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EFFECTIVE BOUNDARY CONDITION METHOD AND APPROXIMATE SECULAR EQUATIONS OF RAYLEIGH WAVES IN ORTHOTROPIC HALF-SPACES COATED BY A THIN LAYER

PHAM CHI VINH AND VU THI NGOC ANH

In this paper, the effective boundary condition method for deriving approximate secular equations of Rayleigh waves propagating in elastic half-spaces coated by a thin layer is introduced. Then, the method is used to obtain approximate secular equations of Rayleigh waves in compressible (incompressible) orthotropic half-spaces covered by a thin incompressible (compressible) orthotropic layer. Approximate secular equations of third order have been derived and it is shown that they have high accuracies. Some numerical examples are carried out to evaluate the effect of incompressibility on the Rayleigh wave propagation. It is shown that the incompressibility affects considerably on the Rayleigh wave velocity.

1. Introduction

The structures of a thin film attached to solids, modeled as half-spaces coated by a thin layer, are widely applied in modern technology. The measurement of mechanical properties of thin films deposited on half-spaces before and during loading is therefore very significant; for examples, see [Makarov et al. 1995; Every 2002] and references therein. Among various measurement methods, the surface/guided wave method is most widely used [Every 2002], because it is non-destructive and it is connected with reduced cost, less inspection time, and greater coverage [Hess et al. 2013], and for which the guided Rayleigh wave is a versatile and convenient tool [Küchler and Richter 1998; Hess et al. 2013].

For the Rayleigh-wave approach, the explicit dispersion relations of Rayleigh waves supported by thin-film/substrate interactions are employed as theoretical bases for extracting the mechanical properties of the thin films from experimental data. They are therefore the main purpose of any investigation of Rayleigh waves propagating in half-spaces covered by a thin layer.

Since the layer is assumed to be thin, i. e. its dimensionless thickness $\varepsilon = k.h$ (h is the thickness of the layer, k is the wave number) satisfies $0 < \varepsilon \ll 1$, it is reasonable that all researchers want to find approximate secular equations of Rayleigh waves that take the form $F = 0$, where F is a polynomial of n-order ($n \geq 1$) in terms of ε and the coefficients of the polynomial are functions of the material parameters and the Rayleigh wave velocity. Tiersten [1969] and Bövik [1996] assumed that the layer and the half-space are both isotropic and the authors derived approximate secular equations of second order. For this case, Vinh and Anh [2014a] obtained a fourth-order approximate secular equation with very high accuracy. Steigmann and Ogden [2007] considered a transversely isotropic layer with residual stress overlying an isotropic half-space and the authors obtained a second-order approximate dispersion relation. In [Vinh and Linh 2012; Vinh et al. 2014b] the layer and the half-space are both assumed to be orthotropic and approximate secular equations of third-order were obtained. In [Vinh and Linh 2013]

Keywords: Rayleigh waves, thin layer, effective boundary condition method, approximate secular equation.

the layer and the half-space are both subjected to homogeneous prestrains and an approximate secular equation of third-order was established which is valid for any prestrain and for a general strain energy function.

In the mentioned above investigations, the contact between the layer and the half-space is assumed to be perfectly bonded. For the case of sliding contact, Achenbach and Keshava [1967] derived an approximate secular equation of third-order. However, this approximate secular equation includes the shear coefficient, originating from Mindlin's plate theory [1951], whose usage should be avoided as noted by Touratier [1991], Muller and Touratier [1995] and Stephen [1997]. Recently, Vinh, Anh and Thanh [Vinh et al. 2014a] derived a fourth-order approximate secular equation with very high accuracy for the isotropic case. For the orthotropic case, an approximate secular equation of third order was established recently by Vinh and Anh [2014b].

To derive approximate secular equations, all researchers replace approximately the effect of the thin layer on the half-space by the so-called effective boundary conditions on the interface and the Rayleigh wave is then considered as a surface wave propagating in the half-space (without the coating layer) that is subjected to the effective boundary conditions. We call this approach "the effective boundary condition method".

For obtaining the effective boundary conditions, Achenbach and Keshava [1967] and Tiersten [1969] replaced the thin layer by a plate, while Bövik [1996] expanded the stresses at the top surface of the layer into Taylor series of its thickness up to the first-order, and expressed the first-order normal derivative in terms of the tangential and time derivatives by using the basic equations in component form. With this technique Bövik [1996] derived the first-order effective boundary conditions. It should be noted that this first-order effective boundary conditions contain some second-order terms as indicated in Remark 2. In order to obtain higher-order effective boundary conditions we need to get expressions of higher-order normal derivative in terms of the tangential and time derivatives. However, it is not easy to have them if we start from the basic equations in component form. Recently, starting from the basic equations in matrix form, Vinh and Linh [2012; 2013], Vinh and Anh [2014a; 2014b], and Vinh et al. [2014a; 2014b] obtained the effective boundary conditions of third and fourth orders. Nevertheless, in these papers the effective boundary condition method based on the Taylor expansion technique and the basic equations in matrix form has not been presented in detail. Furthermore, in these works (and in all other previous papers) both the layer and the half-space were assumed to be either compressible or incompressible. The case when the half-space is compressible (incompressible) and the layer incompressible (compressible) has never been under investigation.

The main aim of this paper is first to present in detail "the effective boundary condition method" that is based on the Taylor expansion technique and the basic equations in matrix form. Then, the method is employed to derive approximate secular equations of:

(i) Rayleigh waves propagating in a compressible orthotropic half-space covered by a thin incompressible orthotropic layer.

(ii) Rayleigh waves propagating in an incompressible half-space covered by a thin compressible layer.

Approximate secular equations of third order have been derived and it is shown that they have high accuracies. Some numerical examples are carried out to investigate the effect of incompressibility on the

Rayleigh wave propagation. It is shown that the incompressibility affects considerably on the Rayleigh wave velocity.

Note that, for Rayleigh waves propagating in compressible (incompressible) orthotropic half-spaces coated by a thin compressible (incompressible) orthotropic layer, the third-order approximate secular equations have been derived in [Vinh and Linh 2012; Vinh et al. 2014b].

It should be noted that there has been recently progress in studying not only a free surface wave and its speed, but also the case of loading, e.g., in application to moving loads; see, e.g., [Erbaş et al. 2014].

The paper is organized as follows. In Section 2, the effective boundary condition method is presented in detail. In Sections 3 and 4, the propagation of Rayleigh waves in an incompressible (compressible) orthotropic elastic half-space coated by a thin compressible (incompressible) orthotropic elastic layer is considered. The third-order approximate secular equations have been derived using the effective boundary condition method. In Section 5, as an application of the obtained results, the effect of the incompressibility of half-spaces and layers on the Rayleigh wave velocity is evaluated numerically using the obtained approximate secular equations.

2. The effective boundary condition method

Consider an orthotropic elastic homogeneous half-space $x_2 \geq 0$ coated by a thin orthotropic elastic homogeneous layer $-h \leq x_2 \leq 0$. Note that same quantities related to the half-space and the layer have the same symbol but are systematically distinguished by a bar if pertaining to the layer. We are interested in the plane strain so that:

$$\bar{u}_n = \bar{u}_n(x_1, x_2, t), \quad u_n = u_n(x_1, x_2, t), \qquad n = 1, 2,\ \bar{u}_3 \equiv 0,\ u_3 \equiv 0, \tag{1}$$

where $\bar{u}_n$ and u_n are the displacement components, t is the time. The effective boundary condition method based on the Taylor expansion technique and the basic equations in matrix form is carried out as follows.

