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Decomposition of Monomial Algebras: Applications and Algorithms

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ABSTRACT. Considering finite extensions $K[A] \subseteq K[B]$ of positive affine semigroup rings over a field K we have developed in [BEN] an algorithm to decompose $K[B]$ as a direct sum of monomial ideals in $K[A]$. By computing the regularity of homogeneous semigroup rings from the decomposition, we have confirmed the Eisenbud-Goto conjecture in a range of new cases not tractable by standard methods. Here we first illustrate this technique and its implementation in our *Macaulay2* package *MonomialAlgebras* by computing the decomposition and the regularity step by step for an explicit example. We then focus on ring-theoretic properties of simplicial semigroup rings. From the characterizations given in [BEN], we develop and prove explicit algorithms testing various properties, including being Buchsbaum, Cohen-Macaulay, Gorenstein, normal, and seminormal. All algorithms are implemented in our *Macaulay2* package.

INTRODUCTION. Let B be a positive affine semigroup, that is, B is a finitely generated subsemigroup of $\mathbb{N}^m$ for some m. Let K be a field and $K[B]$ the affine semigroup ring associated to B, which can be identified with the subring of $K[t_1,\ldots,t_m]$ generated by monomials $t^u := t_1^{u_1}\cdots t_m^{u_m}$, where $u = (u_1,\ldots,u_m) \in B$. Denote by $C(B)$ and by $G(B)$ the cone and the group generated by B. From now on let $A \subseteq B$ be positive affine semigroups with $C(A) = C(B)$. We will now discuss the decomposition of $K[B]$ into a direct sum of monomial ideals in $K[A]$. Observe that

$$K[B] = \bigoplus\nolimits_{g\in G} K\cdot\left\{t^b \mid b \in B\cap g\right\},$$

where $G := G(B)/G(A)$. We have $C(A) = C(B)$ if and only if $K[B]$ is a finitely generated $K[A]$-module. From this it follows that G is finite, and we can compute the above decomposition since all summands are finitely generated. Moreover, there are shifts $h_g \in G(B)$ such that $I_g := K\cdot\left\{t^{b-h_g} \mid b \in B\cap g\right\}$ is a monomial ideal in $K[A]$. Thus, $K[B] \cong \bigoplus_{g\in G} I_g(-h_g)$ as $\mathbb{Z}^m$-graded $K[A]$-modules (with $\deg t^b = b$). A detailed formulation of the algorithm computing the ideals I_g and shifts h_g and a more general version of the decomposition in the setup of cancellative abelian semigroup rings over an integral domain can be found in [BEN, Algorithm 1, Theorem 2.1].

Our original motivation for developing this decomposition was to provide a fast algorithm to compute the Castelnuovo-Mumford regularity $\operatorname{reg} K[B]$ of a homogeneous semigroup ring in order to test the Eisenbud-Goto conjecture [EG, Conjecture p. 93]. Recall that the ***Castelnuovo-Mumford regularity*** $\operatorname{reg} M$ of a finitely generated graded module M over a standard graded polynomial ring $R = K[x_1,\ldots,x_n]$ is defined as the smallest integer m such that every j-th syzygy module of M is generated by elements of degree at most $m+j$. Moreover, B is called a ***homogeneous semigroup*** if there exists a group homomorphism $\deg\colon G(B) \to \mathbb{Z}$ with $\deg b_i = 1$ for $i = 1,\ldots,n$, where

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$\mathrm{Hilb}(B) = \{b_1,\dots,b_n\}$ is the minimal generating set of B; by $\operatorname{reg} K[B]$ we mean its regularity with respect to the R-module structure which is given by the K-algebra homomorphism $R \twoheadrightarrow K[B]$, $x_i \mapsto t^{b_i}$.

The toric Eisenbud-Goto conjecture can be formulated as follows: let K be a field and B a homogeneous semigroup, then $\operatorname{reg} K[B] \leq \deg K[B] - \operatorname{codim} K[B]$, where $\deg K[B]$ denotes the degree and $\operatorname{codim} K[B] := \dim_K K[B]_1 - \dim K[B]$ the codimension. Even this special case of the Eisenbud-Goto conjecture is largely open; for references on known results see [BEN, §4]. The regularity of $K[B]$ is usually computed from a minimal graded free resolution. If n is large this computation is very expensive, and hence it is impossible to test the conjecture systematically in high codimension using this method. However, choosing A to be generated by minimal generators $e_1,\dots,e_d$ of $C(B)$ of degree 1 the regularity can be computed as $\operatorname{reg} K[B] = \max\{\operatorname{reg} I_g + \deg h_g \mid g \in G\}$, where $\operatorname{reg} I_g$ denotes the regularity of I_g with respect to the canonical $T = K[x_1,\dots,x_d]$-module structure given by $T \twoheadrightarrow K[A]$, $x_i \mapsto t^{e_i}$. Since the free resolution of every ideal I_g appearing has length at most $d-1$, this computation is typically much faster than the traditional approaches. This enabled us to test the conjecture for a large class of homogeneous semigroup rings by using our regularity algorithm. See [BEN, §4] for details.

