using this theory which contains only one higher order material parameter [Roque et al. 2013; Sahmani et al. 2013; Şimşek et al. 2013; Şimşek and Reddy 2013]. For example, using the modified couple stress theory, Abbasnejad et al. [2013] examined the dynamic and static pull-in voltage of a FGM microbeam under electrostatic load and demonstrated the effect of the distribution of materials in beam's thickness on the pull-in voltage. Tadi Beni et al. [2014] employed the modified couple stress theory to examine the pull-in instability of rotational nano-mirror and to demonstrate the effect of the Casimir force on size-dependent pull-in instability. By considering nonlinear geometry and using the modified couple stress theory, Ansari et al. [2014] studied the dynamic behavior of functionally graded rectangular Mindlin microplate and discussed the effects of parameters such as length scale parameter and boundary conditions on the rectangular microplate. Based on the modified couple stress theory, Zeighampour et al. [2014b] investigated vibration and instability of double-walled carbon nanotube conveying fluid using

Donnell's shell model. They found that the effects of parameters such as length and length scale parameter are more considerable in the modified couple stress theory than in the classical continuum theory.

Since using an appropriate model is essential for studying the behavior of structures, the investigation of the nanoscaled structure requires an appropriate model. So far, many studies are carried out to investigate the mechanical behavior of nanostructures using the beam model [Ansari et al. 2013; Mohammad Abadi and Daneshmehr 2014; Şimşek 2010]. However, it should be noted that in these investigations, nanoelements such as CNTs are also usually modeled as beams. Considering the geometry of many nanostructure elements, which are conical or cylindrical, usage the shell model will certainly results in correct prediction of the behavior of such structures. Thus, researchers are induced to investigate the mechanical and static behavior of conical shell, which is one of the appealing structural elements to many researchers due to its unique geometric properties and numerous applications [Dung et al. 2013; Sofiyev 2009; Viola et al. 2014; Su et al. 2014]. Many studies, such as [Zhao and Liew 2011; Malekzadeh and Daraie 2014], investigated the free vibration and dynamic behavior of functionally graded conical shell panels and truncated conical shell respectively, have to date conducted on conical shell. Therefore, today, considering the increasing growth of nanoscience and the high potentials of the shell model due to its unique characteristics, it is highly essential to investigate static and dynamic behaviors of the conical nanoshell. For that reason, researchers such as Zeighampour and Tadi Beni [2014a; 2015] examined the behavior of single-walled carbon nanocones (SWCNC) using the modified couple stress theory.

Therefore, this study attempts to gain a better understanding of conical nanoshells by examining the dynamic behavior of FGM conical nanoshells using the modified couple stress theory. In this paper, the equations of the FG conical nanoshell are developed using Love's thin shell theory and the modified couple stress theory, proposing considerable advantages in the modeling of nanostructures. These advantages include modeling the size effect with only one higher order material constant using the modified couple stress theory for nanoscale calculations; modeling FGMs which expands the formulation to the FG conical nanoshell and which can be extended for isotropic materials in special cases; and the ability to obtain more precise results using the shell model for structure behavior in the nanoscale. To achieve this goal, considering the above discussion, in this paper the modified couple stress theory and the thin shell model are used to investigate the free vibration of FG truncated conical nanoshell. Using the Hamilton's principle, the governing equations and the boundary conditions are derived, and, the free vibration of the simply supported FGM conical thin shell are studied in the special case using the Galerkin method. Finally, the effects of parameters such as length, dimensionless length scale parameter and apex angle on the vibration of this nanostructure are investigated as well.

2. Governing equations

Figure 1 illustrates an FG truncated conical shell of length L, radius R_1, thickness h, and apex angle 2α, and the coordinate system (x, θ, z) is located on the middle surface of the conical shell.

Based on the power law distribution, the material properties of the FG conical shell are assumed to make up a mixture of metal and ceramic, are variable along shell thickness, and, based on the volume fraction of elements, these properties are considered continuously variable from the inner shell surface to outer one. The volume fraction of FG components is

$$V_m = (\tilde{z}/h)^N, \quad V_c = 1 - V_m, \tag{1}$$

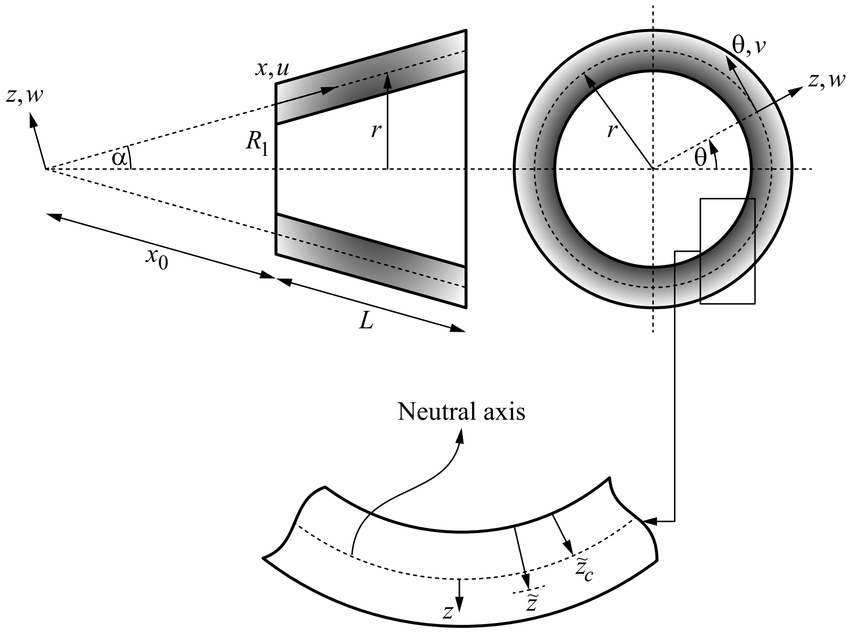

Figure 1. Schematic view of truncated conical shell.

where N is defined as the power index whose variation takes place in the $0 \leq N \leq \infty$ interval and according to Figure 1, $\tilde{z}$ is the distance of an arbitrary surface from the inner one of the conical shell. Therefore, the material properties of the conical shell are defined as

$$\begin{aligned} E(\tilde{z}) &= (E_m - E_c)\left(\frac{\tilde{z}}{h}\right)^N + E_c, \\ \rho(\tilde{z}) &= (\rho_m - \rho_c)\left(\frac{\tilde{z}}{h}\right)^N + \rho_c, \\ \nu(\tilde{z}) &= (\nu_m - \nu_c)\left(\frac{\tilde{z}}{h}\right)^N + \nu_c, \end{aligned} \tag{2}$$

where E_c, ρ_c and ν_c respectively represent Young's modulus, density, and Poisson's ratio of ceramics when $\tilde{z} = 0$, and, E_m, ρ_m and ν_m represent Young's modulus, density, and Poisson's ratio of metals, respectively, when $\tilde{z} = h$.

According to the modified couple stress theory introduced by Yang et al. [2002], the strain energy is expressed as a function of strain tensor and curvature tensor as

$$U = \frac{1}{2}\int_{\Omega} (\boldsymbol{\sigma} : \boldsymbol{\epsilon} + \boldsymbol{m} : \boldsymbol{\chi})\, dV, \tag{3}$$

where $\boldsymbol{\epsilon}$ stands for the strain tensor, $\boldsymbol{\sigma}$ represents the Cauchy stress tensor, $\boldsymbol{\chi}$ is the symmetric curvature tensor and $\boldsymbol{m}$ is the deviatoric part of couple stress tensor, and, the components of the introduced tensors are defined as

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{4}$$

$$\chi_{ij} = \tfrac{1}{4}(e_{ipq}\eta_{jpq} + e_{jpq}\eta_{ipq}), \tag{5}$$

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}, \tag{6}$$

$$m_{ij} = 2l^2\mu(\tilde{z})\chi_{ij}, \tag{7}$$

where u_i, e_{ipq} and η_{ipq} are the components of the displacement vector, permutation symbol and deviatoric stretch gradient tensor, respectively, and l stands for the material length scale parameter. It should be noted that this study is ignored the variation of parameter l with material coordinates, and the value of l in the whole of the FG material is considered constant.