Step 1. From the basic equations in component form governing the plane motions (1) of the layer including the equations of motions (without body forces):

$$\bar{\sigma}_{11,1} + \bar{\sigma}_{12,2} = \bar{\rho}\ddot{\bar{u}}_1, \quad \bar{\sigma}_{12,1} + \bar{\sigma}_{22,2} = \bar{\rho}\ddot{\bar{u}}_2 \tag{2}$$

(a comma indicates differentiation with respect to x_k, a dot signifies differentiation with respect to t), and the strain-stress relations (Equation (55) with bars for compressible materials, Equation (23) with bars for incompressible ones) we establish a matrix equation that is of the form

$$\begin{bmatrix} \bar{\boldsymbol{u}}' \\ \bar{\boldsymbol{t}}' \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} \bar{\boldsymbol{u}} \\ \bar{\boldsymbol{t}} \end{bmatrix}, \quad -h \leq x_2 \leq 0, \tag{3}$$

where

$$\bar{\boldsymbol{u}} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}, \quad \bar{\boldsymbol{t}} = \begin{bmatrix} \bar{\sigma}_{12} \\ \bar{\sigma}_{22} \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_2 \\ \boldsymbol{M}_3 & \boldsymbol{M}_4 \end{bmatrix}, \qquad \boldsymbol{M}_4 = \boldsymbol{M}_1^T, \tag{4}$$

where $\bar{\sigma}_{mn}$ are the components of the stress tensor, $\bar{\rho}$ is the mass density of the layer, the prime signifies differentiation with respect to x_2, $\boldsymbol{M}_k$ are 2×2-matrices whose entries are expressed in terms of the material parameters of the layer, the derivatives with respect to x_1 and t (they do not depend on the derivatives with respect to x_2), the symbol ^{T} indicates the transpose of a matrix. For the compressible elastic materials, the matrix Equation (3) is derived by using Equations (1), (2) and (55). For the incompressible elastic materials, in addition to Equations (1), (2) and (23), the incompressibility condition is taken into account, and to obtain the matrix Equation (3) we have to eliminate from the basic equations the Lagrange multiplier p (also called the hydrostatic pressure) associated with the incompressibility constraint.

Remark 1. (i) Equation (3) expresses the normal derivative $\partial/\partial x_2$ in terms of the tangential derivative $\partial/\partial x_1$ and the time-derivative $\partial/\partial t$.

(ii) From the matrix (3) we arrive immediately at the Stroh formalism [1962].
From (3) we have:

$$\begin{bmatrix} \bar{\boldsymbol{u}}^{(n)} \\ \bar{\boldsymbol{t}}^{(n)} \end{bmatrix} = \boldsymbol{M}^n \begin{bmatrix} \bar{\boldsymbol{u}} \\ \bar{\boldsymbol{t}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_1^{(n)} & \boldsymbol{M}_2^{(n)} \\ \boldsymbol{M}_3^{(n)} & \boldsymbol{M}_4^{(n)} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{u}} \\ \bar{\boldsymbol{t}} \end{bmatrix}, \quad -h \le x_2 \le 0, \tag{5}$$

where $\bar{\boldsymbol{u}}^{(n)}$ and $\bar{\boldsymbol{t}}^{(n)}$ are the derivative of n-order with respect to x_2 of $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{t}}$. Taking $x_2 = 0$ in (5) provides

$$\begin{bmatrix} \bar{\boldsymbol{u}}^{(n)}(0) \\ \bar{\boldsymbol{t}}^{(n)}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_1^{(n)} & \boldsymbol{M}_2^{(n)} \\ \boldsymbol{M}_3^{(n)} & \boldsymbol{M}_4^{(n)} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{u}}(0) \\ \bar{\boldsymbol{t}}(0) \end{bmatrix}, \quad n = 1, 2, \dots. \tag{6}$$

Here $\bar{\boldsymbol{u}}(0)$ and $\bar{\boldsymbol{t}}(0)$ are the value of $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{t}}$ at the bottom plane $x_2 = 0$ of the layer. From (6) it follows that

$$\bar{\boldsymbol{t}}^{(k)}(0) = \boldsymbol{M}_3^{(k)} \bar{\boldsymbol{u}}(0) + \boldsymbol{M}_4^{(k)} \bar{\boldsymbol{t}}(0), \quad k = 1, 2, \dots. \tag{7}$$

Step 2. To expand the traction vector at the top surface of the layer into Taylor series of its thickness up to n-order:

$$\bar{\boldsymbol{t}}(-h) = \bar{\boldsymbol{t}}(0) + \bar{\boldsymbol{t}}'(0)\frac{(-h)}{1!} + \bar{\boldsymbol{t}}''(0)\frac{(-h)^2}{2!} + \dots + \bar{\boldsymbol{t}}^{(n)}(0)\frac{(-h)^n}{n!}. \tag{8}$$

Suppose that the top surface of the layer is free from traction, i.e., $\bar{\boldsymbol{t}}(-h) = 0$.

Step 3. To derive the so-called pre-effective boundary condition of n-order in matrix form by substituting the expressions (7) into (8). It is of the form:

$$\bar{\boldsymbol{t}}(0) + \sum_{k=1}^{n} \left[\boldsymbol{M}_3^{(k)} \bar{\boldsymbol{u}}(0) + \boldsymbol{M}_4^{(k)} \bar{\boldsymbol{t}}(0) \right] \frac{(-h)^k}{k!} = \boldsymbol{0}. \tag{9}$$

Step 4. To obtain the effective boundary condition by using the pre-effective boundary condition (9) and the contact conditions between the layer and the half-space.

If the layer and the half-space are in welded contact with each other, then the displacement vector and the traction vector are continuous through the plane $x_2 = 0$, i. e.,

$$\bar{\boldsymbol{u}}(0) = \boldsymbol{u}(0), \quad \bar{\boldsymbol{t}}(0) = \boldsymbol{t}(0). \tag{10}$$

From (10) one can see that the derivative of any order of the displacement and traction vectors with respect to x_1 and t are also continuous through the plane $x_2 = 0$. From these facts it implies:

$$\boldsymbol{M}_3^{(k)}\bar{\boldsymbol{u}}(0) = \boldsymbol{M}_3^{(k)}\boldsymbol{u}(0), \quad \boldsymbol{M}_4^{(k)}\bar{\boldsymbol{t}}(0) = \boldsymbol{M}_4^{(k)}\boldsymbol{t}(0), \qquad k = 1, 2, \dots. \tag{11}$$

From (9)–(11) we have

$$\boldsymbol{t}(0) + \sum_{k=1}^{n}\left[\boldsymbol{M}_3^{(k)}\boldsymbol{u}(0) + \boldsymbol{M}_4^{(k)}\boldsymbol{t}(0)\right]\frac{(-h)^k}{k!} = \boldsymbol{0}. \tag{12}$$

This is the effective boundary condition of n-order in matrix form for the case of welded contact. Note that this effective boundary condition can be used not only for the plane wave problems but also for any dynamic problem.

For the case of sliding contact, where the normal displacement and stress components are continuous through the plane $x_2 = 0$, the horizontal displacement component is discontinuous through this plane, the tangential stresses vanish at it [Vinh and Anh 2014b; Vinh et al. 2014a], the situation is rather different. The effective boundary condition (in matrix form) can not be obtained directly from the pre-effective boundary condition (9) due to the discontinuity of the horizontal displacement component. In order to derive the effective boundary condition for this case we restrict ourselves to the plane wave motions [Vinh and Anh 2014b; Vinh et al. 2014a] and eliminate the horizontal displacement component from the pre-effective boundary condition (9).

Step 5. To derive the approximate secular equation (of n-order) by considering the Rayleigh wave as a surface wave propagating in the half-space (with wave number k and velocity c), without the coating layer, that is subjected the effective boundary condition. It is of the form

$$D_0(x) + D_1(x)\varepsilon + D_2(x)\frac{\varepsilon^2}{2!} + \cdots + D_n(x)\frac{\varepsilon^n}{n!} + O(\varepsilon^{n+1}) = 0, \tag{13}$$

where x is the dimensionless squared velocity of Rayleigh waves, $\varepsilon = k.h$ (the dimensionless thickness of the layer) is assumed to be much smaller than the unit, $D_n(x)$ are explicit functions of x and the dimensionless material parameters of the layer and the half-space. The error of the approximate secular equation (13) in comparison with the exact secular equation is $O(\varepsilon^{n+1})$.

In the next two sections, we apply the effective boundary condition method to two cases:

(i) An incompressible elastic half-space covered by a thin compressible elastic layer.

(ii) A compressible elastic half-space covered by a thin incompressible elastic layer.

The layer and the half-space are assumed to be in welded contact with each other.

