

Computing inclusions of Schur modules

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ABSTRACT. We describe a software package for constructing minimal free resolutions of graded $\mathrm{GL}_n(\mathbb{Q})$ -equivariant modules M over $\mathbb{Q}[x_1, \dots, x_n]$ such that, for all i , the i -th syzygy module of M is generated in a single degree. We do so by describing some algorithms for manipulating polynomial representations of $\mathrm{GL}_n(\mathbb{Q})$ following ideas of Olver and Eisenbud-Fløystad-Weyman.

INTRODUCTION. This article describes the *Macaulay 2* package *PieriMaps*, which defines maps of representations of the general linear group $\mathrm{GL}_n(\mathbb{Q})$ of the form $\mathbf{S}_\mu(\mathbb{Q}^n) \rightarrow \mathbf{S}_{(d)}(\mathbb{Q}^n) \otimes \mathbf{S}_\lambda(\mathbb{Q}^n)$, and presents them as degree 0 maps $A \otimes \mathbf{S}_\mu(\mathbb{Q}^n)(-d) \rightarrow A \otimes \mathbf{S}_\lambda(\mathbb{Q}^n)$ of free modules over the polynomial ring $A = \mathrm{Sym}(\mathbb{Q}^n)$. Here dominant weights of $\mathrm{GL}_n(\mathbb{Q})$ are identified with weakly decreasing sequences λ of length n , and $\mathbf{S}_\lambda(\mathbb{Q}^n)$ denotes the irreducible representation of highest weight λ . Such maps are of general importance, and appeared recently in the work of Eisenbud, Fløystad and Weyman [EFW]. The package also describes certain related maps in characteristic p .

Let K be a field, and $A := K[x_1, \dots, x_n]$ the polynomial ring in n variables. In a recent paper of Eisenbud and Schreyer [ES], a theorem regarding the “shape” of minimal free resolutions of Cohen-Macaulay A -modules was established. Given a Cohen-Macaulay A -module M , the **Betti diagram** $\beta_{i,j}(M)$ is the number of generators of degree j in the i th syzygy module of a minimal free resolution of M . A Betti diagram is **pure** if for each i , $\beta_{i,j}(M) \neq 0$ for at most one j . In this case, we let (d_1, \dots, d_r) be the degree sequence of β : that is, $\beta_{i,d_i}(M) \neq 0$ for all i . The theorem mentioned above states that any Betti diagram of a Cohen-Macaulay module is a rational linear combination of pure Betti diagrams. The Herzog-Kühl equations [HK, Theorem 1] show that each strictly increasing degree sequence determines the corresponding Betti diagram up to a rational multiple. For $\mathrm{char} K = 0$, Eisenbud, Fløystad, and Weyman [EFW] constructed pure free resolutions for each degree sequence which live in the category of $\mathrm{GL}_n(K)$ representations. Eisenbud and Schreyer give a characteristic free construction which gives different rational multiples of the Betti diagrams in general. It is still not completely known which multiples can and cannot come from the Betti diagram of a graded module. For example, it is an interesting open problem to determine for a given degree sequence the smallest integer multiple given by the Herzog-Kühl equations which actually comes from a module.

It is the goal of this article to describe how these resolutions can be represented concretely in *Macaulay 2* [M2]. One can find \mathbb{Z} -forms for these maps and work in positive characteristic, and one such \mathbb{Z} -form is implemented, but in general it will not produce pure resolutions. We should mention that this choice of \mathbb{Z} -form is not unique. It would be interesting to investigate how often the characteristic p resolutions will be pure, and to construct \mathbb{Z} -forms which give equivariant pure resolutions in positive characteristic.

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The rest of the article is organized as follows. In Section 1, we review a construction for representations of $\mathrm{GL}_n(\mathbb{Q})$. In Section 2, we give the construction of Eisenbud, Fløystad, and Weyman, and its extension to characteristic p . In Section 3, we describe the differentials in terms of bases, in the way that it is implemented in *PieriMaps*, and illustrate an example. Finally, in Section 4 we give some examples of *Macaulay 2* code which show how one can use this package.

1. REPRESENTATIONS. In this section, we present a construction for irreducible polynomial representations of the rational algebraic group $\mathrm{GL}_n(\mathbb{Q})$ which is convenient for our purposes.