With the assumption of plane stress, the stress-strain equations for the FGM are defined as

$$\sigma_{xx} = \frac{E(\tilde{z})}{1-\nu^2(\tilde{z})}(\epsilon_{xx} + \nu(\tilde{z})\epsilon_{\theta\theta}), \quad \sigma_{\theta\theta} = \frac{E(\tilde{z})}{1-\nu^2(\tilde{z})}(\epsilon_{\theta\theta} + \nu(\tilde{z})\epsilon_{xx}), \quad \sigma_{x\theta} = 2\mu(\tilde{z})\epsilon_{x\theta}, \tag{8}$$

where $\mu(\tilde{z})$ is the shear modulus and $E(\tilde{z})$ and $\nu(\tilde{z})$ are Young's modulus and Poisson's ratio, respectively. The displacement field of the conical shell based on Love's thin shell along x, θ, z represented respectively by u, v, w is defined [Leissa 1993] as

$$\begin{aligned} u(x,\theta,z,t) &= U(x,\theta,t) - z\frac{\partial W(x,\theta,t)}{\partial x}, \\ v(x,\theta,z,t) &= V(x,\theta,t) - \frac{z}{x\sin\alpha}\left(\frac{\partial W(x,\theta,t)}{\partial\theta} - V(x,\theta,t)\cos\alpha\right), \\ w(x,\theta,z,t) &= W(x,\theta,t), \end{aligned} \tag{9}$$

where $U(x,\theta,t)$, $V(x,\theta,t)$, and $W(x,\theta,t)$ are neutral surface displacement of the conical shell, and the position of this plate $(\tilde{z}_c)$ is determined by using the equilibrium of longitudinal forces [Barber 2011] as

$$\int_A \sigma_{xx}\,dA = \int_A \frac{E(\tilde{z})}{1-\nu^2(\tilde{z})}\left(z\frac{\partial^2 w}{\partial x^2}\right)dA = 0, \tag{10}$$

where

$$z = \tilde{z} - \tilde{z}_c. \tag{11}$$

By substituting (11) into (10), we obtain

$$\tilde{z}_c = \frac{\displaystyle\int_A \frac{E(\tilde{z})}{1-\nu^2(\tilde{z})}\tilde{z}\,dA}{\displaystyle\int_A \frac{E(\tilde{z})}{1-\nu^2(\tilde{z})}\,dA}. \tag{12}$$

The classic and nonclassical strain tensors in orthogonal coordinate system are obtained by using the equations [Eringen 1980; Zhao and Pedroso 2008]

$$\epsilon_{(j)}^{(i)}=\epsilon_j^i\sqrt{\frac{g_{\underline{ii}}}{g_{\underline{jj}}}}=\frac{1}{2}\sqrt{\frac{g_{\underline{ii}}}{g_{\underline{jj}}}}\bigg[\bigg(\bigg(\frac{u^{(i)}}{\sqrt{g_{\underline{ii}}}}\bigg)_{,j}+\Gamma_{mj}^i\frac{u^{(m)}}{\sqrt{g_{\underline{mm}}}}\bigg)+g_{nj}g^{im}\bigg(\bigg(\frac{u^{(n)}}{\sqrt{g_{\underline{nn}}}}\bigg)_{,m}+\Gamma_{qm}^n\frac{u^{(q)}}{\sqrt{g_{\underline{qq}}}}\bigg)\bigg], \quad (13)$$

$$\eta_{(i)(j)}^{(k)}=\eta_{ij}^k\sqrt{\frac{g_{\underline{kk}}}{g_{\underline{ii}}g_{\underline{jj}}}}=\frac{1}{2}\sqrt{\frac{g_{\underline{kk}}}{g_{\underline{ii}}g_{\underline{jj}}}}(u^k|_{ij}+u^k|_{ji}), \quad (14)$$

$$u^k|_{lm}=\bigg(\frac{u^{(k)}}{\sqrt{g_{\underline{kk}}}}\bigg)_{,lm}+\Gamma_{ql}^k\bigg(\frac{u^{(q)}}{\sqrt{g_{\underline{qq}}}}\bigg)_{,m}+\Gamma_{qm}^k\bigg(\frac{u^{(q)}}{\sqrt{g_{\underline{qq}}}}\bigg)_{,l}-\Gamma_{ml}^q\bigg(\frac{u^{(k)}}{\sqrt{g_{\underline{kk}}}}\bigg)_{,q} \\ +\big((\Gamma_{lp}^k)_{,m}+\Gamma_{qm}^k\Gamma_{pl}^q-\Gamma_{pq}^k\Gamma_{ml}^q\big)\bigg(\frac{u^{(p)}}{\sqrt{g_{\underline{pp}}}}\bigg), \quad (15)$$

where $u^{(i)}$, $\epsilon_{(j)}^{(i)}$ and $\eta_{(i)(j)}^{(k)}$ are the physical components of displacement vector u^i, displacement gradient ϵ_{ij} and higher-order displacement gradient η_{ij}^k, and g_{ii} and Γ_{jk}^i stand for components of the metric tensor and Christoffel symbols of the second kind. Underscores placed under the indices indicate lack of addition on them. In the cylindrical coordinate system, the components of metric tensor and Christoffel symbol are expressed as

$$\begin{aligned} &g_{xx}=1, \quad g_{\theta\theta}=(x\sin\alpha(1+z/x\tan\alpha))^2,\\ &g_{zz}=1, \quad g_{kl}=0\ (k\neq l),\\ &\Gamma_{z\theta}^\theta=\Gamma_{\theta z}^\theta=\frac{1}{x\tan\alpha(1+z/x\tan\alpha)},\\ &\Gamma_{\theta\theta}^z=-x\sin\alpha\cos\alpha(1+z/x\tan\alpha),\\ &\Gamma_{x\theta}^\theta=\Gamma_{\theta x}^\theta=\frac{1}{x(1+z/x\tan\alpha)},\\ &\Gamma_{\theta\theta}^x=-x\sin^2\alpha(1+z/x\tan\alpha). \end{aligned} \quad (16)$$

Classical strain tensors components are obtained by substituting (16) into (13) as

$$\begin{aligned} &\epsilon_{zz}=\frac{\partial w}{\partial z}, \quad \epsilon_{\theta\theta}=\frac{1}{x\sin\alpha(1+z/x\tan\alpha)}\bigg[\frac{\partial v}{\partial\theta}+u\sin\alpha+w\cos\alpha\bigg], \quad \epsilon_{xx}=\frac{\partial u}{\partial x},\\ &\epsilon_{x\theta}=\epsilon_{\theta x}=\frac{1}{2x\sin\alpha(1+z/x\tan\alpha)}\bigg[\frac{\partial u}{\partial\theta}+x\sin\alpha(1+z/x\tan\alpha)\frac{\partial v}{\partial x}-v\sin\alpha\bigg],\\ &\epsilon_{\theta z}=\epsilon_{z\theta}=\frac{1}{2x\sin\alpha(1+z/x\tan\alpha)}\bigg[\frac{\partial w}{\partial\theta}+x\sin\alpha(1+z/x\tan\alpha)\frac{\partial v}{\partial z}-v\cos\alpha\bigg],\\ &\epsilon_{zx}=\epsilon_{xz}=\frac{1}{2}\bigg[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\bigg], \end{aligned} \quad (17)$$

and the components of the deviatoric stretch gradient tensor are obtained by substituting (15) and (16) into (14) as

$$\eta_{xxx}=\frac{\partial^2 u}{\partial x^2}, \quad \eta_{zzz}=\frac{\partial^2 w}{\partial z^2}, \quad \eta_{xx\theta}=\frac{\partial^2 v}{\partial x^2}, \quad \eta_{zz\theta}=\frac{\partial^2 v}{\partial z^2},$$