In first section, we illustrate a step by step decomposition and regularity computation for an explicit example using our *Macaulay2* [M2] package *MonomialAlgebras*. We say that $K[B]$ is a *simplicial semigroup ring* if the cone $C(B)$ is simplicial. In second section, we focus on simplicial semigroup rings $K[B]$. Based on the characterizations of ring-theoretic properties given in [BEN, Proposition 3.1] we develop explicit algorithms for testing whether $K[B]$ is Buchsbaum, Cohen-Macaulay, Gorenstein, seminormal, or normal. We also discuss that, by known results, all these ring-theoretic properties imply the Eisenbud-Goto conjecture. The algorithms mentioned are implemented in our *Macaulay2* package.

DECOMPOSITION AND REGULARITY. Our *Macaulay2* package can be loaded by

```
Macaulay2, version 1.6
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone
```

```
i1 : needsPackage "MonomialAlgebras";
--loading configuration for package "MonomialAlgebras" from file .../init-MonomialAlgebras.m2
--loading configuration for package "FourTiTwo" from file .../init-FourTiTwo.m2
```

We discuss the decomposition and computation of the regularity at the example of the homogeneous semigroup $B \subset \mathbb{N}^3$ specified by a list of generators

```
i2 : B = {{4,0,0},{2,2,0},{2,0,2},{0,2,2},{0,3,1},{3,1,0},{1,1,2}};
```

As an input for our algorithms we encode this data in a multigraded polynomial ring

```
i3 : K = ZZ/101;
i4 : S = K[x_1..x_7, Degrees => B];
```

The usual approach to computing $\operatorname{reg} K[B]$ is to obtain it from a minimal graded free resolution of the toric ideal I_B with respect to the standard grading:

```
i5 : IB = binomialIdeal S;
o5 : Ideal of S
i6 : R = newRing(ring IB, Degrees => {7:1});
i7 : betti res sub(IB,R)
```

```
             0 1  2  3 4 5
o7 = total: 1 8 15 13 6 1
         0: 1 .  .  . . .
         1: . 6  8  3 . .
         2: . 2  3  . . .
         3: . .  4 10 6 1

o7 : BettiTally
```

Hence, we observe that $\operatorname{reg} K[B] = 3$. On the other hand, using the decomposition of $K[B]$, we can proceed as follows: The command

```
i8 : dc = decomposeMonomialAlgebra S

o8 = HashTable{| -1 | => {ideal (x , x ), | -1 |}}
               | 1  |           1   3    | 1  |
               | 0  |                    | 0  |
               0 => {ideal 1, 0}

o8 : HashTable
```

decomposes $K[B]$ over $K[A]$ where $A \subseteq B$ is generated by minimal generators of $C(B)$ with minimal coordinate sum; so in the example $A = \langle (4,0,0),(2,2,0),(2,0,2),(0,2,2),(0,3,1)\rangle$. The keys of the hash table represent the elements of G and the values are the tuples (I_g, h_g), hence

$$K[B] \cong \langle \bar{x}_1, \bar{x}_3 \rangle \, (-(-1,1,0)) \oplus K[A] \qquad (\dagger)$$

as $\mathbb{Z}^3$-graded $K[A]$-modules; here we write $K[A] \cong T/J$ with $T = K[x_1,x_2,x_3,x_4,x_5]$ and $\bar{x}_i$ for the class of x_i. The on-screen output of *Macaulay2* does not distinguish between the class and the representative. With respect to the standard grading $\deg u = (u_1+u_2+u_3)/4$, Equation (†) yields $\operatorname{reg} K[B] = \max\{\operatorname{reg}\langle \bar{x}_1, \bar{x}_3\rangle + (-1+1+0)/4,\ \operatorname{reg} K[A] + 0\}$. We compute $\operatorname{reg} K[A]$:

```
i9 : KA = ring first first values dc;

i10 : T = newRing(ring ideal KA, Degrees => {5:1});

i11 : J = sub(ideal KA,T);

o11 : Ideal of T

i12 : betti res J

              0 1 2
o12 = total: 1 3 2
          0: 1 . .
          1: . 1 .
          2: . 2 2

o12 : BettiTally
```