3. Rayleigh waves in incompressible half-spaces coated by a thin compressible layer

3.1. ***Effective boundary conditions.*** Consider an incompressible orthotropic homogeneous half-space $x_2 \geq 0$ coated by a thin compressible orthotropic homogeneous layer $-h \leq x_2 \leq 0$. The layer is assumed to be perfectly bonded to the half-space. The matrix equation for the layer is of the form (3) in which matrices $\boldsymbol{M}_k$ are given by [Vinh and Linh 2012]

$$\boldsymbol{M}_1 = \begin{bmatrix} 0 & -\partial_1 \\ -\dfrac{\bar{c}_{12}}{\bar{c}_{22}}\partial_1 & 0 \end{bmatrix}, \qquad \boldsymbol{M}_2 = \begin{bmatrix} \dfrac{1}{\bar{c}_{66}} & 0 \\ 0 & \dfrac{1}{\bar{c}_{22}} \end{bmatrix},$$
$$\boldsymbol{M}_3 = \begin{bmatrix} \dfrac{\bar{c}_{12}^2 - \bar{c}_{11}\bar{c}_{22}}{\bar{c}_{22}}\partial_1^2 + \bar{\rho}\partial_t^2 & 0 \\ 0 & \bar{\rho}\partial_t^2 \end{bmatrix}, \quad \boldsymbol{M}_4 = \boldsymbol{M}_1^T. \tag{14}$$

Here we use the notations $\partial_1 = \partial/\partial x_1$, $\partial_1^2 = \partial^2/\partial x_1^2$, $\partial_t^2 = \partial^2/\partial t^2$. The pre-effective boundary condition of n-order in matrix form is (9). Since the layer and the half-space are in welded contact with each other, the effective boundary condition of n-order in matrix form is therefore (12). For $n = 3$, in component form, (12) is written as [Vinh and Linh 2012]

$$\sigma_{12} + h(r_1\sigma_{22,1} - r_3 u_{1,11} - \bar{\rho}\ddot{u}_1) + \frac{h^2}{2}\Big[r_2\sigma_{12,11} + \frac{\bar{\rho}}{\bar{c}_{66}}\ddot{\sigma}_{12} - r_3 u_{2,111} - \bar{\rho}(1+r_1)\ddot{u}_{2,1}\Big]$$
$$+\frac{h^3}{6}\Big(r_4\sigma_{22,111} + \bar{\rho}r_5\ddot{\sigma}_{22,1} - r_6 u_{1,1111} - \bar{\rho}r_7\ddot{u}_{1,11} - \frac{\bar{\rho}^2}{\bar{c}_{66}}\ddot{u}_{1,tt}\Big) = 0, \quad \text{at } x_2 = 0, \tag{15}$$

and

$$\sigma_{22} + h(\sigma_{12,1} - \bar{\rho}\ddot{u}_2) + \frac{h^2}{2}\Big[r_1\sigma_{22,11} + \frac{\bar{\rho}}{\bar{c}_{22}}\ddot{\sigma}_{22} - r_3 u_{1,111} - \bar{\rho}(1+r_1)\ddot{u}_{1,1}\Big]$$
$$+\frac{h^3}{6}\Big[r_2\sigma_{12,111} + \bar{\rho}r_8\ddot{\sigma}_{12,1} - r_3 u_{2,1111} - \bar{\rho}(1+2r_1)\ddot{u}_{2,11} - \frac{\bar{\rho}^2}{\bar{c}_{22}}\ddot{u}_{2,tt}\Big] = 0, \quad \text{at } x_2 = 0, \tag{16}$$

where

$$r_1 = \frac{\bar{c}_{12}}{\bar{c}_{22}}, \quad r_2 = r_1 + \frac{r_3}{\bar{c}_{66}}, \quad r_3 = \frac{\bar{c}_{12}^2 - \bar{c}_{11}\bar{c}_{22}}{\bar{c}_{22}}, \quad r_4 = r_1 r_2 + \frac{r_3}{\bar{c}_{22}},$$
$$r_5 = \frac{1+r_1}{\bar{c}_{22}} + \frac{r_1}{\bar{c}_{66}}, \quad r_6 = (r_1+r_2)r_3, \quad r_7 = r_1^2 + 2r_2, \quad r_8 = \frac{1+r_1}{\bar{c}_{66}} + \frac{1}{\bar{c}_{22}} \tag{17}$$

Remark 2. (i) From the traction-free boundary condition $\bar{\boldsymbol{t}}(-h) = 0$, by expanding $\bar{\boldsymbol{t}}(0)$ into Taylor series at $x_2 = -h$ one can see that $\bar{\boldsymbol{t}}(0) = O(h)$, consequently $\boldsymbol{t}(0) = O(h)$ due to the continuity of stresses at the interface.

(ii) From Equations (15), (16) and the fact: $hr_1\sigma_{22,1} = O(h^2)$, $h\sigma_{12,1} = O(h^2)$ that is implied from (i), the first-order approximate effective boundary conditions are

$$\sigma_{12} - h(r_3 u_{1,11} + \bar{\rho}\ddot{u}_1) = 0, \quad \sigma_{22} - h\bar{\rho}\ddot{u}_2 = 0, \qquad \text{at } x_2 = 0. \tag{18}$$

These first-order approximate effective boundary conditions recover the ones for the isotropic case obtained by Tiersten [1969] and Dai, Kaplunov and Prikazchikov [Dai et al. 2010].

(iii) Also from the fact: $hr_1\sigma_{22,1} = O(h^2)$, $h\sigma_{12,1} = O(h^2)$ one can see that the first-order approximate effective boundary conditions derived by Bövik [1996] include some $O(h^2)$ terms.

(iv) Again from the statement (i) it follows

$$\frac{h^3}{6}(r_4\sigma_{22,111}+\bar{\rho}r_5\ddot{\sigma}_{22,1})=O(h^4),\quad \frac{h^3}{6}(r_2\sigma_{12,111}+\bar{\rho}r_8\ddot{\sigma}_{12,1})=O(h^4). \tag{19}$$

These terms therefore can be excluded from the third-order conditions (15) and (16).

Now we consider a Rayleigh wave with velocity c (>0) and the wave number k (>0) traveling with the x_1-direction and decaying in the x_2-direction. Then its displacements and the stresses are sought in the form

$$\begin{gathered} u_1=U_1(y)e^{ik(x_1-ct)},\quad u_2=iU_2(y)e^{ik(x_1-ct)} \\ \sigma_{12}=-k\Sigma_1(y)e^{ik(x_1-ct)},\quad \sigma_{22}=ik\Sigma_2(y)e^{ik(x_1-ct)},\quad y=kx_2 \end{gathered} \tag{20}$$

Introducing (20) into (15) and (16) and taking into account (19) yield

$$\begin{aligned} &\Sigma_1(0)+\varepsilon[r_1\Sigma_2(0)-(r_3+\bar{\rho}c^2)U_1(0)] \\ &\quad+\frac{\varepsilon^2}{2}\left\{[r_3+\bar{\rho}c^2(1+r_1)]U_2(0)-\left(r_2+\frac{\bar{\rho}c^2}{\bar{c}_{66}}\right)\Sigma_1(0)\right\} \\ &\quad+\frac{\varepsilon^3}{6}\left(r_6+r_7\bar{\rho}c^2+\frac{\bar{\rho}^2c^4}{\bar{c}_{66}}\right)U_1(0)=0, \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\Sigma_2(0)+\varepsilon[\bar{\rho}c^2U_2(0)-\Sigma_1(0)] \\ &\quad+\frac{\varepsilon^2}{2}\left\{[r_3+\bar{\rho}c^2(1+r_1)]U_1(0)-\left(r_1+\frac{\bar{\rho}c^2}{\bar{c}_{22}}\right)\Sigma_2(0)\right\} \\ &\quad-\frac{\varepsilon^3}{6}\left[r_3+\bar{\rho}c^2(1+2r_1)+\frac{\bar{\rho}^2c^4}{\bar{c}_{22}}\right]U_2(0)=0. \end{aligned} \tag{22}$$

Since $0<\varepsilon\ll 1$, the relations (21) and (22) are the third-order approximate effective boundary conditions whose error is $O(\varepsilon^4)$.

3.2. ***Approximate secular equation of third-order.*** Now we can ignore the layer and consider the propagation of Rayleigh waves in the half-space, without the coating layer, that is subjected to the effective boundary conditions (21) and (22).