$$
\begin{aligned}
\eta_{x\theta} = \eta_{\theta x\theta} &= \frac{\dfrac{\partial^2 v}{\partial x\,\partial\theta} - \dfrac{1}{x(1+z/x\tan\alpha)}\dfrac{\partial v}{\partial\theta} - \dfrac{u\sin\alpha + w\cos\alpha}{x(1+z/x\tan\alpha)} + \cos\alpha\dfrac{\partial w}{\partial x} + \sin\alpha\dfrac{\partial u}{\partial x}}{x\sin\alpha(1+z/x\tan\alpha)},\\
\eta_{z\theta\theta} = \eta_{\theta z\theta} &= \frac{\dfrac{\partial^2 v}{\partial z\,\partial\theta} - \dfrac{1}{x\tan\alpha(1+z/x\tan\alpha)}\dfrac{\partial v}{\partial\theta} - \dfrac{w\cos\alpha + u\sin\alpha}{x\tan\alpha(1+z/x\tan\alpha)} + \cos\alpha\dfrac{\partial w}{\partial z} + \sin\alpha\dfrac{\partial u}{\partial z}}{x\sin\alpha(1+z/x\tan\alpha)},\\
\eta_{\theta xx} = \eta_{x\theta x} &= \frac{\dfrac{\partial^2 u}{\partial x\,\partial\theta} - \dfrac{1}{x(1+z/x\tan\alpha)}\dfrac{\partial u}{\partial\theta} - \sin\alpha\dfrac{\partial v}{\partial x} + \dfrac{v\sin\alpha}{x(1+z/x\tan\alpha)}}{x\sin\alpha(1+z/x\tan\alpha)},\\
\eta_{\theta zz} = \eta_{z\theta z} &= \frac{\dfrac{\partial^2 w}{\partial z\,\partial\theta} - \dfrac{1}{x\tan\alpha(1+z/x\tan\alpha)}\dfrac{\partial w}{\partial\theta} - \cos\alpha\dfrac{\partial v}{\partial z} + \dfrac{v\cos\alpha}{x\tan\alpha(1+z/x\tan\alpha)}}{x\sin\alpha(1+z/x\tan\alpha)},\\
\eta_{\theta\theta x} &= \frac{\dfrac{\partial^2 u}{\partial\theta^2} - u\sin^2\alpha - w\sin\alpha\cos\alpha - 2\sin\alpha\dfrac{\partial v}{\partial\theta} + x\sin\alpha(1+z/x\tan\alpha)\left(\cos\alpha\dfrac{\partial u}{\partial z} + \sin\alpha\dfrac{\partial u}{\partial x}\right)}{(x\sin\alpha(1+z/x\tan\alpha))^2},\\
\eta_{\theta\theta z} &= \frac{\dfrac{\partial^2 w}{\partial\theta^2} - w\cos^2\alpha - u\sin\alpha\cos\alpha - 2\cos\alpha\dfrac{\partial v}{\partial\theta} + x\sin\alpha(1+z/x\tan\alpha)\left(\cos\alpha\dfrac{\partial w}{\partial z} + \sin\alpha\dfrac{\partial w}{\partial x}\right)}{(x\sin\alpha(1+z/x\tan\alpha))^2},\\
\eta_{\theta\theta\theta} &= \frac{\dfrac{\partial^2 v}{\partial\theta^2} - v + x\sin^2\alpha(1+z/x\tan\alpha)\dfrac{\partial v}{\partial x} + 2\sin\alpha\dfrac{\partial u}{\partial\theta} + 2\cos\alpha\dfrac{\partial w}{\partial\theta} + x\cos\alpha\sin\alpha(1+z/x\tan\alpha)\dfrac{\partial v}{\partial z}}{(x\sin\alpha(1+z/x\tan\alpha))^2},\\
\eta_{x\theta z} = \eta_{\theta xz} &= \frac{\dfrac{\partial^2 w}{\partial x\,\partial\theta} - \dfrac{1}{x(1+z/x\tan\alpha)}\dfrac{\partial w}{\partial\theta} - \cos\alpha\dfrac{\partial v}{\partial x} + \dfrac{v\cos\alpha}{x(1+z/x\tan\alpha)}}{x\sin\alpha(1+z/x\tan\alpha)},\\
\eta_{z\theta x} = \eta_{\theta zx} &= \frac{\dfrac{\partial^2 u}{\partial z\,\partial\theta} - \dfrac{1}{x\tan\alpha(1+z/x\tan\alpha)}\dfrac{\partial u}{\partial\theta} - \sin\alpha\dfrac{\partial v}{\partial z} + \dfrac{v\cos\alpha}{x(1+z/x\tan\alpha)}}{x\sin\alpha(1+z/x\tan\alpha)},\\
\eta_{xxz} &= \frac{\partial^2 w}{\partial x^2}, \quad \eta_{zzx} = \frac{\partial^2 u}{\partial z^2}, \quad \eta_{zx\theta} = \eta_{xz\theta} = \frac{\partial^2 v}{\partial x\,\partial z}.
\end{aligned}
\tag{18}
$$

Now, by substituting the displacement field according to (9) into (17) and (18) and using (5) and the assumptions of Love's thin shell theory as $(1 \pm z/x\tan\alpha) \approx 1$ and $(z/x\tan\alpha)^2 \approx 0$, the classical and nonclassical nonzero strain components are respectively obtained as

$$
\begin{aligned}
\epsilon_{xx} &= \frac{\partial U}{\partial x} - z\frac{\partial^2 W}{\partial x^2},\\
\epsilon_{\theta\theta} &= \frac{1}{x\sin\alpha}\left[\frac{\partial V}{\partial\theta} + U\sin\alpha + W\cos\alpha - \frac{z}{x\sin\alpha}\frac{\partial^2 W}{\partial\theta^2} - z\sin\alpha\frac{\partial W}{\partial x}\right],\\
\epsilon_{x\theta} = \epsilon_{\theta x} &= \frac{1}{2x\sin\alpha}\left[\frac{\partial U}{\partial\theta} + x\sin\alpha\frac{\partial V}{\partial x} - V\sin\alpha - 2z\frac{\partial^2 W}{\partial x\,\partial\theta} + \frac{2z}{x}\frac{\partial W}{\partial\theta}\right],
\end{aligned}
\tag{19}
$$

and

$$\chi_{zz} = \frac{1}{2x^2\sin^2\alpha}\left[\cos\alpha\frac{\partial U}{\partial\theta} + x\sin\alpha\cos\alpha\left[\frac{\partial V}{\partial x} - \frac{V}{x}\right] - 2z\cos\alpha\frac{\partial^2 W}{\partial x\,\partial\theta} + \frac{2z\cos\alpha}{x}\frac{\partial W}{\partial\theta}\right],$$

$$\chi_{xx} = \frac{1}{x\sin\alpha}\left[\cos\alpha\left[\frac{V}{x} - \frac{\partial V}{\partial x}\right] + \frac{\partial^2 W}{\partial x\,\partial\theta} - \frac{1}{x}\frac{\partial W}{\partial\theta}\right],$$

$$\chi_{\theta\theta} = \frac{1}{2x\sin\alpha}\left[\cos\alpha\left[\frac{\partial V}{\partial x} - \frac{V}{x}\right] - \frac{1}{x\tan\alpha}\frac{\partial U}{\partial\theta} - 2\frac{\partial^2 W}{\partial x\,\partial\theta} + \frac{2}{x}\frac{\partial W}{\partial\theta}\right],$$

$$\chi_{x\theta} = \chi_{\theta x} = \frac{1}{2x^2\sin^2\alpha}\left[\frac{\partial^2 W}{\partial\theta^2} - x^2\sin^2\alpha\frac{\partial^2 W}{\partial x^2} + x\sin^2\alpha\frac{\partial W}{\partial x} - \cos\alpha\frac{\partial V}{\partial\theta}\right],$$

$$\chi_{z\theta} = \chi_{\theta z} = \frac{1}{4x^2\sin^2\alpha}\left[x\sin\alpha\frac{\partial^2 V}{\partial x\,\partial\theta} + \sin\alpha\frac{\partial V}{\partial\theta} - \frac{\partial^2 U}{\partial\theta^2} + 2x\sin\alpha\cos\alpha\frac{\partial W}{\partial x}\right],$$

$$\chi_{xz} = \chi_{zx} = \frac{1}{4}\left[\frac{1}{x}\frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial x^2} - \frac{V}{x^2} + \frac{1}{x\sin\alpha}\left[\frac{1}{x}\frac{\partial U}{\partial\theta} - \frac{\partial^2 U}{\partial x\,\partial\theta}\right]\right]. \tag{20}$$

Now, by substituting Equations (19) and (20) into Equations (6) and (7) and determining the components of classical and nonclassical stresses, the strain energy of the conical shell is obtained using (3) as

$$\begin{aligned}
U_s = \frac{1}{2}\int_0^{2\pi}\int_{x_0}^{x_0+L}\Bigg[&[N_{xx}]\frac{\partial U}{\partial x} + \left[\frac{N_{x\theta}}{x\sin\alpha} - \frac{Y_{\theta\theta}\cos\alpha}{2x^2\sin^2\alpha} + \frac{Y_{zz}\cos\alpha}{2x^2\sin^2\alpha} + \frac{Y_{xz}}{2x^2\sin\alpha}\right]\frac{\partial U}{\partial\theta} \\
&- \left[\frac{Y_{z\theta}}{2x^2\sin^2\alpha}\right]\frac{\partial^2 U}{\partial\theta^2} - \left[\frac{Y_{zx}}{2x\sin\alpha}\right]\frac{\partial^2 U}{\partial x\,\partial\theta} + \left[\frac{N_{\theta\theta}}{x} - \frac{N_{xx}}{x}\right]U + \left[\frac{Y_{z\theta}}{2x\sin\alpha}\right]\frac{\partial^2 V}{\partial x\,\partial\theta} \\
&+ \left[\frac{Y_{zx}}{2}\right]\frac{\partial^2 V}{\partial x^2} + \left[N_{x\theta} - \frac{Y_{xx}}{x\tan\alpha} + \frac{Y_{zz}}{2x\tan\alpha} + \frac{Y_{\theta\theta}}{2x\tan\alpha} + \frac{Y_{xz}}{2x}\right]\frac{\partial V}{\partial x} \\
&+ \left[\frac{N_{\theta\theta}}{x\sin\alpha} - \frac{Y_{x\theta}\cos\alpha}{x^2\sin^2\alpha} + \frac{Y_{z\theta}}{2x^2\sin\alpha}\right]\frac{\partial V}{\partial\theta} \\
&+ \left[-\frac{N_{x\theta}}{x} + \frac{Y_{xx}}{x^2\tan\alpha} - \frac{Y_{zz}}{2x^2\tan\alpha} - \frac{Y_{\theta\theta}}{2x^2\tan\alpha} - \frac{Y_{xz}}{2x^2}\right]V \\
&+ \left[\frac{2M_{xx}}{x} - \frac{M_{\theta\theta}}{x} + \frac{Y_{z\theta}}{x\tan\alpha} + \frac{Y_{x\theta}}{2x}\right]\frac{\partial W}{\partial x} - [M_{xx} + Y_{x\theta}]\frac{\partial^2 W}{\partial x^2} + \left[\frac{N_{\theta\theta}}{x\tan\alpha}\right]W \\
&- \left[\frac{M_{\theta\theta}}{x^2\sin^2\alpha} - \frac{Y_{x\theta}}{x^2\sin^2\alpha}\right]\frac{\partial^2 W}{\partial\theta^2} + \left[\frac{2M_{x\theta}}{x^2\sin\alpha} - \frac{Y_{xx}}{x^2\sin\alpha} + \frac{Y_{\theta\theta}}{x^2\sin\alpha} + \frac{T_{zz}\cos\alpha}{x^3\sin^2\alpha}\right]\frac{\partial W}{\partial\theta} \\
&- \left[\frac{2M_{x\theta}}{x\sin\alpha} - \frac{Y_{xx}}{x\sin\alpha} + \frac{Y_{\theta\theta}}{x\sin\alpha} + \frac{T_{zz}\cos\alpha}{x^2\sin^2\alpha}\right]\frac{\partial^2 W}{\partial x\,\partial\theta}\Bigg]x\sin\alpha\,dx\,d\theta,
\end{aligned} \tag{21}$$