Hence $\operatorname{reg} K[A] = 2$. We can compute $\operatorname{reg}\langle \bar{x}_1, \bar{x}_3\rangle$ as follows:

```
i13 : I1 = first (values dc)#0

o13 = ideal (x , x )
              1   3

o13 : Ideal of KA

i14 : g = matrix entries sub(gens I1, T);

               1      2
o14 : Matrix T  <--- T

i15 : betti res image map(coker gens J, source g, g)
```

```
              0 1 2 3
o15 = total: 2 5 4 1
          1: 2 2 . .
          2: . . . .
          3: . 3 4 1

o15 : BettiTally
```

Thus, $\operatorname{reg}\langle\overline{x_1},\overline{x_3}\rangle = 3$, and therefore we see again that $\operatorname{reg} K[B] = 3$. Observe that the resolution of $K[B]$ has length 5, whereas the ideals I_g have resolutions of length at most 3. The command

```
i16 : regularityMA S

o16 = {3, {{ideal (x , x ), | -1 |}}}
                    1   3   | 1  |
                            | 0  |

o16 : List
```

provides an implementation of this approach, also returning the tuples (I_g, h_g) where the maximum is achieved. By [BEN, Proposition 4.1] we have $\deg K[B] = \#G \cdot \deg K[A] = 10$ since

```
i17 : degree J

o17 = 5
```

Moreover, $\operatorname{codim} K[B] = 4$ since $\dim K[B] = \dim C(B) = 3$. Hence the ring $K[B]$ satisfies the Eisenbud-Goto bound.

ALGORITHMS FOR RING THEORETIC PROPERTIES. In this section, we focus on simplicial semigroup rings $K[B]$. Based on the characterizations given in [BEN, Proposition 3.1], we develop and prove explicit algorithms for testing whether $K[B]$ is Buchsbaum, Cohen-Macaulay, Gorenstein, seminormal, or normal. In the simplicial case, all these properties are independent of K and they imply the Eisenbud-Goto conjecture by results of [SV, Corollary p. 307], [T, Corollary 2.3 and Proposition 2.2], and [N, Theorem 3.16]. As an example, consider the following homogeneous simplicial semigroup $B \subset \mathbb{N}^3$ specified by the generators

```
i18 : B = {{4,0,0},{0,4,0},{0,0,4},{1,0,3},{0,2,2},{3,0,1},{1,2,1}};
```

We compute the decomposition of $K[B]$ over $K[A]$, where $A = \langle(4,0,0),(0,4,0),(0,0,4)\rangle \subset B$ is generated again by minimal generators of $C(B)$ with minimal coordinate sum.

```
i19 : S = K[x_1..x_7, Degrees => B];

i20 : decomposeMonomialAlgebra S

o20 = HashTable{| -1 | => {ideal 1, | 3 |}          }
                | 0  |              | 0 |
                | 1  |              | 1 |

                | -1 | => {ideal 1, | 3 |}
                | 2  |              | 2 |
                | -1 |              | 3 |

                | 0 | => {ideal 1, | 0 |}
                | 2 |              | 2 |
                | 2 |              | 2 |

                | 1  | => {ideal 1, | 1 |}
                | 0  |              | 0 |
                | -1 |              | 3 |
```

```
| 1 | => {ideal 1, | 1 |}
| 2 |              | 2 |
| 1 |              | 1 |
| 2 | => {ideal (x , x , x ), | 2 |}
| 0 |             3   1   2   | 0 |
| 2 |                         | 2 |
| 2 | => {ideal 1, | 2 |}
| 2 |              | 2 |
| 0 |              | 4 |
0 => {ideal 1, 0}
o20 : HashTable
```

Hence, $K[B] \cong K[A] \oplus K[A](-1)^4 \oplus K[A](-2)^2 \oplus \langle x_1, x_2, x_3 \rangle (-1)$ with respect to the standard grading induced by $\deg u = (u_1 + u_2 + u_3)/4$. It follows that depth $K[B] = 1$; thus, $K[B]$ is not Cohen-Macaulay. Hence, $K[B]$ is also not normal by [H, Theorem 1]. We can test seminormality via Algorithm 1.

Algorithm 1 (Seminormality test)**.**
Input: A simplicial semigroup $B \subseteq \mathbb{N}^m$.
Output: `true` if $K[B]$ is seminormal, `false` otherwise.