Since the half-space is made of incompressible orthotropic materials, the strain-stress relations are (see [Ogden and Vinh 2004])

$$\sigma_{11}=-p+c_{11}u_{1,1}+c_{12}u_{2,2},\quad \sigma_{22}=-p+c_{12}u_{1,1}+c_{22}u_{2,2},\quad \sigma_{12}=c_{66}(u_{1,2}+u_{2,1}), \tag{23}$$

where σ_{ij} and p are respectively the stress, the hydrostatic pressure associated with the incompressibility constraint. The material constants c_{11}, c_{22}, c_{12}, c_{66} satisfy the inequalities [Ogden and Vinh 2004]

$$c_{kk}>0\quad (k=1,2,6),\qquad c_{11}+c_{22}-2c_{12}>0. \tag{24}$$

In the absent of body forces, the equations of motion are

$$\sigma_{11,1}+\sigma_{12,2}=\rho\ddot{u}_1,\quad \sigma_{12,1}+\sigma_{22,2}=\rho\ddot{u}_2. \tag{25}$$

For an incompressibility material, we have

$$u_{1,1} + u_{2,2} = 0. \tag{26}$$

In addition to Equations (23), (25) and (26) are required the effective boundary conditions (15) and (16) at $x_2 = 0$ and the decay condition at $x_2 = +\infty$, namely

$$u_n = 0, \quad n = 1, 2, \qquad \text{at } x_2 = +\infty. \tag{27}$$

Since the Rayleigh wave propagates in the x_1-direction with velocity c, wave number k and decays in the x_2-direction, according to [Ogden and Vinh 2004], its displacement components satisfying the decay condition (27) are given by $(20)_1$ in which

$$U_1(y) = b_1 B_1 e^{-b_1 y} + b_2 B_2 e^{-b_2 y}, \quad U_2(y) = B_1 e^{-b_1 y} + B_2 e^{-b_2 y}, \tag{28}$$

where B_1 and B_2 are constants to be determined and b_1 and b_2 are the roots of the equation

$$\gamma b^4 - (2\beta - X)b^2 + (\gamma - X) = 0, \tag{29}$$

with positive real parts (for ensuring the decay conditions), $X = \rho c^2$, and

$$\gamma = c_{66}, \quad \beta = (\delta - 2\gamma)/2, \quad \delta = c_{11} + c_{22} - 2c_{12}. \tag{30}$$

From (29) it follows

$$b_1^2 + b_2^2 = \frac{2\beta - X}{\gamma} := S, \quad b_1^2 b_2^2 = \frac{\gamma - X}{\gamma} := P. \tag{31}$$

It is not difficult to verify that if the Rayleigh wave exists (this implies that the real parts of b_1, b_2 must be positive), then

$$0 < X < c_{66}, \tag{32}$$

and

$$b_1 b_2 = \sqrt{P}, \quad b_1 + b_2 = \sqrt{S + 2\sqrt{P}}. \tag{33}$$

Substituting $(20)_1$, (28) into (23) and using (25) lead to that the stresses are given by $(20)_2$ in which

$$\Sigma_1(y) = \beta_1 B_1 e^{-b_1 y} + \beta_2 B_2 e^{-b_2 y}, \quad \Sigma_2(y) = \gamma_1 B_1 e^{-b_1 y} + \gamma_2 B_2 e^{-b_2 y}, \tag{34}$$

where

$$\beta_n = c_{66}(b_n^2 + 1), \quad \gamma_n = (X - \delta + \beta_n) b_n, \qquad n = 1, 2. \tag{35}$$

Introducing Equations (28) and (34) into Equations (21) and (22) provides a homogeneous system linear equations for B_1, B_2, namely,

$$\begin{cases} f(b_1)B_1 + f(b_2)B_2 = 0, \\ F(b_1)B_1 + F(b_2)B_2 = 0, \end{cases} \tag{36}$$

in which

$$\begin{aligned} f(b_n) &= \beta_n + \varepsilon[r_1\gamma_n - (r_3+\bar{X})b_n] + \frac{\varepsilon^2}{2}\left[r_3 + \bar{X}(1+r_1) - \left(r_2 + \frac{\bar{X}}{\bar{c}_{66}}\right)\beta_n\right] \\ &\qquad + \frac{\varepsilon^3}{6}\left(r_6 + r_7\bar{X} + \frac{\bar{X}^2}{\bar{c}_{66}}\right)b_n, \\ F(b_n) &= \gamma_n + \varepsilon(\bar{X} - \beta_n) + \frac{\varepsilon^2}{2}\left\{[r_3 + \bar{X}(1+r_1)]b_n - \left(r_1 + \frac{\bar{X}}{\bar{c}_{22}}\right)\gamma_n\right\} \\ &\qquad - \frac{\varepsilon^3}{6}\left[r_3 + \bar{X}(1+2r_1) + \frac{\bar{X}^2}{\bar{c}_{22}}\right], \quad n = 1, 2,\ \bar{X} = \bar{\rho}c^2. \end{aligned} \tag{37}$$

For a nontrivial solution, the determinant of the matrix of the system (36) must vanish, i.e,

$$f(b_1)F(b_2) - f(b_2)F(b_1) = 0. \tag{38}$$

Using (37) into (38), after algebraically lengthy but straightforward calculations, we arrive at the approximate secular equation of third order of Rayleigh waves, namely

$$A_0 + A_1\varepsilon + A_2\frac{\varepsilon^2}{2} + A_3\frac{\varepsilon^3}{6} + O(\varepsilon^4) = 0, \tag{39}$$

where

$$\begin{aligned} A_0 &= c_{66}[(X-\delta)\sqrt{P} + X], \\ A_1 &= c_{66}[\bar{X} + (r_3+\bar{X})\sqrt{P}]\sqrt{S + 2\sqrt{P}}, \\ A_2 &= -\left(\frac{r_3}{\bar{c}_{66}} + \frac{\bar{X}}{\bar{c}_{22}} + \frac{\bar{X}}{\bar{c}_{66}}\right)A_0 + c_{66}[r_3 + \bar{X}(1-r_1)](1-\sqrt{P}) \\ &\qquad - [r_3 + \bar{X}(1-r_1)][X - \delta + c_{66}(S + \sqrt{P} + 1)] - 2\bar{X}(r_3 + \bar{X}), \\ A_3 &= c_{66}\left\{2r_3 - \bar{X}\left[2(r_1-1) + \frac{3r_3}{\bar{c}_{66}}\right] - \bar{X}\left(\frac{\bar{X}}{\bar{c}_{22}} + \frac{3\bar{X}}{\bar{c}_{66}}\right)\right. \\ &\qquad \left. - \left[r_6 - \bar{X}\left(3r_1^2 - \frac{3r_3}{\bar{c}_{22}} - r_7\right) + \bar{X}\left(\frac{3\bar{X}}{\bar{c}_{22}} + \frac{\bar{X}}{\bar{c}_{66}}\right)\right]\sqrt{P}\right\}\sqrt{S + 2\sqrt{P}}. \end{aligned} \tag{40}$$

Equation (39) in which A_k given by (40) is the desired approximate secular of third order and it is fully explicit. It is useful to convert the secular equations (39) into a dimensionless equation. For this end we use the following dimensionless parameters

$$\begin{aligned} &x = \frac{X}{c_{66}}, \quad \bar{e}_1 = \frac{\bar{c}_{11}}{\bar{c}_{66}}, \quad \bar{e}_2 = \frac{\bar{c}_{66}}{\bar{c}_{22}}, \quad \bar{e}_3 = \frac{\bar{c}_{12}}{\bar{c}_{66}}, \\ &e_\delta = \frac{\delta}{c_{66}}, \quad \bar{e}_d = \bar{e}_1 - \bar{e}_2\bar{e}_3^2, \quad r_\mu = \frac{\bar{c}_{66}}{c_{66}}, \quad r_v = \frac{c_2}{\bar{c}_2}, \quad c_2 = \sqrt{\frac{c_{66}}{\rho}}, \quad \bar{c}_2 = \sqrt{\frac{\bar{c}_{66}}{\bar{\rho}}}. \end{aligned} \tag{41}$$