where the classical and nonclassical force and momentum are expressed as

$$N_{ij}=\int_{-\tilde{z}_c}^{h-\tilde{z}_c}\sigma_{ij}\,dz,\quad M_{ij}=\int_{-\tilde{z}_c}^{h-\tilde{z}_c}\sigma_{ij}z\,dz,\quad Y_{ij}=\int_{-\tilde{z}_c}^{h-\tilde{z}_c}m_{ij}\,dz,\quad T_{ij}=\int_{-\tilde{z}_c}^{h-\tilde{z}_c}m_{ij}z\,dz, \tag{22}$$

and the kinetic energy of the conical thin shell is defined as

$$T=\frac{1}{2}\int_{\Omega}\rho(\tilde{z})\left[\left(\frac{\partial U}{\partial t}\right)^2+\left(\frac{\partial V}{\partial t}\right)^2+\left(\frac{\partial W}{\partial t}\right)^2\right]x\sin\alpha\,dx\,d\theta\,dz, \tag{23}$$

where $\rho(\tilde{z})$ stands for the density of the conical shell.

Also, the work done by external forces acting on the conical shell is expressed as

$$w_e=w_d+w_b, \tag{24}$$

$$w_d=\int_\theta\int_x(f_xU+f_\theta V+f_zW)x\sin\alpha\,dx\,d\theta, \tag{25}$$

$$\begin{aligned}w_b=&\int_\theta\left[\overline{N}_x^uU+\overline{N}_x^vV+\overline{N}_x^wW+\hat{P}_x^{vh}\frac{\partial V}{\partial x}+\overline{M}_x^w\frac{\partial W}{\partial x}\right]x\sin\alpha\,d\theta\Big|_{x_0}^{x_0+L},\\&+\int_x\left[\overline{N}_\theta^uU+\overline{N}_\theta^vV+\overline{N}_\theta^wW+\hat{P}_\theta^{uh}\frac{\partial U}{\partial\theta}+\overline{M}_\theta^w\frac{\partial W}{\partial\theta}\right]x\sin\alpha\,dx\Big|_0^{\theta_0},\end{aligned} \tag{26}$$

where f_x, f_θ and f_z represent volume distributed forces and $\overline{N}_x^u$, $\overline{N}_x^v$, $\overline{N}_x^w$, $\hat{P}_x^{vh}$, $\overline{M}_x^w$, $\overline{N}_\theta^u$, $\overline{N}_\theta^v$, $\overline{N}_\theta^w$, $\hat{P}_\theta^{uh}$, $\overline{M}_\theta^w$ are the classical and nonclassical force and momentum applied to typical ($x=\text{constant}$) and ($\theta=\text{constant}$) lines and edge.

Now, the Hamilton's principle is

$$\int_{t_1}^{t_2}(\delta T-\delta U_s+\delta W_e)\,dt=0. \tag{27}$$

By substituting the work done by external forces applied as well as strain energy and kinetic energy of the shell in Hamilton's principle, and using variations method and performing direct, lengthy mathematical calculations, the governing equations and boundary conditions are derived as

$$\begin{aligned}\delta U:&-\frac{D_{1,0}}{x}\frac{\partial U}{\partial x}+\frac{D_{1,0}}{x^2}U-D_{1,0}\frac{\partial^2U}{\partial x^2}+\frac{D_{5,0}}{x^2\sin^2\alpha}\left[-1+\frac{l^2}{4x^2}-\frac{l^2}{x^2\tan^2\alpha}\right]\frac{\partial^2U}{\partial\theta^2}-\frac{D_{5,0}l^2}{4x^3\sin^2\alpha}\frac{\partial^3U}{\partial x\,\partial\theta^2}\\&+\frac{D_{5,0}l^2}{4x^4\sin^4\alpha}\frac{\partial^4U}{\partial\theta^4}+\frac{D_{5,0}l^2}{4x^2\sin^2\alpha}\frac{\partial^4U}{\partial x^2\partial\theta^2}+\frac{1}{x^2\sin\alpha}\left[D_{1,0}+D_{5,0}\left[1-\frac{l^2}{4x^2}\right]\right]\frac{\partial V}{\partial\theta}-\frac{D_{5,0}l^2}{2x^2\sin\alpha}\frac{\partial^3V}{\partial x^2\partial\theta}\\&-\frac{1}{x\sin\alpha}\left[D_{3,0}+D_{5,0}\left[1-\frac{l^2}{4x^2}\right]\right]\frac{\partial^2V}{\partial x\,\partial\theta}-\frac{D_{5,0}l^2}{4x^3\sin^3\alpha}\frac{\partial^4V}{\partial x\,\partial\theta^3}-\frac{D_{5,0}l^2}{4x\sin\alpha}\frac{\partial^4V}{\partial x^3\partial\theta}-\frac{D_{5,0}l^2}{4x^4\sin^3\alpha}\frac{\partial^3V}{\partial\theta^3}\\&+\frac{1}{x^3\sin^2\alpha}\left[\frac{D_{5,0}l^2}{x\tan\alpha}-D_{1,1}-2D_{5,1}-\frac{D_{5,1}l^2}{x^2\tan^2\alpha}-D_{3,1}\right]\frac{\partial^2W}{\partial\theta^2}+\frac{D_{1,0}}{x^2\tan\alpha}W-\frac{1}{x}\left[\frac{D_{3,0}}{\tan\alpha}+\frac{D_{1,1}}{x}\right]\frac{\partial W}{\partial x}\\&-\frac{1}{x^2\sin^2\alpha}\left[\frac{3D_{5,0}l^2}{2x\tan\alpha}-D_{3,1}-2D_{5,1}-\frac{D_{5,1}l^2}{x^2\tan^2\alpha}\right]\frac{\partial^3W}{\partial x\,\partial\theta^2}+D_{1,1}\frac{\partial^3W}{\partial x^3}+\frac{D_{1,1}}{x}\frac{\partial^2W}{\partial x^2}-f_x+I_{1,0}\frac{\partial^2U}{\partial t^2}=0,\end{aligned} \tag{28}$$