1: Let $e_1, \ldots, e_d \in B$ be minimal generators of $C(B)$ with minimal coordinate sum, and set $A := \langle e_1, \ldots, e_d \rangle$.
2: Compute $B_A := \{x \in B \mid x \notin B + (A \setminus \{0\})\}$ as described in [BEN, Algorithm 1, Step 1].
3: **for all** $x \in B_A$ **do**
4: Solve the linear system of equations $\sum_{i=1}^{d} \lambda_i e_i = x$ for $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Q}^d$.
5: **if** $\|\lambda\|_\infty > 1$ **then return** `false`.
6: **end for**
7: **return** `true`.

Here, $\|-\|_\infty$ denotes the maximum norm. Note that all λ_i are non-negative since $C(B)$ is a simplicial cone. Verifying in Step 5 the condition $\|\lambda\|_\infty \geq 1$ instead, results in an algorithm which tests normality. Using our package we observe that, in the example above, $K[B]$ is not seminormal:

```
i21 : isSeminormalMA B
o21 = false
```

The Buchsbaum property can be tested by Algorithm 2. We denote by $K[A]_+$ the homogeneous maximal ideal of $K[A]$.

Algorithm 2 (Buchsbaum test)**.**
Input: A simplicial semigroup $B = \langle b_1, \ldots, b_n \rangle \subseteq \mathbb{N}^m$.
Output: `true` if $K[B]$ is Buchsbaum, `false` otherwise.

1: Let $e_1, \ldots, e_d \in B$ be minimal generators of $C(B)$ with minimal coordinate sum, and set $A := \langle e_1, \ldots, e_d \rangle$.
2: Using the (minimal) generators $e_1, \ldots, e_d$ of A decompose $K[B] \cong \bigoplus_{g \in G} I_g(-h_g)$, where $I_g \subseteq K[A]$, $h_g \in G(B)$ and $G = G(B)/G(A)$ by [BEN, Algorithm 1].
3: **if** there exists $g \in G$ with $I_g \neq K[A]$ and $I_g \neq K[A]_+$ **then return** `false`.
4: $H := \{h_g \mid g \in G \text{ with } I_g = K[A]_+\}$.
5: $C := \{b_1, \ldots, b_n\} \setminus \{0, e_1, \ldots, e_d\}$.

6: $H+C := \left\{h_g + b_i \mid h_g \in H,\, b_i \in C\right\}$.
7: **return** `true` if $(H+C)\cap H = \varnothing$ and `false` otherwise.

Proof of Algorithm 2. By [BEN, Proposition 3.1], the ring $K[B]$ is Buchsbaum iff each ideal I_g is either equal to $K[A]$, or to $K[A]_+$ and $h_g + b \in B$ for all $b \in \mathrm{Hilb}(B)$. So, Step 3 is correct and we may now assume that $I_g = K[A]$ or $I_g = K[A]_+$ for all $g \in G$. Recall that $I_g = \{t^{v-h_g} \mid v \in \Gamma_g\}K[A]$ where $\Gamma_g = \{x \in B_A \mid x \in g\}$. Moreover, $\{t^{v-h_g} \mid v \in \Gamma_g\}$ is always a minimal generating set of I_g and $h_g = \sum_{k=1}^{d} \min\left\{\lambda_k^v \mid v \in \Gamma_g\right\} e_k$ where $v = \sum_{k=1}^{d} \lambda_k^v e_k$ with $\lambda_k^v \in \mathbb{Q}$. Since $h + e_k \in B_A$ for all $h \in H$ and all $k = 1,\ldots,d$, we have $H \cap B = \varnothing$. In case that $(H+C)\cap H \neq \varnothing$, we obtain $h + b \notin B$ for some $h \in H$ and some $b \in B \setminus \{0\}$, that is $h + \mathrm{Hilb}(B) \not\subseteq B$. Hence, $K[B]$ is not Buchsbaum.