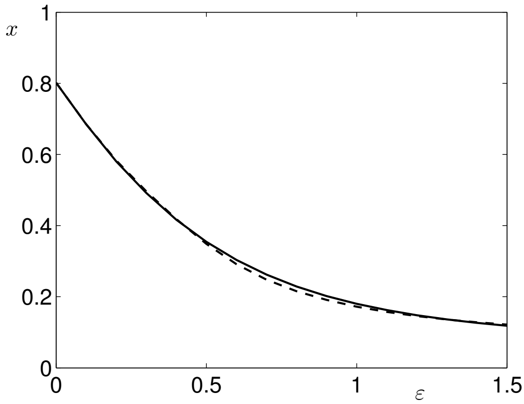

Figure 1. Dimensionless Rayleigh wave velocity $x(\varepsilon)$ in the interval [0 1.5], calculated by the exact secular equation (solid line) and by the approximate secular equation (42) (dashed line). Here we take $e_\delta = 2.6,\ \bar{e}_1 = 2,\ \bar{e}_2 = 1,\ \bar{e}_3 = 0.6,\ r_v = 2.8,\ r_\mu = 0.6.$

By dividing two sides of (39) by $(c_{66})^2$ we have

$$D_0 + D_1\varepsilon + D_2\frac{\varepsilon^2}{2} + D_3\frac{\varepsilon^3}{6} + O(\varepsilon^4) = 0, \tag{42}$$

where

$$\begin{aligned}
D_0 &= (x - e_\delta)\sqrt{P} + x,\\
D_1 &= r_\mu[r_v^2 x + (r_v^2 x - \bar{e}_d)\sqrt{P}]\sqrt{S + 2\sqrt{P}},\\
D_2 &= [\bar{e}_d - r_v^2 x(1+\bar{e}_2)]D_0 + 2r_\mu[\bar{e}_d - r_v^2 x(1-\bar{e}_2\bar{e}_3)]\sqrt{P} - 2r_\mu[\bar{e}_d - r_v^2 x(1-\bar{e}_2\bar{e}_3 + r_\mu\bar{e}_d) + r_\mu r_v^4 x^2],\\
D_3 &= -r_\mu\big\{2\bar{e}_d + r_v^2 x(2\bar{e}_2\bar{e}_3 - 2 - 3\bar{e}_d) + r_v^4 x^2(\bar{e}_2 + 3)\\
&\quad + \big[\bar{e}_d(\bar{e}_d - 2\bar{e}_2\bar{e}_3) - r_v^2 x(2\bar{e}_2^2\bar{e}_3^2 - 2\bar{e}_2\bar{e}_3 + 2\bar{e}_d + 3\bar{e}_2\bar{e}_d) + r_v^4 x^2(1+3\bar{e}_2)\big]\sqrt{P}\big\}\sqrt{S + 2\sqrt{P}},\\
&\qquad S = e_\delta - 2 - x,\ P = 1 - x.
\end{aligned} \tag{43}$$

It is clear from (42) and (43) that the squared dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ depends on 7 dimensionless parameters $e_\delta,\ \bar{e}_1,\ \bar{e}_2,\ \bar{e}_3,\ r_\mu,\ r_v$ and ε. Note that $e_\delta > 0,\ \bar{e}_1 > 0,\ \bar{e}_2 > 0$ and $\bar{e}_1 - \bar{e}_2\bar{e}_3^2 > 0$, according to the inequalities (24) and (56).

When $\varepsilon = 0$, from (42) and the first of (43) it implies

$$(x - e_\delta)\sqrt{1 - x} + x = 0. \tag{44}$$

That is the secular equation of Rayleigh waves in an incompressible orthotropic elastic half-space [Ogden and Vinh 2004].

Figure 1 presents the dependence on $\varepsilon = k.h \in [0\,1.5]$ of the dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ that is calculated by the exact secular equation (solid line) and by the approximate secular (42) (dashed line). The dimensionless parameters are taken as $e_\delta = 2.6,\ \bar{e}_1 = 2,\ \bar{e}_2 = 1,\ \bar{e}_3 = 0.6,\ r_v = 2.8$, $r_\mu = 0.6$. Note that the exact secular equation is of the form of a 6×6 determinant that is established from the traction-free conditions at the layer surface and the continuity conditions of displacements and stresses at the interface between the layer and the half-space. It is similar to Equation (19) in [Farnell and Adler 1972, p. 48]. As it is rather cumbersome it need not be written down. It is shown from Figure 1 that the exact and approximate curves of Rayleigh wave velocity almost totally coincide with each other for the values of $\varepsilon \in [0\,1.5]$. The maximum absolute error in the interval $[0\,1.5]$ is 0.0141 at $\varepsilon = 0.7$. This says that the approximate secular (42) has high accuracy.

3.3. ***Isotropic case.*** When the layer is isotropic and the half-space is transversely isotropic (with the plane of isotropy being the (x_1x_2)-plane), we have

$$\bar{c}_{11} = \bar{c}_{22} = \bar{\lambda} + 2\bar{\mu}, \quad \bar{c}_{12} = \bar{\lambda}, \quad \bar{c}_{66} = \bar{\mu}, \quad c_{11} = c_{22}, \quad c_{11} - c_{12} = 2c_{66}, \tag{45}$$

consequently

$$\bar{e}_1 = 1/\bar{g}, \quad \bar{e}_2 = \bar{g}, \quad \bar{e}_3 = 1/\bar{g} - 2, \quad \bar{e}_d = 4(1-\bar{g}), \quad e_\delta = 4, \quad S = 2 - x, \tag{46}$$

where $\bar{g} = \bar{\mu}/(\bar{\lambda} + 2\bar{\mu})$. Taking into account (46), the expressions (43) of D_k are simplified to

$$\begin{aligned}
D_0 &= (x-4)\sqrt{1-x} + x,\\
D_1 &= r_\mu[r_v^2 x + (r_v^2 x - 4 + 4\bar{g})\sqrt{1-x}](1+\sqrt{1-x}),\\
D_2 &= [4(1-\bar{g}) - (1+\bar{g})r_v^2 x]D_0 + 4r_\mu[2(1-\bar{g}) - \bar{g}r_v^2 x]\sqrt{1-x}\\
&\qquad -2r_\mu[4(1-\bar{g}) - 2(2r_\mu - 2r_\mu\bar{g} + \bar{g})r_v^2 x + r_\mu r_v^4 x^2],\\
D_3 &= -r_\mu\big\{8(1-\bar{g}) + 4(2\bar{g}-3)r_v^2 x + (3+\bar{g})r_v^4 x^2\\
&\qquad +\big[8(1-\bar{g}) + 4(\bar{g}^2-2)r_v^2 x + (1+3\bar{g})r_v^4 x^2\big]\sqrt{P}\big\}(1+\sqrt{1-x}).
\end{aligned} \tag{47}$$

When the layer and the half-space are both isotropic, the expressions (47) are unchanged, but in which: $x = \rho c^2/\mu$, $r_\mu = \bar{\mu}/\mu$, $\bar{\mu}$ and μ are the shear moduli.

4. Rayleigh waves in compressible half-spaces coated by a thin incompressible layer

4.1. ***Effective boundary conditions.*** Consider a compressible orthotropic homogeneous half-space $x_2 \geq 0$ coated by a thin incompressible orthotropic homogeneous layer $-h \leq x_2 \leq 0$. The layer and the half-space are assumed to be in perfectly bonded contact to each other. For this case, the matrix equation for the layer is (3) in which matrices $\boldsymbol{M}_k$ are given by [Vinh et al. 2014b]

$$\boldsymbol{M}_1 = \begin{bmatrix} 0 & -\partial_1 \\ -\partial_1 & 0 \end{bmatrix}, \quad \boldsymbol{M}_2 = \begin{bmatrix} \dfrac{1}{\bar{c}_{66}} & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{M}_3 = \begin{bmatrix} -\bar{\delta}\partial_1^2 + \bar{\rho}\partial_t^2 & 0 \\ 0 & \bar{\rho}\,\partial_t^2 \end{bmatrix}, \quad \boldsymbol{M}_4 = \boldsymbol{M}_1, \tag{48}$$

where $\bar{\delta} = \bar{c}_{11} - 2\bar{c}_{12} + \bar{c}_{22}$. The pre-effective boundary condition of n-order in matrix form is (9). Since the layer is perfectly bonded to the half-space, the effective boundary condition of n-order in matrix form is (12). For $n=3$, (12) is written in component form as (see [Vinh et al. 2014b])