$$
\begin{aligned}
\delta V :& -\frac{1}{x^2\sin\alpha}\left[D_{1,0}+D_{5,0}\left[1-\frac{3l^2}{4x^2}\right]\right]\frac{\partial U}{\partial\theta}-\frac{1}{x\sin\alpha}\left[D_{3,0}+D_{5,0}\left[1+\frac{3l^2}{4x^2}\right]\right]\frac{\partial^2 U}{\partial x\,\partial\theta}\\
&+\frac{D_{5,0}l^2}{2x^2\sin\alpha}\frac{\partial^3 U}{\partial x^2\partial\theta}+\frac{3D_{5,0}l^2}{4x^4\sin^3\alpha}\frac{\partial^3 U}{\partial\theta^3}-\frac{D_{5,0}l^2}{4x\sin\alpha}\frac{\partial^4 U}{\partial x^3\partial\theta}-\frac{D_{5,0}l^2}{4x^3\sin^3\alpha}\frac{\partial^4 U}{\partial x\,\partial\theta^3}\\
&+\frac{D_{5,0}}{x^2}\left[1-\frac{3l^2}{x^2\tan^2\alpha}-\frac{3l^2}{4x^2}\right]V-\frac{D_{5,0}}{x}\left[1-\frac{3l^2}{x^2\tan^2\alpha}-\frac{3l^2}{4x^2}\right]\frac{\partial V}{\partial x}+\frac{D_{5,0}l^2}{2x}\frac{\partial^3 V}{\partial x^3}\\
&-D_{5,0}\left[1+\frac{3l^2}{x^2\tan^2\alpha}+\frac{3l^2}{4x^2}\right]\frac{\partial^2 V}{\partial x^2}+\frac{D_{5,0}l^2}{4}\frac{\partial^4 V}{\partial x^4}-\frac{1}{x^2\sin^2\alpha}\left[D_{1,0}+\frac{D_{5,0}l^2}{x^2}\left[\frac{1}{\tan^2\alpha}+\frac{3}{4}\right]\right]\frac{\partial^2 V}{\partial\theta^2}\\
&-\frac{D_{5,0}l^2}{4x^3\sin^2\alpha}\frac{\partial^3 V}{\partial x\,\partial\theta^2}+\frac{D_{5,0}l^2}{4x^2\sin^2\alpha}\frac{\partial^4 V}{\partial x^2\partial\theta^2}-\frac{1}{x^2\sin\alpha}\left[\frac{3D_{5,0}l^2}{x\tan\alpha}-D_{1,1}+\frac{2D_{5,1}l^2}{x^2\tan^2\alpha}\right]\frac{\partial^2 W}{\partial x\,\partial\theta}\\
&+\frac{1}{x^3\sin^3\alpha}\left[\frac{D_{5,0}l^2}{x\tan\alpha}+D_{1,1}\right]\frac{\partial^3 W}{\partial\theta^3}-\frac{1}{x^2\sin^2\alpha}\left[D_{1,0}\cos\alpha-\frac{3D_{5,0}l^2\cos\alpha}{x^2}\right]\frac{\partial W}{\partial\theta}\\
&+\frac{1}{x\sin\alpha}\left[\frac{5D_{5,0}l^2}{2x\tan\alpha}+D_{3,1}+2D_{5,1}+\frac{D_{5,1}l^2}{x^2\tan^2\alpha}\right]\frac{\partial^3 W}{\partial x^2\partial\theta}-f_\theta+I_{1,0}\frac{\partial^2 V}{\partial t^2}=0,
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\delta W :&\frac{1}{x}\left[\frac{D_{3,0}}{\tan\alpha}+\frac{D_{1,1}}{x}\right]\frac{\partial U}{\partial x}+\frac{1}{x^2}\left[\frac{D_{1,0}}{\tan\alpha}-\frac{D_{1,1}}{x}\right]U-\frac{1}{x^3\sin^2\alpha}\left[\frac{2D_{5,0}l^2}{x\tan\alpha}-\frac{2D_{5,1}l^2}{x^2\tan^2\alpha}+D_{1,1}\right]\frac{\partial^2 U}{\partial\theta^2}\\
&+\frac{1}{x^2\sin^2\alpha}\left[\frac{3D_{5,0}l^2}{2x\tan\alpha}-\frac{D_{5,1}l^2}{x^2\tan^2\alpha}-D_{3,1}-2D_{5,1}\right]\frac{\partial^3 U}{\partial x\,\partial\theta^2}-D_{1,1}\frac{\partial^3 U}{\partial x^3}-\frac{2D_{1,1}}{x}\frac{\partial^2 U}{\partial x^2}\\
&+\frac{1}{x^2\sin\alpha}\left[\frac{D_{1,0}}{\tan\alpha}-\frac{2D_{5,0}l^2}{x^2\tan\alpha}+\frac{2D_{5,1}}{x}-\frac{2D_{5,1}l^2}{x^3\tan^2\alpha}-\frac{D_{1,1}}{x}-\frac{2D_{3,1}}{x}\right]\frac{\partial V}{\partial\theta}-\frac{1}{x\sin\alpha}\left[\frac{5D_{5,0}l^2}{2x\tan\alpha}\right.\\
&\left.+\frac{D_{5,1}l^2}{x^2\tan^2\alpha}+D_{3,1}+2D_{5,1}\right]\frac{\partial^3 V}{\partial x^2\partial\theta}-\frac{1}{x^3\sin^3\alpha}\left[\frac{D_{5,0}l^2}{x\tan\alpha}+D_{1,1}\right]\frac{\partial^3 V}{\partial\theta^3}\\
&+\frac{1}{x^2\sin\alpha}\left[\frac{2D_{5,0}l^2}{x\tan\alpha}+\frac{2D_{5,1}l^2}{x^2\tan^2\alpha}+D_{1,1}\right]\frac{\partial^2 V}{\partial x\,\partial\theta}+\frac{1}{x^2\tan\alpha}\left[\frac{D_{1,0}}{\tan\alpha}-\frac{D_{1,1}}{x}\right]W\\
&+\frac{1}{x^2\sin^2\alpha}\left[\frac{2D_{5,2}l^2}{x^2\tan^2\alpha}+2D_{3,2}+4D_{5,2}+2D_{5,0}l^2\right]\frac{\partial^4 W}{\partial x^2\partial\theta^2}+\frac{2}{x}[D_{1,2}+D_{5,0}l^2]\frac{\partial^3 W}{\partial x^3}\\
&+\frac{1}{x^3\sin^2\alpha}\left[\frac{2D_{1,2}}{x}+\frac{2D_{3,2}}{x}+\frac{4D_{5,2}}{x}+\frac{4D_{5,0}l^2}{x}+\frac{6D_{5,2}l^2}{x^3\tan^2\alpha}-\frac{2D_{1,1}}{\tan\alpha}\right]\frac{\partial^2 W}{\partial\theta^2}\\
&-\frac{1}{x^3\sin^2\alpha}\left[\frac{6D_{5,2}l^2}{x^2\tan^2\alpha}+2D_{3,2}+4D_{5,2}+2D_{5,0}l^2\right]\frac{\partial^3 W}{\partial x\,\partial\theta^2}+\frac{1}{x^3}\left[D_{1,2}+D_{5,0}l^2\left[1+\frac{1}{\tan^2\alpha}\right]\right]\frac{\partial W}{\partial x}\\
&-\frac{1}{x}\left[\frac{D_{1,2}}{x}+\frac{D_{5,0}l^2}{x}\left[1+\frac{1}{\tan^2\alpha}\right]+\frac{2D_{3,1}}{\tan\alpha}\right]\frac{\partial^2 W}{\partial x^2}+[D_{1,2}+D_{5,0}l^2]\frac{\partial^4 W}{\partial x^4}\\
&+\frac{1}{x^4\sin^4\alpha}[D_{1,2}+D_{5,0}l^2]\frac{\partial^4 W}{\partial\theta^4}-f_w+I_{1,0}\frac{\partial^2 W}{\partial t^2}=0.
\end{aligned}
\tag{30}
$$

And the boundary conditions for $x =$ constant are obtained as

$$\int_\theta \left[N_{xx} + \frac{\partial}{\partial\theta}\left[\frac{Y_{zx}}{2x\sin\alpha}\right] + \bar{N}_x^u \right] x\sin\alpha\, d\theta|_x = 0 \quad \text{or} \quad \delta U|_x = 0, \tag{31}$$

$$\int_\theta \Bigg[\left[N_{x\theta} - \frac{Y_{xx}}{x\tan\alpha} + \frac{Y_{zz}}{2x\tan\alpha} + \frac{Y_{\theta\theta}}{2x\tan\alpha} + \frac{Y_{xz}}{2x} \right] - \frac{\partial}{\partial\theta}\left[\frac{Y_{z\theta}}{2x\sin\alpha}\right] - \frac{1}{x}\frac{\partial}{\partial x}\left[\frac{xY_{zx}}{2}\right] + \bar{N}_x^v \Bigg] x\sin\alpha\, d\theta|_x = 0 \quad \text{or} \quad \delta V|_x = 0, \tag{32}$$

$$\int_\theta \left[\left[\frac{Y_{zx}}{2}\right] + \hat{P}_x^{vh} \right] x\sin\alpha\, d\theta\Big|_x = 0 \quad \text{or} \quad \delta\left(\frac{\partial V}{\partial x}\right)\Big|_x = 0, \tag{33}$$

$$\int_\theta \Bigg[\frac{1}{x}\frac{\partial}{\partial x}[xM_{xx} + xY_{x\theta}] + \left[\frac{2M_{xx}}{x} - \frac{M_{\theta\theta}}{x} + \frac{Y_{z\theta}}{x\tan\alpha} + \frac{Y_{x\theta}}{2x} \right] + \frac{\partial}{\partial\theta}\left[\frac{2M_{x\theta}}{x\sin\alpha} - \frac{Y_{xx}}{x\sin\alpha} + \frac{Y_{\theta\theta}}{x\sin\alpha} + \frac{T_{zz}\cos\alpha}{x^2\sin^2\alpha} \right] + \bar{N}_x^w \Bigg] x\sin\alpha\, d\theta\Big|_x = 0 \quad \text{or} \quad \delta W\Big|_x = 0, \tag{34}$$

$$\int_\theta [-[M_{xx} + Y_{x\theta}] + \bar{M}_x^w] d\theta\Big|_x = 0 \quad \text{or} \quad \delta\left(\frac{\partial W}{\partial x}\right)\Big|_x = 0. \tag{35}$$