In case that $K[B]$ is not Buchsbaum, there is an $h \in H$ and some $b \in \mathrm{Hilb}(B)$ such that $h + b \notin B$. It is now sufficient to show that $b \in C$ and $h + b \in H$. By the above argument, $b \in C$. Let $m_k = h + b + e_k$ for $k = 1,\ldots,d$. Suppose that $m_i \notin B_A$ for some $i \in \{1,\ldots,d\}$. Since $m_k - e_k \notin B$ for all $k = 1,\ldots,d$, necessarily $m_i - e_j \in B$ for some $j \neq i$. Consider $y = m_j - \sum_{k=1}^{d} n_k e_k \in B$ with $n_k \in \mathbb{N}$ such that $\sum_{k=1}^{d} n_k$ is maximal. By construction $y \in B_A$, moreover, $n_j = 0$ since $m_j - e_j \notin B$. In the same way if $x = m_i - e_j - \sum_{k=1}^{d} n_k e_k \in B$ with $\sum_{k=1}^{d} n_k$ maximal, then $x \in B_A$. Since $m_i, m_j \in g$ for some $g \in G$, we also have $x, y \in g$. Since $e_1,\ldots,e_d$ are linearly independent, we have $\lambda_j^y - \lambda_j^x \geq 2$. Moreover, since $t^{y-h_g}, t^{x-h_g} \in K[A]$ we get that t^{y-h_g} is not a linear form. Hence $I_g \neq K[A]$ and $I_g \neq K[A]_+$, thus, $m_k \in B_A$ for all $k = 1,\ldots,d$. We have $\#\Gamma_g \in \{1,d\}$ by minimality, hence $\Gamma_g = \{m_1,\ldots,m_d\}$. By construction, $h_g = h + b$ and $I_g = K[A]_+$, therefore $h + b \in H$. □

In Step 2 of Algorithm 2, the shifts h_g and hence the ideals I_g are uniquely determined since $e_1,\ldots,e_d$ are linearly independent. This is not true for arbitrary generating sets. Continuing the example, by [SV, Corollary p. 307] and

```
i22 : isBuchsbaumMA B
o22 = true
```

it follows that $K[B]$ satisfies the Eisenbud-Goto conjecture. We can also read off from the decomposition the regularity and the Eisenbud-Goto bound: we have $\operatorname{reg} K[A] = 0$ and $\operatorname{reg}\langle x_1,x_2,x_3\rangle = 1$, therefore $\operatorname{reg} K[B] = \max\{0,1,2,1+1\} = 2$. Moreover, $\deg K[B]$ is the number of ideals which occur in the decomposition, hence $\deg K[B] - \operatorname{codim} K[B] = 8 - 4 = 4$.

When B is Buchsbaum, the regularity of $K[B]$ is independent of the field K since all ideals in the decomposition are equal to the homogeneous maximal ideal or to $K[A]$.

We finish this section by providing Algorithm 3 for testing the Gorenstein property.

Algorithm 3 (Gorenstein test)**.**
Input: A simplicial semigroup $B \subseteq \mathbb{N}^m$.
Output: `true` if $K[B]$ is Gorenstein, `false` otherwise.
1: Let $e_1,\ldots,e_d \in B$ be minimal generators of $C(B)$ with minimal coordinate sum, and set $A := \langle e_1,\ldots,e_d\rangle$.
2: Using the (minimal) generators $e_1,\ldots,e_d$ of A decompose $K[B] \cong \bigoplus_{g\in G} I_g(-h_g)$ where $I_g \subseteq K[A]$, $h_g \in G(B)$ and $G = G(B)/G(A)$ by [BEN, Algorithm 1].
3: **if** there exists $g \in G$ with $I_g \neq K[A]$ **then return** `false`.
4: $H := \left\{h_g \mid g \in G\right\}$.
5: **if** $h \in H$ with maximal coordinate sum is not unique **then return** `false`.

6: Let $h \in H$ with maximal coordinate sum.
7: **while** $H \neq \varnothing$ **do**
8: Let $h_g \in H$.
9: **if** $h - h_g \notin H$ **then return** `false`.
10: $H := H \setminus \{h_g,\, h - h_g\}$.
11: **end while**
12: **return** `true`.

Proof of Algorithm 3. By [BEN, Proposition 3.1] the ring $K[B]$ is Gorenstein if and only if $I_g = K[A]$ for all $g \in G$ and H has a unique maximal element with respect to $\leq$ given by $x \leq y$ if there is a $z \in B$ such that $x + z = y$. Note that $H = B_A$ since $I_g = K[A]$ for all $g \in G$. If there is a maximal element $h \in H$, then this element has maximal coordinate sum. If H has more than one element with maximal coordinate sum, then H does not have a unique maximal element. To complete the proof we need to show that an element $h_g \in H$ satisfies $h_g \leq h$ iff $h - h_g \in H$. But this follows from the fact that if $x \notin B_A$ then $x + y \notin B_A$ for all $x, y \in B$. ☐

Performing Steps 1–3 of Algorithm 3 (and returning `true` afterwards) also gives a test for the Cohen-Macaulay property.

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