$$\sigma_{12}+h(\sigma_{22,1}+\bar{\delta}u_{1,11}-\bar{\rho}\ddot{u}_1)+\frac{h^2}{2}\left(r_9\sigma_{12,11}+\frac{\bar{\rho}}{\bar{c}_{66}}\ddot{\sigma}_{12}+\bar{\delta}u_{2,111}-2\bar{\rho}\ddot{u}_{2,1}\right)$$
$$+\frac{h^3}{6}\left(r_9\sigma_{22,111}+\frac{\bar{\rho}}{\bar{c}_{66}}\ddot{\sigma}_{22,1}-r_{10}u_{1,1111}-\bar{\rho}r_{11}\ddot{u}_{1,11}-\frac{\bar{\rho}^2}{\bar{c}_{66}}\ddot{u}_{1,tt}\right)=0, \quad \text{at } x_2=0, \tag{49}$$

$$\sigma_{22}+h(\sigma_{12,1}-\bar{\rho}\ddot{u}_2)+\frac{h^2}{2}(\sigma_{22,11}+\bar{\delta}u_{1,111}-2\bar{\rho}\ddot{u}_{1,1})$$
$$+\frac{h^3}{6}\left(r_9\sigma_{12,111}+\frac{2\bar{\rho}}{\bar{c}_{66}}\ddot{\sigma}_{12,1}+\bar{\delta}u_{2,1111}-3\bar{\rho}\ddot{u}_{2,11}\right)=0, \quad \text{at } x_2=0, \tag{50}$$

where

$$r_9=1-\frac{\bar{\delta}}{\bar{c}_{66}}, \quad r_{10}=\bar{\delta}\left(\frac{\bar{\delta}}{\bar{c}_{66}}-2\right), \quad r_{11}=2r_9+1. \tag{51}$$

Suppose that the Rayleigh wave travels along surface $x_2=0$ with velocity $c\ (>0)$ and wave number $k\ (>0)$ in the x_1-direction and decays in the x_2-direction. Then, the displacements and stresses are sought in the form

$$\begin{aligned} u_1&=U_1(y)e^{ik(x_1-ct)}, & u_2&=iU_2(y)e^{ik(x_1-ct)} \\ \sigma_{12}&=k\Sigma_1(y)e^{ik(x_1-ct)}, & \sigma_{22}&=ik\Sigma_2(y)e^{ik(x_1-ct)}, \qquad y=kx_2 \end{aligned} \tag{52}$$

Introducing (52) into (49) and (50) and taking into account Remark 2(i) yield

$$\Sigma_1(0)+\varepsilon[(\bar{\rho}c^2-\bar{\delta})U_1(0)-\Sigma_2(0)]+\frac{\varepsilon^2}{2}\left[(\bar{\delta}-2\bar{\rho}c^2)U_2(0)-\left(r_9+\frac{\bar{\rho}c^2}{\bar{c}_{66}}\right)\Sigma_1(0)\right]$$
$$-\frac{\varepsilon^3}{6}\left(r_{10}+r_{11}\bar{\rho}c^2+\frac{\bar{\rho}^2c^4}{\bar{c}_{66}}\right)U_1(0)=0, \tag{53}$$

$$\Sigma_2(0)+\varepsilon(\bar{\rho}c^2U_2(0)+\Sigma_1(0))+\frac{\varepsilon^2}{2}[(2\bar{\rho}c^2-\bar{\delta})U_1(0)-\Sigma_2(0)]+\frac{\varepsilon^3}{6}(\bar{\delta}-3\bar{\rho}c^2)U_2(0)=0. \tag{54}$$

Since $0<\varepsilon\ll 1$, the relations (53) and (54) are the third-order approximate effective boundary conditions with the error being $O(\varepsilon^4)$.

4.2. ***Approximate secular equation of third-order.*** Now we can consider the propagation of Rayleigh waves in the uncoated half-space $x_2\geq 0$ whose surface is subjected to the effective boundary conditions (53) and (54).

Since the half-space is made of compressible orthotropic materials, the strain-stress relations are (see [Vinh and Ogden 2004])

$$\sigma_{11}=c_{11}u_{1,1}+c_{12}u_{2,2}, \quad \sigma_{22}=c_{12}u_{1,1}+c_{22}u_{2,2}, \quad \sigma_{12}=c_{66}(u_{1,2}+u_{2,1}), \tag{55}$$

where the material constants c_{11}, c_{22}, c_{12}, c_{66} satisfy the inequalities [Vinh and Ogden 2004]

$$c_{kk}>0, \quad k=1,2,6, \qquad c_{11}c_{22}-c_{12}^2>0. \tag{56}$$

In the absence of body forces, the equations of motion is (25). In addition to Equations (55) and (25), we require the effective boundary conditions (49) and (50) at $x_2 = 0$ and the decay condition

$$u_n = 0, \quad n = 1, 2, \text{ at } x_2 = +\infty. \tag{57}$$

Suppose that the Rayleigh wave travels along surface $x_2 = 0$ with velocity c and wave number k in the x_1-direction and decays in the x_2-direction. According to [Vinh and Ogden 2004], the Rayleigh wave displacement components satisfying the decay condition (57) are given by $(52)_1$ in which

$$U_1(y) = B_1 e^{-b_1 y} + B_2 e^{-b_2 y}, \quad U_2(y) = \beta_1 B_1 e^{-b_1 y} + \beta_2 B_2 e^{-b_2 y}, \tag{58}$$

where B_1 and B_2 are constant to be determined and b_1, b_2 are roots of the equation

$$c_{22}c_{66}b^4 + [(c_{12}+c_{66})^2 + c_{22}(X - c_{11}) + c_{66}(X - c_{66})]b^2 + (c_{11} - X)(c_{66} - X) = 0, \tag{59}$$

whose real parts are positive to ensure to the decay condition, $X = \rho c^2$, and

$$\beta_k = \frac{b_k(c_{12}+c_{66})}{c_{22}b_k^2 - c_{66} + \rho c^2} = \frac{c_{11} - \rho c^2 - c_{66}b_k^2}{(c_{12}+c_{66})b_k}, \quad k = 1, 2. \tag{60}$$

From (59) we have

$$\begin{aligned} b_1^2 + b_2^2 &= -\frac{(c_{12}+c_{66})^2 + c_{22}(X - c_{11}) + c_{66}(X - c_{66})}{c_{22}c_{66}} := S, \\ b_1^2 b_2^2 &= \frac{(c_{11} - X)(c_{66} - X)}{c_{22}c_{66}} := P. \end{aligned} \tag{61}$$

It is not difficult to verify that if the Rayleigh wave exists (this follows that the real parts of b_1, b_2 must be positive), then

$$0 < X < \min\{c_{11}, c_{66}\}, \tag{62}$$

and $b_1 b_2$, $b_1 + b_2$ are given by (33). Introducing $(52)_1$ and (58) into (55) yields that the stresses are given by $(52)_2$ in which

$$\begin{aligned} \Sigma_1(y) &= -c_{66}\big[(b_1+\beta_1)B_1 e^{-b_1 y} + (b_2+\beta_2)B_2 e^{-b_2 y}\big], \\ \Sigma_2(y) &= (c_{12} - c_{22}b_1\beta_1)B_1 e^{-b_1 y} + (c_{12} - c_{22}b_2\beta_2)B_2 e^{-b_2 y}. \end{aligned} \tag{63}$$

Using (58), (63) into the effective boundary conditions (53) and (54) leads to the system (36) with

$$\begin{aligned} f(b_n) &= -c_{66}(b_n+\beta_n) + \varepsilon[(\bar{X} - \bar{\delta}) - (c_{12} - c_{22}b_n\beta_n)] \\ &\quad + \frac{\varepsilon^2}{2}\left[c_{66}\left(r_9 + \frac{\bar{X}}{\bar{c}_{66}}\right)(b_n+\beta_n) + (\bar{\delta} - 2\bar{X})\beta_n\right] - \frac{\varepsilon^3}{6}\left(r_{10} + r_{11}\bar{X} + \frac{\bar{X}^2}{\bar{c}_{66}}\right), \\ F(b_n) &= c_{12} - c_{22}b_n\beta_n + \varepsilon[\bar{X}\beta_n - c_{66}(b_n+\beta_n)] \\ &\quad + \frac{\varepsilon^2}{2}[(2\bar{X} - \bar{\delta}) - (c_{12} - c_{22}b_n\beta_n)] + \frac{\varepsilon^3}{6}(\bar{\delta} - 3\bar{X})\beta_n, \quad n = 1, 2, \ \bar{X} = \bar{\rho}c^2. \end{aligned} \tag{64}$$