The boundary conditions for $\theta =$ constant are

$$\int_x \Bigg[\left[\frac{N_{x\theta}}{x\sin\alpha} - \frac{Y_{\theta\theta}\cos\alpha}{2x^2\sin^2\alpha} + \frac{Y_{zz}\cos\alpha}{2x^2\sin^2\alpha} + \frac{Y_{xz}}{2x^2\sin\alpha} \right] + \frac{\partial}{\partial\theta}\left[\frac{Y_{z\theta}}{2x^2\sin^2\alpha}\right] + \frac{1}{x}\frac{\partial}{\partial x}\left[\frac{Y_{zx}}{2\sin\alpha}\right] + \bar{N}_\theta^u \Bigg] x\sin\alpha\, dx\Big|_\theta = 0 \quad \text{or} \quad \delta U\Big|_\theta = 0, \tag{36}$$

$$\int_x \left[-\left[\frac{Y_{z\theta}}{2x^2\sin^2\alpha}\right] + \hat{P}_\theta^{uh} \right] x\sin\alpha\, dx\Big|_\theta = 0 \quad \text{or} \quad \delta\left(\frac{\partial U}{\partial\theta}\right)\Big|_\theta = 0, \tag{37}$$

$$\int_x \left[\left[\frac{N_{\theta\theta}}{x\sin\alpha} - \frac{Y_{x\theta}\cos\alpha}{x^2\sin^2\alpha} + \frac{Y_{z\theta}}{2x^2\sin\alpha} \right] - \frac{1}{x}\frac{\partial}{\partial x}\left[\frac{Y_{z\theta}}{2\sin\alpha}\right] + \bar{N}_\theta^v \right] x\sin\alpha\, dx\Big|_\theta = 0 \quad \text{or} \quad \delta V\Big|_\theta = 0, \tag{38}$$

$$\int_x \Bigg[\left[\frac{2M_{x\theta}}{x^2\sin\alpha} - \frac{Y_{xx}}{x^2\sin\alpha} + \frac{Y_{\theta\theta}}{x^2\sin\alpha} + \frac{T_{zz}\cos\alpha}{x^3\sin^2\alpha} \right] + \frac{1}{x}\frac{\partial}{\partial x}\left[\frac{2M_{x\theta}}{\sin\alpha} - \frac{Y_{xx}}{\sin\alpha} + \frac{Y_{\theta\theta}}{\sin\alpha} + \frac{T_{zz}\cos\alpha}{x\sin^2\alpha} \right] + \frac{\partial}{\partial\theta}\left[\frac{M_{\theta\theta}}{x^2\sin^2\alpha} - \frac{Y_{x\theta}}{x^2\sin^2\alpha} \right] + \bar{N}_\theta^w \Bigg] x\sin\alpha\, dx\Big|_\theta = 0 \quad \text{or} \quad \delta W\Big|_\theta = 0, \tag{39}$$

$$\int_x \left[-\left[\frac{M_{\theta\theta}}{x^2\sin^2\alpha} - \frac{Y_{x\theta}}{x^2\sin^2\alpha} \right] + \bar{M}_\theta^w \right] x\sin\alpha\, dx\Big|_\theta = 0 \quad \text{or} \quad \delta\left(\frac{\partial W}{\partial\theta}\right)\Big|_\theta = 0. \tag{40}$$

In the equations above (equations of motion and boundary conditions) parameters $D_{1,i}$, $D_{3,i}$, $D_{5,i}$ and $I_{1,i}$ are defined as

$$
\begin{aligned}
(D_{1,i}) &= \int_{-\tilde{z}_c}^{h-\tilde{z}_c} \frac{E(\tilde{z})}{1-\nu^2(\tilde{z})}(z^i)\,dz, \;\; (i=0,1,2),\\
(D_{3,i}) &= \int_{-\tilde{z}_c}^{h-\tilde{z}_c} \frac{E(\tilde{z})\nu(\widehat{z})}{1-\nu^2(\tilde{z})}(z^i)\,dz, \;\; (i=0,1,2),\\
(D_{5,i}) &= \int_{-\tilde{z}_c}^{h-\tilde{z}_c} \mu(\tilde{z})(z^i)\,dz, \;\; (i=0,1,2),\\
(I_{1,i}) &= \int_{-\tilde{z}_c}^{h-\tilde{z}_c} \rho(\tilde{z})(z^i)\,dz, \; (i=0,1,2).
\end{aligned}
\tag{41}
$$

Now, it can be argued that (28)–(30) are the equations of motion of the FG conical shell based on the modified couple stress theory. Also, (31)–(40) are the classical and nonclassical boundary conditions of the FG conical shell based on the modified couple stress theory. In special cases, these equations of motion and boundary conditions are reduced to the following equations:

Case 1: By setting the length scale parameter to zero ($l = 0$), the equations obtained in this paper are reduced to the equations of motion and boundary conditions in the classical continuum theory based on the thin shell model for FG materials.

Case 2: By assuming constant mechanical properties (Young's modulus, Poisson's ratio and density), the equations obtained in this paper are reduced to the equations of motion and classical and nonclassical boundary conditions based on the modified couple stress theory and the thin shell model for the isotropic homogeneous structure.

Case 3: By assuming cases (1) and (2), the equations obtained in this paper are reduced to the equations of motion and boundary conditions in the classical continuum theory based on the thin shell model.

3. Case study

In this section, the free vibration of the simply supported FG conical shell in the special case is examined so as to evaluate the equations derived. The equations governing the simply supported FG conical shell are similar to (28)–(30); hence, it suffices to investigate the boundary conditions. It should be noted that the boundary conditions in (36)–(40) are satisfied due to the variability of the θ angle from 0 to 2π edges; thus, the boundary conditions in $x = x_0$ and $x = x_0 + L$ edges must be investigated, which are (31)–(35). By substituting (22) into (31)–(35), the boundary conditions for the simply supported FG conical shell are obtained as

$$
\begin{aligned}
V|_{x=x_0,x_0+L} &= 0,\\
W|_{x=x_0,x_0+L} &= 0,
\end{aligned}
\tag{42}
$$

$$\Bigg[D_{1,0}\frac{\partial U}{\partial x}-D_{1,1}\frac{\partial^2 W}{\partial x^2}+\frac{D_{3,0}}{x}U-\frac{D_{3,1}}{x^2\sin^2\alpha}\frac{\partial^2 W}{\partial\theta^2}-\frac{D_{3,1}}{x}\frac{\partial W}{\partial x}-\frac{D_{5,0}l^2}{4x^2\sin^2\alpha}\frac{\partial^3 U}{\partial x\,\partial\theta^2}$$
$$+\frac{D_{5,0}l^2}{4x^3\sin^2\alpha}\frac{\partial^2 U}{\partial\theta^2}+\frac{1}{x\sin\alpha}\left[D_{3,0}-\frac{D_{5,0}l^2}{4x^2}\right]\frac{\partial V}{\partial\theta}+\frac{D_{5,0}l^2}{4x\sin\alpha}\frac{\partial^3 V}{\partial x^2\partial\theta}+\frac{D_{5,0}l^2}{4x^2\sin\alpha}\frac{\partial^2 V}{\partial x\,\partial\theta}$$
$$+\frac{D_{3,0}}{x\tan\alpha}W\Bigg]\Bigg|_{x=x_0,x_0+L}=0, \tag{43}$$

$$\left[-\frac{D_{5,0}l^2}{4x\sin\alpha}\frac{\partial^2 U}{\partial x\,\partial\theta}+\frac{D_{5,0}l^2}{4x^2\sin\alpha}\frac{\partial U}{\partial\theta}+\frac{D_{5,0}l^2}{4}\frac{\partial^2 V}{\partial x^2}+\frac{D_{5,0}l^2}{4x}\frac{\partial V}{\partial x}-\frac{D_{5,0}l^2}{4x^2}V\right]\Bigg|_{x=x_0,x_0+L}=0, \tag{44}$$

$$\Bigg[[D_{1,2}+D_{5,0}l^2]\frac{\partial^2 W}{\partial x^2}+\frac{1}{x}[D_{3,2}-D_{5,0}l^2]\frac{\partial W}{\partial x}+\frac{1}{x^2\sin^2\alpha}[D_{3,2}-D_{5,0}l^2]\frac{\partial^2 W}{\partial\theta^2}$$
$$-D_{1,1}\frac{\partial U}{\partial x}+\frac{1}{x\sin\alpha}\left[\frac{D_{5,0}l^2}{x\tan\alpha}-D_{3,1}\right]\frac{\partial V}{\partial\theta}-\frac{D_{3,1}}{x}U-\frac{D_{3,1}}{x\tan\alpha}W\Bigg]\Bigg|_{x=x_0,x_0+L}=0. \tag{45}$$

Hence, (28)–(30) as well as (42)–(45) are the governing equations and boundary conditions for the simply supported FG thin conical shell, which must be simultaneously solved in order to investigate the free vibration.