Let $\lambda := (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ be a partition and $m := |\lambda| = \lambda_1 + \dots + \lambda_n$. The **Young diagram** of λ is a pictorial representation of λ . We draw n rows of boxes with λ_i boxes in the i -th row, making sure that each row is left-justified. The notation (i, j) refers to the box in the i -th row and j -th column. A **filling** of shape λ is an assignment of the numbers $\{1, \dots, n\}$ (repetitions allowed) to the boxes of the Young diagram of λ . By picking some order on the boxes of λ , we get an action of the symmetric group \mathfrak{S}_m on the fillings of λ . We'll say that $\sigma \in \mathfrak{S}_m$ is **row-preserving** if it permutes the rows of λ amongst themselves. The **Schur module**¹ $\mathbf{S}_\lambda(\mathbb{Q}^n)$ is the rational vector space with basis given by the fillings T of λ together with the following relations:

(Symmetry) $T = \sigma \cdot T$ for any row-preserving permutation σ .

(Shuffle) For i and j such that (i, j) and $(i+1, j)$ are boxes of λ , let

$$B = \{(i, k) \mid j \leq k \leq \lambda_i\} \cup \{(i+1, k) \mid 1 \leq k \leq j\}.$$

Then $\sum \sigma \cdot T = 0$, where the sum is over all permutations which fix all boxes not in B .

For more details, the reader is referred to [Wey, Proposition 2.1.15] where our notion of Schur module is called a Weyl functor, and is denoted by K_λ . The above presentation implicitly replaces the use of divided powers with symmetric powers, but this distinction is irrelevant in characteristic 0.

A filling is a **semistandard tableau** if the numbers are weakly increasing from left to right along rows, and strictly increasing from top to bottom along columns. A basis for $\mathbf{S}_\lambda(\mathbb{Q}^n)$ (over \mathbb{Q}) is given by the semistandard tableaux of shape λ . We define an action of $\mathrm{GL}_n(\mathbb{Q})$ on $\mathbf{S}_\lambda(\mathbb{Q}^n)$ as follows. Given a filling T , let (j_1, \dots, j_m) be its entries (in some order). For $g = (g_{i,j}) \in \mathrm{GL}_n(\mathbb{Q})$, set $g \cdot T = \sum_I g_{i_1, j_1} \dots g_{i_m, j_m} T_I$ where the sum is over all index sets $I = (i_1, \dots, i_m) \in \{1, \dots, n\}^m$, and T_I is the filling obtained by replacing each j_k by i_k .

We will need **Pieri's formula**: if (d) is a partition with one row, then

$$\mathbf{S}_{(d)}(\mathbb{Q}^n) \otimes_{\mathbb{Q}} \mathbf{S}_\lambda(\mathbb{Q}^n) \cong \bigoplus_{\mu} \mathbf{S}_\mu(\mathbb{Q}^n)$$

where μ ranges over all partitions with at most n parts obtained from λ by adding d boxes, no two of which are in the same column. Thus there are inclusions (unique up to scalar multiple) $\mathbf{S}_\mu(\mathbb{Q}^n) \rightarrow \mathbf{S}_{(d)}(\mathbb{Q}^n) \otimes_{\mathbb{Q}} \mathbf{S}_\lambda(\mathbb{Q}^n)$, which we will call **Pieri inclusions**. This direct sum decomposition is only valid in characteristic 0; it is the main barrier to extending the setup of this article to positive characteristic.

2. EQUIVARIANT PURE FREE RESOLUTIONS. Fix a degree sequence $d = (d_0, \dots, d_n)$. Define a partition $\alpha(d, 0) = \lambda$ by $\lambda_i = d_n - d_i - n + i$, and for $1 \leq j \leq n$, define partitions

$$\alpha(d, j) = (\lambda_1 + d_1 - d_0, \lambda_2 + d_2 - d_1, \dots, \lambda_j + d_j - d_{j-1}, \lambda_{j+1}, \lambda_{j+2}, \dots, \lambda_n).$$

¹Actually, we are defining the **Weyl module** of highest weight λ , but in characteristic 0, it is isomorphic to what is usually called the Schur module.