Setting to zero the determinant of the coefficients of the system (36)–(64) yields the equation

$$A_0 + A_1\varepsilon + \tfrac{1}{2}A_2\varepsilon^2 + \tfrac{1}{6}A_3\varepsilon^3 + O(\varepsilon^4) = 0, \tag{65}$$

where

$$
\begin{aligned}
A_0 &= c_{66}[(c_{12}^2 - c_{11}c_{22} + c_{22}X)b_1b_2 + (c_{11} - X)X],\\
A_1 &= c_{66}[\overline{X}(c_{11} - X) + c_{22}(\overline{X} - \bar{\delta})b_1b_2](b_1 + b_2),\\
A_2 &= \left(\frac{\bar{\delta}}{\bar{c}_{66}} - \frac{\overline{X}}{\bar{c}_{66}}\right)A_0 + 2[c_{66}\bar{\delta} + \overline{X}(\overline{X} - \bar{\delta})](X - c_{11}) + 2c_{66}[c_{12}\bar{\delta} - \overline{X}(\overline{X} - \bar{\delta})]\, b_1b_2,\\
A_3 &= c_{66}\left\{\left[3\overline{X}\left(\frac{\bar{\delta}}{\bar{c}_{66}} - \frac{\overline{X}}{\bar{c}_{66}}\right) - 2\bar{\delta}\right](c_{11} - X) - c_{22}\left[r_2 + \overline{X}\left(\frac{\overline{X}}{\bar{c}_{66}} - \frac{2\bar{\delta}}{\bar{c}_{66}}\right)\right]b_1b_2\right\}(b_1 + b_2),
\end{aligned}
\tag{66}
$$

in which b_1b_2 and $b_1 + b_2$ are given by (33) and (61). Equation (65) is the desired third-order approximate secular equations and it is fully explicit. In dimensionless form (65) takes the form

$$
D_0 + D_1\varepsilon + D_2\frac{\varepsilon^2}{2} + D_3\frac{\varepsilon^3}{6} + O(\varepsilon^4) = 0,
\tag{67}
$$

where

$$
\begin{aligned}
D_0 &= (e_2x - e_d)b_1b_2 + (e_1 - x)x,\\
D_1 &= r_\mu[r_v^2x(e_1 - x) + e_2(r_v^2x - \bar{e}_\delta)\, b_1b_2](b_1 + b_2),\\
D_2 &= -(r_v^2x - \bar{e}_\delta)D_0 + 2r_\mu[\bar{e}_\delta + r_\mu r_v^2x(r_v^2x - \bar{e}_\delta)](x - e_1) + 2r_\mu[e_3\bar{e}_\delta - r_\mu r_v^2x(r_v^2x - \bar{e}_\delta)]\, b_1b_2,\\
D_3 &= r_\mu\big\{[3r_v^2x(\bar{e}_\delta - r_v^2x) - 2\bar{e}_\delta](e_1 - x) - e_2[\bar{e}_\delta(\bar{e}_\delta - 2) + r_v^2x(r_v^2x - 2\bar{e}_\delta)]\, b_1b_2\big\}(b_1 + b_2),\\
b_1b_2 &= \sqrt{P}, \quad b_1 + b_2 = \sqrt{S + 2\sqrt{P}}, \quad P = \frac{(1 - x)(e_1 - x)}{e_2},\\
S &= \frac{e_2(e_1 - x) + 1 - x - (1 + e_3)^2}{e_2}.
\end{aligned}
\tag{68}
$$

Here we use the following dimensionless parameters

$$
\begin{aligned}
&x = \frac{X}{c_{66}}, \quad e_1 = \frac{c_{11}}{c_{66}}, \quad e_2 = \frac{c_{22}}{c_{66}}, \quad e_3 = \frac{c_{12}}{c_{66}}, \qquad e_d = e_1e_2 - e_3^2,\\
&\bar{e}_\delta = \frac{\bar{\delta}}{\bar{c}_{66}}, \quad r_\mu = \frac{\bar{c}_{66}}{c_{66}}, \quad r_v = \frac{c_2}{\bar{c}_2}, \quad c_2 = \sqrt{\frac{c_{66}}{\rho}}, \quad \bar{c}_2 = \sqrt{\frac{\bar{c}_{66}}{\bar{\rho}}}.
\end{aligned}
\tag{69}
$$

It is clear from (67) and (68) that the squared dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ depends on 7 dimensionless parameters e_1, e_2, e_3, $\bar{e}_\delta$, r_μ, r_v and ε. Note that $e_1 > 0$, $e_2 > 0$, $e_1e_2 - e_3^2 > 0$, $\bar{e}_\delta > 0$ according to the inequalities (24) and (56).

If $\varepsilon = 0$, from (67) and the first of (68) it follows

$$
(e_2x - e_d)\sqrt{P} + (e_1 - x)x = 0.
\tag{70}
$$

Equation (70) is the secular equation of Rayleigh waves propagating in an orthotropic elastic half-space [Chadwick 1976; Vinh and Ogden 2004].

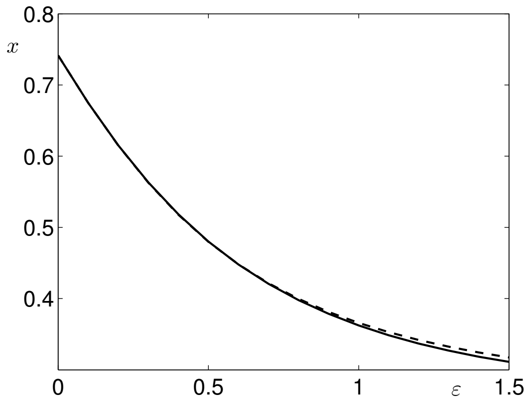

Figure 2. The Rayleigh wave velocity curves drawn by solving the exact dispersion (solid line) and by the approximate secular equation (67) (dashed line) with $r_\mu = 1.2$, $r_v = 1.8,\ e_1 = 2.5,\ e_2 = 2.8,\ e_3 = 1.3,\ \bar{e}_\delta = 3.2$.

Figure 2 presents the dependence on $\varepsilon = k.h \in [0\,1.5]$ of the dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ that is calculated by the exact secular equation (of the form of a 6×6 determinant as noted above, solid line) and by the approximate secular (67) (dashed line) with $r_\mu = 1.2,\ r_v = 1.8,\ e_1 = 2.5,$ $e_2 = 2.8,\ e_3 = 1.3,\ \bar{e}_\delta = 3.2$. It is shown from Figure 2 that the exact velocity curve and the third-order approximate velocity curve are very close to each other for the values of $\varepsilon \in [0\,1.5]$. The maximum absolute error in the interval $[0\,1.5]$ is 0.0062 at $\varepsilon = 1.5$. This means that the approximate secular equation (67) have very high accuracy.

4.3. ***Isotropic case.*** When the layer is transversely isotropic (with the plane of isotropy being the (x_1x_2)-plane) and the half-space is isotropic, i.e: $c_{11} = c_{22} = \lambda + 2\mu,\ c_{12} = \lambda,\ c_{66} = \mu,\ \bar{c}_{11} = \bar{c}_{22},\ \bar{c}_{11} - \bar{c}_{12} = 2\bar{c}_{66}$, from (60), (61) and (69), it is easy to verify that

$$\bar{e}_\delta = 4, \quad b_1 = \sqrt{1 - gx}, \quad b_2 = \sqrt{1 - x}, \quad \beta_1 = b_1, \quad \beta_2 = \frac{1}{b_2}, \tag{71}$$

where $g = \mu/(\lambda + 2\mu)$. With the help of (71), the secular equation (67) is simplified to

$$\overline{D}_0 + \overline{D}_1\varepsilon + \tfrac{1}{2}\overline{D}_2\varepsilon^2 + \tfrac{1}{6}\overline{D}_3\varepsilon^3 = 0, \tag{72}$$

in which

$$\begin{aligned}
\overline{D}_0 &= (x-2)^2 - 4\sqrt{1-x}\sqrt{1-gx},\\
\overline{D}_1 &= r_\mu x[(r_v^2 x - 4)\sqrt{1-x} + r_v^2 x\sqrt{1-gx}],\\
\overline{D}_2 &= -(r_v^2 x - 4)\overline{D}_0 - 2r_\mu[4(2 - x + 2b_1b_2) + r_\mu r_v^2 x(r_v^2 x - 4)(1 - b_1b_2)],\\
\overline{D}_3 &= r_\mu x\big\{b_1[3r_v^2 x(4 - r_v^2 x) - 8] - b_2[8 + r_v^2 x(r_v^2 x - 8)]\big\}.
\end{aligned} \tag{73}$$

Here $r_\mu = \bar{c}_{66}/\mu$. When the layer and the half-space are both isotropic, $\overline{D}_k$, $(k = 0, 1, 2, 3)$ are also calculated by (73), but in which $x = \rho c^2/\mu,\ r_\mu = \bar{\mu}/\mu,\ \bar{\mu}$ and μ are the shear moduli.