In order to solve the above equations and considering boundary conditions, the following approximate solutions are used [Dung et al. 2014]:

$$U(x,\theta,t)=U_0\cos\left(\frac{m\pi(x-x_0)}{L}\right)\sin(n\theta)\sin(\omega t),\qquad V(x,\theta,t)=U_0\sin\left(\frac{m\pi(x-x_0)}{L}\right)\cos(n\theta)\sin(\omega t),$$
$$W(x,\theta,t)=W_0\sin\left(\frac{m\pi(x-x_0)}{L}\right)\sin(n\theta)\sin(\omega t). \tag{46}$$

In the above equations ω, n and m stand for the natural frequency of the nanoshell, circumferential and axial wave numbers, respectively. Given the above assumption, the majority of boundary conditions in (42)–(45) are satisfied although some of them are not fully satisfied. However, in the references, it is common that for a complicated formulation, like that one above, not all boundary conditions be satisfied. Therefore, to investigate the free vibration of the nanoshell, it is necessary to solve the equations of motion in (28)–(30). For this purpose, the Galerkin method is employed. To simplify integration, (28)–(29) are multiplied by x^4 and (30) is multiplied by x^6, and, finally, using the Galerkin method and the equations derived, the following equation is obtained:

$$\begin{gathered}\int_{x_0}^{x_0+L}\int_0^{2\pi}\psi_1 x\cos\left(\frac{m\pi(x-x_0)}{L}\right)\sin\alpha\,dx\,d\theta=0,\\ \int_{x_0}^{x_0+L}\int_0^{2\pi}\psi_2 x\sin\left(\frac{m\pi(x-x_0)}{L}\right)\sin\alpha\,dx\,d\theta=0,\\ \int_{x_0}^{x_0+L}\int_0^{2\pi}\psi_3 x\sin\left(\frac{m\pi(x-x_0)}{L}\right)\sin\alpha\,dx\,d\theta=0,\end{gathered} \tag{47}$$

	ρ (kg/m^3)	ν	E (GPa)
Aluminum	70	2702	0.3
Ceramic	427	3100	0.17

Table 1. Material properties of aluminum and ceramic.

where the parameters ψ_1, ψ_2 and ψ_3 are obtained as

$$\begin{aligned}\psi_1 &= x^4[A_{11}(U_0)+A_{12}(V_0)+A_{13}(W_0)],\\ \psi_2 &= x^4[A_{21}(U_0)+A_{22}(V_0)+A_{23}(W_0)],\\ \psi_3 &= x^6[A_{31}(U_0)+A_{32}(V_0)+A_{33}(W_0)].\end{aligned} \tag{48}$$

In the above equation, A_{ij} are values obtained by substituting Equation (47) into Equations (28)–(30). Hence, the matrix form of Equation (48) is

$$([K]-\omega^2[M])\begin{Bmatrix} U_0 \\ V_0 \\ W_0 \end{Bmatrix} = 0. \tag{49}$$

Considering the eigenvalue problem, in order to obtain a nontrivial solution to Equation (49), the determinant of the coefficients must be set to zero, and, by solving the derived equation, one can determine the frequency of nanoshell.

4. Results and discussion

This section is devoted to the investigation of the free vibration of the simply supported FG conical thin shell using the modified couple stress theory. The effect of parameters such as dimensionless length scale parameter, apex angle, dimensionless length parameter and circumferential and axial wave numbers on the dimensionless natural frequency are studied and the results are compared with those obtained based on the classical continuum theory. As mentioned before, assuming $l=0$, the resulting equations are obtained based on the classical continuum theory.

The geometric features of the conical shell are assumed to be $h=0.34$ nm, $L/R_1=2$ and $h/R_1=0.04$, and the FG conical shell is assumed to consist of aluminum or ceramic with the material properties in Table 1 [Sahmani et al. 2013].

4.1. ***Comparison of results.*** Since according to the authors studies, so far, no study is carried out on the FG conical nanoshell, in order to verify the results, first the correctness of the results are shown by comparing the obtained results from the homogeneous conical shell based on the classical continuum theory and assuming $N=0$ with that of [Irie et al. 1984; Jin et al. 2014; Lam and Hua 1999a] in Tables 2 and 3. Afterwards, as shown in Figure 2, the correctness of using the size effect, is assessed by comparing the results from an isotropic homogeneous conical nanoshell with that of [Zeighampour and Tadi Beni 2014a] based on the modified couple stress theory. Besides, according to Table 2, the size effect which leads to increase in natural frequency are shown correctly. By way of comparison, the dimensionless natural frequency is calculated based on $\Omega=\omega R\sqrt{\rho(1-\nu^2)/E}$.

n	[Irie et al. 1984]	[Jin et al. 2014]	Present	Present $l = h$
1	0.5462	0.5462	0.5988	0.6791
2	0.631	0.6309	0.6371	0.6594
3	0.5065	0.5063	0.5101	0.5511
4	0.3947	0.3943	0.4044	0.4852
5	0.3348	0.3340	0.3436	0.4975
6	0.3248	0.3239	0.3290	0.5748
7	0.3524	0.3213	0.3264	0.6833
8	0.4033	0.4022	0.3934	0.8386
9	0.4684	0.4673	0.4635	1.0227

Table 2. Comparison of frequency parameter Ω for the conical shell. ($E = 211\,\text{GPa}$, $\nu = 0.3$, $\rho = 7800\,\text{kg m}^{-3}$, $R_2 = 1\,\text{m}$, $h/R_2 = 0.01$, $\alpha = 45$, $L\sin(\alpha)/R_2 = 0.5$).

	$\alpha = 30°$			$\alpha = 45°$			$\alpha = 60°$		
n	A†	B††	Present	A†	B††	Present	A†	B††	Present
2	0.7910	0.8420	0.8297	0.6879	0.7655	0.7534	0.5722	0.6348	0.6301
3	0.7284	0.7376	0.7329	0.6973	0.7212	0.7135	0.6001	0.6238	0.6208
4	0.6352	0.6362	0.6365	0.6664	0.6739	0.6699	0.6054	0.6145	0.6130
5	0.5531	0.5528	0.5552	0.6304	0.6323	0.6309	0.6077	0.6111	0.6102
6	0.4949	0.4950	0.4972	0.6032	0.6035	0.6029	0.6159	0.6171	0.6158
7	0.4653	0.4661	0.4662	0.5918	0.5921	0.5905	0.6343	0.6350	0.6322
8	0.4654	0.4660	0.4629	0.5992	0.6001	0.5962	0.6650	0.6660	0.6607
9	0.4892	0.4916	0.4848	0.6257	0.6273	0.6202	0.7084	0.7101	0.7016

Table 3. Comparison of frequency parameter Ω for the conical shell. ($E = 211\,\text{GPa}$, $\nu = 0.3$, $\rho = 7800\,\text{kgm}^{-3}$, $R_2 = 1\,\text{m}$, $h/R_2 = 0.01$, $L\sin(\alpha)/R_2 = 0.25$). †Values in column A are from [Irie et al. 1984]. ††Values in B are from [Lam and Hua 1999a].

As is visible, according to [Hua 2000; Lam and Hua 1997; 1999b; Sofiyev et al. 2009] using the Galerkin method, the comparison study is carried out for $n \geq 2$ in order to establish convergent validity; therefore, in this study according to Table 2, the obtained results have good consistency for $n \geq 2$; besides, in order to gain confidence on the accuracy of the current study, an additional comparison is carried out in different apex angle according to Table 3. As is clear, the results have appropriate precision, according to Table 2, Table 3 and Figure 2.

4.2. ***Effects of size parameter and apex angle on natural frequency.*** Figure 3 illustrates the effect of dimensionless length scale parameter on natural frequency for $\alpha = 30°$ according to the modified couple stress theory. As can be seen in Figure 3, decrease in the dimensionless length scale parameter is accompanied by increase in natural frequency according to the modified couple stress theory, and this increase is intensified with the increase in the gradient index N. In fact, decrease in h/l stiffens the conical shell, leading to increase in natural frequency. Also, as $N = 0$ is according to the aluminum shell

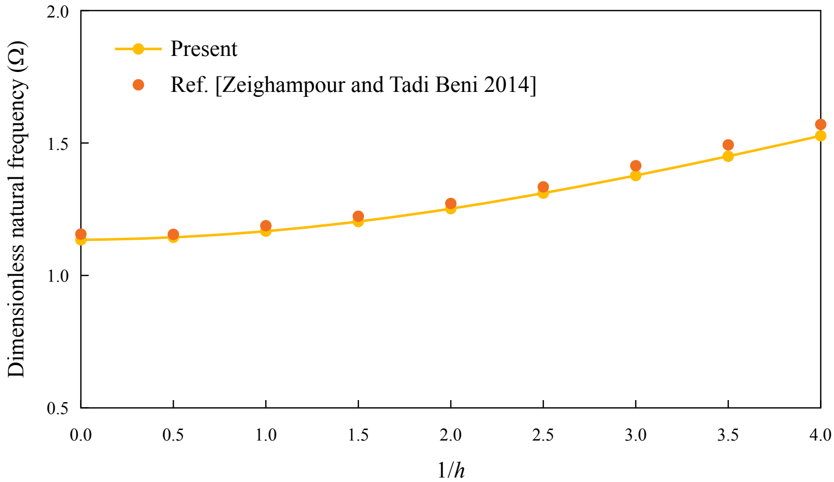

Figure 2. Comparison of dimensionless natural frequency versus dimensionless length scale parameter for SWCNC.