Let $A := \mathbb{Q}[x_1, \dots, x_n]$, and define A -modules $\mathbf{F}(d)_i := A(-d_i) \otimes_{\mathbb{Q}} \mathbf{S}_{\alpha(d,i)}(\mathbb{Q}^n)$ for $0 \leq i \leq n$ (here $A(a)$ denotes a grading shift by a). The natural action of $\mathrm{GL}_n(\mathbb{Q})$ on $A = \bigoplus_{i \geq 0} \mathbf{S}_{(i)}(\mathbb{Q}^n)$ and on $\mathbf{S}_{\alpha(d,i)}(\mathbb{Q}^n)$ gives an action of $\mathrm{GL}_n(\mathbb{Q})$ on $\mathbf{F}(d)_i$. Note that $|\alpha(d,i)| - |\alpha(d,i-1)| = d_i - d_{i-1}$, and that $\alpha(d,i)$ is obtained from $\alpha(d,i-1)$ by adding boxes only in the i -th row, so there exists a Pieri inclusion $\phi_i: \mathbf{S}_{\alpha(d,i)}(\mathbb{Q}^n) \rightarrow \mathbf{S}_{(d_i-d_{i-1})}(\mathbb{Q}^n) \otimes_{\mathbb{Q}} \mathbf{S}_{\alpha(d,i-1)}(\mathbb{Q}^n)$. Identifying $\mathbf{S}_{(d_i-d_{i-1})}(\mathbb{Q}^n)$ with $\mathrm{Sym}^{d_i-d_{i-1}}(\mathbb{Q}^n)$ gives a degree 0 map $\partial_i: \mathbf{F}(d)_i \rightarrow \mathbf{F}(d)_{i-1}$ given by $p(x) \otimes v \mapsto p(x)\phi_i(v)$.

Theorem (Eisenbud-Fløystad-Weyman). *With the notation above,*

$$0 \rightarrow \mathbf{F}(d)_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} \mathbf{F}(d)_1 \xrightarrow{\partial_1} \mathbf{F}(d)_0$$

is a $\mathrm{GL}_n(\mathbb{Q})$ -equivariant minimal graded free resolution of $M(d) = \mathrm{coker} \partial_1$, which is pure of degree d . Furthermore, $M(d)$ is isomorphic, as a $\mathrm{GL}_n(\mathbb{Q})$ representation, to the direct sum of all irreducible summands of $A \otimes_{\mathbb{Q}} \mathbf{S}_{\lambda}(\mathbb{Q}^n)$ corresponding to the partitions that do not contain $\alpha(d,1)$, and in particular is a module of finite length.

Proof. See [EFW, Theorem 3.2]. We should emphasize that our notation for partitions differs from loc. cit. in that the notions of rows and columns are interchanged. \square

An implementation of the map ∂_1 is given in the method `pureFree` in *PieriMaps*. One can then compute the remaining maps and compose them, or compute a minimal free resolution using *Macaulay 2*. The extension of this construction to characteristic p in general does not produce pure resolutions. The idea is to clear the denominators in the matrix giving a Pieri inclusion, remove the torsion from the cokernel, and then reduce coefficients modulo p , for a given prime. This is also implemented in the method `pureFree`; the user need only specify a characteristic.

3. COMBINATORIAL DESCRIPTION OF THE PIERI INCLUSION. We wish to describe how the Pieri inclusions work in terms of the bases of semistandard tableaux of the Schur modules $\mathbf{S}_{\lambda}(\mathbb{Q}^n)$. The Pieri inclusion which induces the map $\partial_i: \mathbf{F}(d)_i \rightarrow \mathbf{F}(d)_{i-1}$ is given by

$$\mathbf{S}_{\alpha(d,i)}(\mathbb{Q}^n) \rightarrow \mathbf{S}_{(d_i-d_{i-1})}(\mathbb{Q}^n) \otimes_{\mathbb{Q}} \mathbf{S}_{\alpha(d,i-1)}(\mathbb{Q}^n),$$

and $\alpha(d,i-1)$ is obtained from $\alpha(d,i)$ by removing $d_i - d_{i-1}$ boxes from the i -th row. So to describe this map, we first describe the case $d_i - d_{i-1} = 1$. In this case, the map was described by Olver in [Olv, §6]. For the general case, one can iterate this process of removing one box at a time and compose the maps. It needs to be proved (though it is not hard), that such a composition is the desired map, and in fact, if one removes boxes from multiple rows, the order in which the boxes are removed is irrelevant (up to nonzero scalar multiple).