5. Numerical examples

In this section, as an example of application of the obtained approximate secular equations, we consider numerically the effect of the incompressibility on the Rayleigh wave velocity. For this aim we consider four examples. In the first example, a compressible half-space is coated either by a compressible layer or by an incompressible layer. Two these layers have the same elastic constants. In the second example, the compressible half-space is replaced by an incompressible. In the third (fourth) example, two different (compressible and incompressible) half-spaces with the same elastic constants are covered with the same compressible (incompressible) layer.

In particular, in the first example, we take $e_1 = 2.8,\ e_2 = 2.6,\ e_3 = 1.2$ for the half-space and $\bar{e}_1 = 4.5,\ \bar{e}_2 = 1,\ \bar{e}_3 = 1$ for the layers and $r_\mu = 0.5,\ r_v = 2.8$.

In the second example, we choose $\bar{e}_1 = 3.5,\ \bar{e}_2 = 1.5,\ \bar{e}_3 = 1$ for the layers and $e_\delta = 3.5$ for the half-space and $r_\mu = 1,\ r_v = 1.6$.

In the third example, the dimensionless parameters are taken as $\bar{e}_1 = 2.2,\ \bar{e}_2 = 1,\ \bar{e}_3 = 0.5$ for the layer and $e_1 = 2.5,\ e_2 = 2.8,\ e_3 = 1$ for the half-spaces and $r_\mu = 0.5,\ r_v = 2.8$.

In the last one, they are $\bar{e}_\delta = 3$ for the layer and $e_1 = 2.8,\ e_2 = 3,\ e_3 = 1.5$ for the half-spaces and $r_\mu = 1,\ r_v = 1.5$.

The numerical results of the first, second, third and fourth examples are presented in Figures 3, 4, 5 and 6, respectively. To establish the wave velocity curves, the approximate secular equations (42), (67) and Equation (29) in [Vinh and Linh 2012], Equation (3.14) in [Vinh et al. 2014b] are employed.

Figures 3–6 show:

(i) The incompressibility affects strongly the Rayleigh wave velocity.

(ii) The effect of incompressible coating layers is considerably stronger than the one of incompressible half-spaces.

(iii) The incompressibility makes the Rayleigh wave velocity increasing.

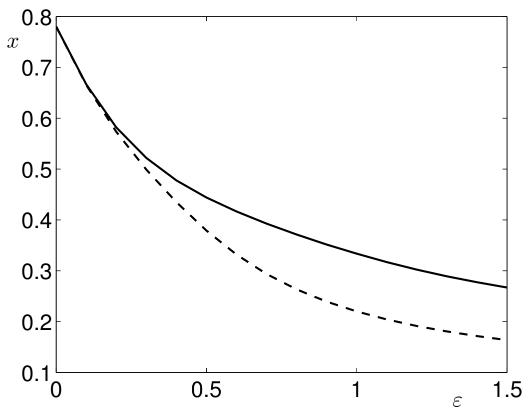

Figure 3. A compressible half-space coated by an incompressible layer (solid line drawn by solving (67)), by an compressible layer (dashed line drawn by solving Eq. (29) in [Vinh and Linh 2012]). Here we take $e_1 = 2.8,\ e_2 = 2.6,\ e_3 = 1.2$ for the half-space and $\bar{e}_1 = 4.5,\ \bar{e}_2 = 1,\ \bar{e}_3 = 1$ for the layers and $r_\mu = 0.5,\ r_v = 2.8$.

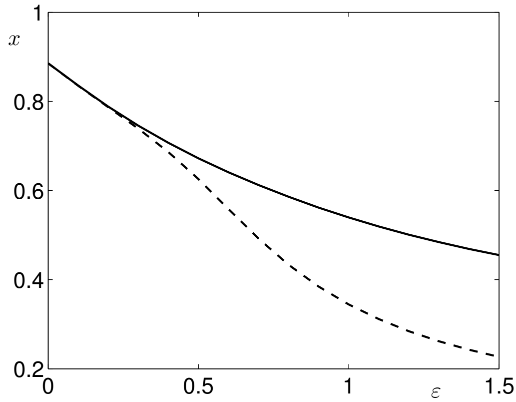

Figure 4. An incompressible half-space coated by an incompressible layer (solid line drawn by solving Equation (3.14) in [Vinh et al. 2014b], by a compressible layer (dashed line drawn by solving (42)). Here we take $\bar{e}_1 = 3.5,\ \bar{e}_2 = 1.5,\ \bar{e}_3 = 1$ for the layers and $e_\delta = 3.5$ for the half-space and $r_\mu = 1,\ r_v = 1.6$.

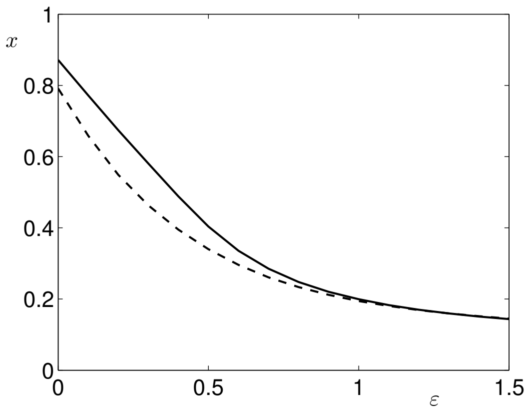

Figure 5. A compressible layer coats a compressible half-space (solid line drawn by solving Equation (29) in [Vinh and Linh 2012], an incompressible half-space (dashed line drawn by solving (42)). Here we take $\bar{e}_1 = 2.2$, $\bar{e}_2 = 1$, $\bar{e}_3 = 0.5$ for the layer and $e_1 = 2.5$, $e_2 = 2.8$, $e_3 = 1$ for the half-spaces and $r_\mu = 0.5$, $r_v = 2.8$.

6. Conclusions

In this paper, the effective boundary condition method is presented in detail. This method is then employed to derive the third-order approximate explicit secular equations of Rayleigh waves propagating in compressible (incompressible) half-space covered by a thin incompressible (compressible) layer. It is shown that they have high accuracies. Numerical examples show that the incompressibility affects considerably on the Rayleigh wave velocity. Since the obtained approximate secular equations and formulas for the Rayleigh wave velocity are totally explicit and have high accuracy, they will be significant in practical applications.

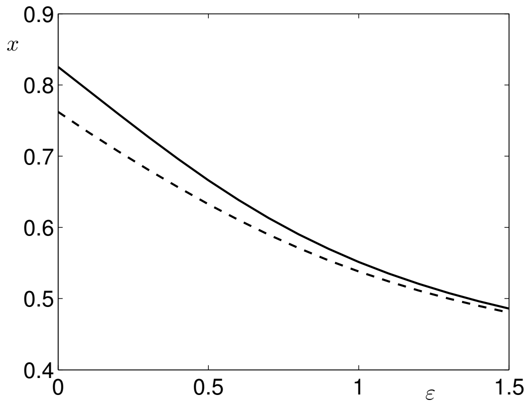

Figure 6. An incompressible layer coats an incompressible half-space (solid line drawn by solving Equation (3.14) in [Vinh et al. 2014b], a compressible half-space (dashed line drawn by solving (67)). Here we take $\bar{e}_\delta = 3$ for the layer and $e_1 = 2.8$, $e_2 = 3$, $e_3 = 1.5$ for the half-spaces and $r_\mu = 1$, $r_v = 1.5$.

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