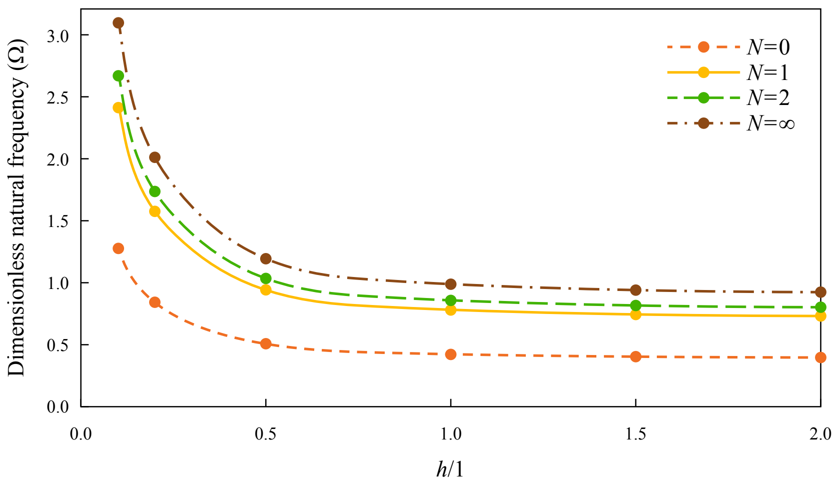

Figure 3. Effects of dimensionless length scale parameter on natural frequency based on modified couple stress theory $\alpha = 30°$.

and $N =$ infinite is according to the ceramic shell, an increase in the gradient index is accompanied by an increase in Young's modulus of FG conical nanoshell, which stiffens the shell and results in increasing the natural frequency.

Figure 4 displays the effect of dimensionless length scale parameter on natural frequency for $\alpha = 60°$ based on the modified couple stress theory. As can be seen in Figures 3 and 4, as the apex angle increases, the natural frequency increases with the decrease in dimensionless length scale parameter, and the effect of increase in the gradient index on the natural frequency can still be seen; however, with the increases of the apex angle, the effects of these two parameters, length scale parameter and gradient index, on the increase of natural frequency are weakened.

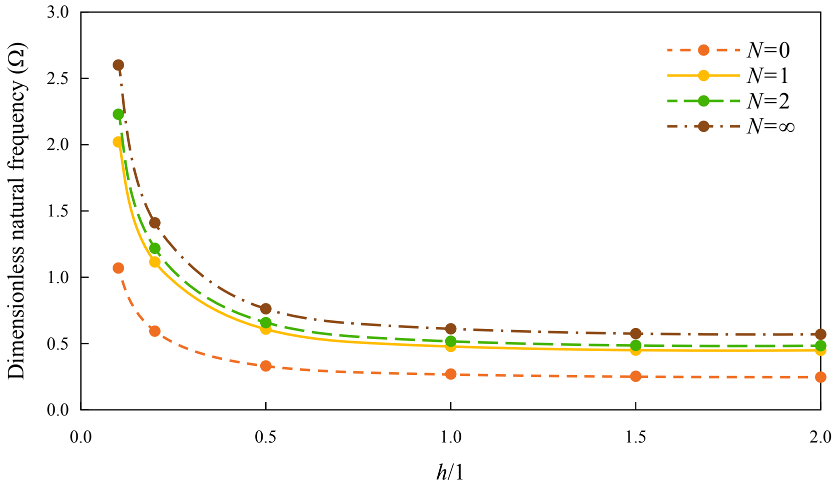

Figure 4. Effects of dimensionless length scale parameter on natural frequency based on modified couple stress theory $\alpha = 60°$.

4.3. ***Effects of length besides circumferential and axial wave number on natural frequency.*** Figures 5, 6 show the effect of length parameter; in addition, the circumferential and axial wave numbers on the natural frequency. As can be seen, an increase in the circumferential and axial wave numbers leads to increase in the natural frequency. On the other hand, the decrease in length parameter induced the decrease in stability in the shell leading to increase in natural frequency intensifies the increasing effect of circumferential and axial wave numbers on the increase in natural frequency. Therefore, in $l = h$, as length parameter decreases from $L = 2R_1$ to $L = R_1$, and the circumferential wave number increases from $n = 1$ to $n = 5$, the natural frequency increases from 1.69 to 2.3 for $L = 2R_1$ and from 3.9 to 9 for $L = R_1$. In addition, according to the illustration, increase in the length scale parameter has an increasing impact on the effect of length parameter and circumferential and axial wave numbers on natural frequency. Thus, in $L = R_1$, as the axial wave number changes from $m = 1$ to $m = 5$, the dimensionless natural frequency increases from 1.95 to 13.6 in $l = h$ and in $l = 2h$, the natural frequency increases from 2.77 to 22.1. Also, in $n = 2$, the decrease in length parameter from $L = 2R_1$ to $L = R_1$ leads to an increase in natural frequency from 1.36 to 3.7 in $l = h$ and from 1.77 to 5.66 in $l = 2h$.

5. Conclusion

Using the modified couple stress theory, a new formulation for the FG truncated conical thin shell is developed in this paper. The size effect is considered using the modified couple stress theory, and material distribution in the FG conical shell is assumed according to the power law distribution as continuously variable along shell thickness. Governing equations as well as classical and nonclassical boundary conditions are derived using Hamilton's principle, and, in the special case, the free vibration of the simply supported FG conical nanoshell is investigated. Also, the effects of parameters such as length scale parameter, apex angle, length, and circumferential and axial wave numbers on natural frequency is studied. Increase in natural frequency induced by increase in length scale parameter is shown based on the modified couple stress theory and compared with the results of classical continuum theory. Moreover,

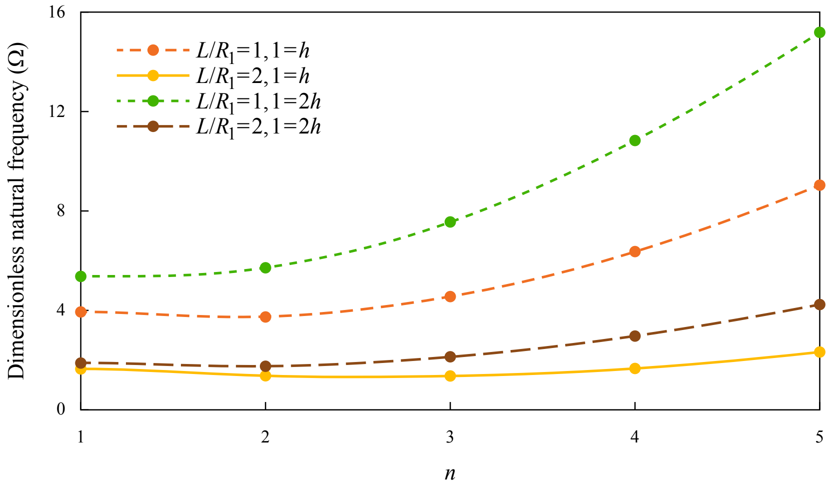

Figure 5. Effects of circumferential wave number and dimensionless length on dimensionless natural frequency, $\alpha = 30°$, $N = 1$.

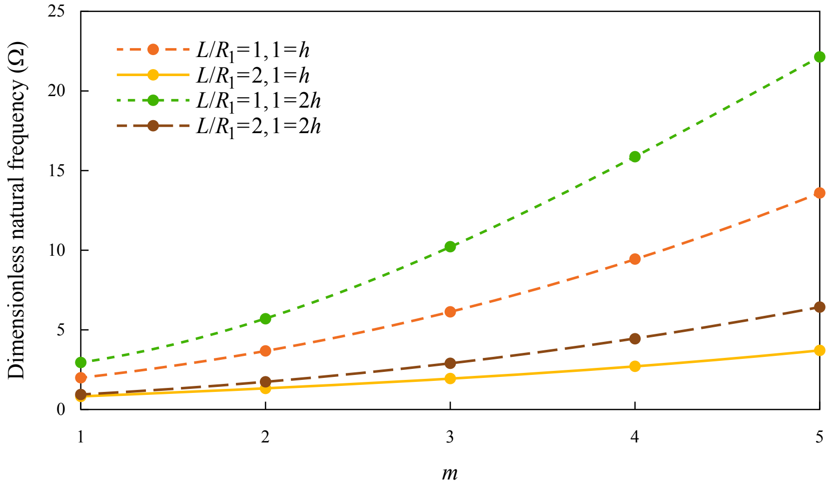

Figure 6. Effects of axial wave number and dimensionless length on dimensionless natural frequency, $\alpha = 30°$, $N = 1$.

the increasing effect of length scale parameter on the natural frequency induced by the decrease in length and increase in circumferential and axial wave numbers is investigated as well.

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