Suppose we have a partition λ with $\lambda_{k-1} > \lambda_k$. Let μ be the partition resulting from adding a box to the k -th row of λ . Set $B_k = \{(j_1, \dots, j_p) \mid 0 = j_1 < \dots < j_p = k\}$, and for $J = (j_1, \dots, j_p) \in B_k$, define $\#J = p$. Given $x^a \otimes T$ where $x^a = x_1^{a_1} \dots x_n^{a_n} \in \mathbb{Q}[x_1, \dots, x_n]$ is a monomial and T is a filling of μ , we first interpret the x^a as a “zeroth” row of T consisting of $a_1 + \dots + a_n$ boxes filled with a_i ’s. We’ll call this a **shape**. Given numbers $0 \leq i < j \leq k$, define $\tau_{i,j}(x^a \otimes T)$ to be the sum of all shapes obtained from $x^a \otimes T$ by removing a box along with its entry (and then the boxes to the right of it get shifted one to the left) from row j and moving it to the end of row i . Then set $\tau_J := \tau_{j_{p-1}, j_p} \circ \dots \circ \tau_{j_2, j_3} \circ \tau_{j_1, j_2}$

and $c_J := \prod_{i=2}^{p-1} (\mu_{j_i} - \mu_k + k - j_i)$. The desired map $\mathbf{S}_\mu(\mathbb{Q}^n) \rightarrow \mathbf{S}_{(1)}(\mathbb{Q}^n) \otimes_{\mathbb{Q}} \mathbf{S}_\lambda(\mathbb{Q}^n)$ is now the alternating sum $\sum_{J \in B_k} \frac{(-1)^{\#J} \tau_J}{c_J}$. We give an example to illustrate all of the above.

Example. Let $\mu = (2, 1, 1)$, $k = 3$, $n = 3$, and consider the element $1 \otimes T$ where $T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}$. We think of T as having a “row 0” which is an empty row on top of T . If $J = (0, 1, 3)$, then

$$\tau_J(T) = \tau_{1,3}(\tau_{0,1}(T)) = \tau_{1,3} \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 2 & 3 & \\ \hline 2 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & \\ \hline 2 & & \\ \hline \end{array}$$

which we really think of as $x_1 \otimes \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 2 & \\ \hline \end{array} + x_2 \otimes \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} = -\frac{1}{2}x_1 \otimes \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array} + x_2 \otimes \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$ in the module $\mathbb{Q}[x_1, x_2, x_3] \otimes_{\mathbb{Q}} \mathbf{S}_{(2,1)} V$. The equality follows from the relations described in Section 1. In this case, $c_J = 2$.

4. AN EXAMPLE OF USING *PieriMaps*. We illustrate some of the main uses of *PieriMaps*. First, we load the package and define a polynomial ring in 3 variables $A = \mathbb{Q}[a, b, c]$:

```
i1 : loadPackage "PieriMaps"
i2 : A = QQ[a,b,c];
```

Now we compute a module whose pure free resolution has degree sequence $\{0, 1, 3, 5\}$.

```
i3 : pureFree({0,1,3,5}, A)
o3 = | 3a 0  b 0  c 0  0 0 0      0 0  0 0 0 0 0 |
     | 0 3a 0  b 0  c 0 0 0      0 0  0 0 0 0 0 |
     | 0 0 2a 0  0 0 2b 0  c      0 0  0 0 0 0 0 |
     | 0 0 0 2a 0  0 0 2b 0      c 0  0 0 0 0 0 |
     | 0 0 0 0 2a 0  0 0  b      0 2c 0 0 0 0 0 |
     | 0 0 0 0 0 2a 0  0  0      b 0  2c 0 0 0 0 |
     | 0 0 0 0 0 0 0  a  -1/2a 0 0  0 3b c 0 0 |
     | 0 0 0 0 0 0 0  0  a  -2a 0 0  2b 2c 0 0 |
           8      15
o3 : Matrix A  <--- A
```

This is the matrix of free A -modules induced by the Pieri inclusion $\mathbf{S}_{3,1}(\mathbb{Q}^3) \rightarrow \mathbf{S}_1(\mathbb{Q}^3) \otimes \mathbf{S}_{2,1}(\mathbb{Q}^3)$. The bases of A^{15} and A^8 can be listed with the commands `standardTableaux(3, {3, 1})` and `standardTableaux(3, {2, 1})`, respectively. For example, the first command has $\{\{0, 0, 0\}, \{1\}\}$ as its first basis element, which is meant to represent the semistandard tableau $\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 1 & & \\ \hline \end{array}$. Alternatively, this map can be produced with the command `pieri({3, 1, 0}, {1}, A)` because the partition $(2, 1, 0)$ is obtained by subtracting 1 from the first entry of $(3, 1, 0)$. The module we are after is the cokernel of this map.

```
i4 : res coker oo
           8      15      10      3
o4 = A  <--- A  <--- A  <--- A  <--- 0
           0      1      2      3      4
```

```
o4 : ChainComplex
```

We can check that this resolution is pure by looking at its Betti table:

```
i5 : betti oo
      0  1  2  3
o5 = total: 8 15 10 3
      0: 8 15  .  .
      1:  .  . 10  .
      2:  .  .  .  3
o5 : BettiTally
```

We can lift the above map to a \mathbb{Z} -form and reduce the coefficients modulo 2:

```
i6 : pieri({3,1},{1},ZZ/2[x,y,z])
o6 = | x 0 y 0 0 z 0 0 0 0 0 0 0 0 0 |
      | 0 x 0 0 y 0 0 z 0 0 0 0 0 0 |
      | 0 0 0 y 0 0 z 0 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 y 0 z 0 0 0 z 0 0 |
      | 0 0 0 0 0 0 y 0 z 0 0 0 0 0 0 |
      | 0 0 0 0 0 0 0 y z 0 0 y 0 0 0 |
      | 0 0 0 0 0 0 0 0 y z 0 0 y 0 0 |
      | 0 0 0 0 0 0 0 0 0 x y 0 z 0 0 |
      | 0 0 0 0 0 0 0 0 0 0 x 0 z 0 0 |
      ZZ      8      ZZ      15
o6 : Matrix (--[x, y, z]) <--- (--[x, y, z])
           2              2
```

However, the resolution of its cokernel is not pure:

```
i7 : betti res coker oo
      0  1  2  3
o7 = total: 8 15 10 3
      0: 8 15  9  3
      1:  .  .  .  .
      2:  .  .  1  .
o7 : BettiTally
```

Now let's look at an example of changing the order of composition of Pieri inclusions. We know that there is a nonzero inclusion of the form $\mathbf{S}_{2,1}(\mathbb{Q}^3) \rightarrow \mathbf{S}_2(\mathbb{Q}^3) \otimes \mathbf{S}_1(\mathbb{Q}^3)$. There are two different ways to get this map with the function `pieri`. We could remove a box from the second row of $(2, 1, 0)$ and then remove a box from the first row of $(2, 0, 0)$ to get the composition

$$\mathbf{S}_{2,1}(\mathbb{Q}^3) \rightarrow \mathbf{S}_1(\mathbb{Q}^3) \otimes \mathbf{S}_2(\mathbb{Q}^3) \xrightarrow{1 \otimes \varphi} \mathbf{S}_1(\mathbb{Q}^3) \otimes \mathbf{S}_1(\mathbb{Q}^3) \otimes \mathbf{S}_1(\mathbb{Q}^3) \xrightarrow{p \otimes 1} \mathbf{S}_2(\mathbb{Q}^3) \otimes \mathbf{S}_1(\mathbb{Q}^3),$$

where φ is a Pieri inclusion and p is the quotient map:

```
i8 : pieri({2,1}, {2,1}, A)
o8 = | ab  ac  1/2b2  1/2bc  1/2bc  1/2c2  0  0 |
      | -a2  0   -1/2ab  1/2ac  -ac   0    bc  1/2c2 |
      | 0   -a2  0      -ab   1/2ab  -1/2ac  -b2  -1/2bc |
      3      8
o8 : Matrix A <--- A
```

Or, we could remove a box from the first row of $(2, 1, 0)$ and then remove a box from the second row of $(1, 1, 0)$ to get the composition

$$\mathbf{S}_{2,1}(\mathbb{Q}^3) \rightarrow \mathbf{S}_1(\mathbb{Q}^3) \otimes \mathbf{S}_{1,1}(\mathbb{Q}^3) \xrightarrow{1 \otimes \varphi} \mathbf{S}_1(\mathbb{Q}^3) \otimes \mathbf{S}_1(\mathbb{Q}^3) \otimes \mathbf{S}_1(\mathbb{Q}^3) \xrightarrow{p \otimes 1} \mathbf{S}_2(\mathbb{Q}^3) \otimes \mathbf{S}_1(\mathbb{Q}^3),$$

which is written as

```
i9 : pieri({2,1}, {1,2}, A)
o9 = | 2ab 2ac b2 bc bc c2 0 0 |
      | -2a2 0 -ab ac -2ac 0 2bc c2 |
      | 0 -2a2 0 -2ab ab -ac -2b2 -bc |
      3      8
o9 : Matrix A <--- A
```

Here, we see that the matrices differ by a scalar multiple of 2. In general, different orders of box removals will yield the same matrix up to nonzero scalar multiple. The differences arise from the denominators c_J .

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