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Zach Teitler

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ZACH TEITLER

ABSTRACT: We describe a new software package for computing multiplier ideals in certain cases, including monomial ideals, monomial curves, generic determinantal ideals, and hyperplane arrangements. In these cases we take advantage of combinatorial formulas for multiplier ideals given by results of Howald, Thompson, and Johnson. The package uses the package *Normaliz*. It is available as a library for *Macaulay2*.

INTRODUCTION. Multiplier ideals have been applied to a number of problems in algebraic geometry in recent years, most spectacularly in recent major advances in the minimal model program [Hacon and McKernan 2007; Birkar et al. 2010] that built on earlier work showing the deformation invariance of plurigenera [Siu 1998]. Other applications include several results on singularities and linear series [Lazarsfeld 2004; Ein and Mustaţă 2006], a bound for symbolic powers [Ein et al. 2001], and applications to algebraic statistics [Watanabe 2009; Zwiernik 2011; Drton et al. 2009, Chapter 5]. New applications of multiplier ideals continue to emerge in topics such as Chow stability [Lee 2008] and singularities in generic liaison [Niu 2014]. With broad and growing interest in multiplier ideals, it is increasingly valuable to compute examples.

For a thorough introduction to multiplier ideals see [Lazarsfeld 2004]. Here is a definition of multiplier ideals in terms of resolution of singularities: Suppose X is a smooth variety over a field $\Bbbk$ (we may assume X is affine, or even just $\Bbbk^n$, since we are primarily interested in local issues), $I \subset \mathcal{O}_X$ is a nonzero ideal sheaf, and $\mu : Y \to X$ is a log resolution of I, so that the total transform $I\mathcal{O}_Y$ defines a divisor F with simple normal crossings support, $F = \sum a_i E_i$, where the E_i are distinct reduced components of F. Then for each real number $c \geq 0$ the c-th multiplier ideal is defined by

$$\mathcal{J}(I^c) = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor c \cdot F \rfloor),$$

where $K_{Y/X}$ is the relative canonical divisor of Y over X, defined locally by the vanishing of the determinant of the Jacobian $d\mu$, and $\lfloor c \cdot F \rfloor$ denotes the component-wise round-down of the $\mathbb{R}$-divisor $c \cdot F$, given by $\lfloor c \cdot F \rfloor = \sum \lfloor c a_i \rfloor E_i$.

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In theory it is algorithmic to compute multiplier ideals by computing a resolution of singularities of I followed by a sheaf pushforward. In practice it is more difficult; see [Frühbis-Krüger 2014].

Shibuta's [2011] algorithm for computing Bernstein–Sato polynomials and multiplier ideals via Gröbner basis methods in Weyl algebras (which he implemented in *Risa/Asir*) was refined and implemented in the *Dmodules* library for *Macaulay2* by Berkesch and Leykin [2010]. The *Dmodules* library can compute multiplier ideals and jumping numbers of arbitrary ideals, but due to the difficulty of the computations can only handle modestly sized examples.

We describe a new software package named *MultiplierIdeals* that computes multiplier ideals of special ideals, including monomial ideals, ideals of monomial curves, generic determinantal ideals, and hyperplane arrangements, via combinatorial methods, using the *Normaliz* software and interface to *Macaulay2* by Bruns, Ichim, and Kämpf [Bruns and Ichim 2010; Bruns and Kämpf 2010]. The combinatorial methods allow computations of somewhat larger examples than can be handled by general methods.

Wherever possible we work over an arbitrary field $\Bbbk$. Since multiplier ideals in our cases are computed by resolutions defined over $\mathbb{Z}$ (or over the $\mathbb{Z}$-algebra generated by the coefficients of the defining equations of the input data), we may work in arbitrary characteristic.

Our package also computes certain quantities associated to multiplier ideals: the log canonical thresholds and jumping numbers. Because of the round-down operation, $\mathcal{J}(I^{c+\epsilon}) = \mathcal{J}(I^c)$ for sufficiently small $\epsilon > 0$. A real number $c \geq 0$ is a jumping number of I if $\mathcal{J}(I^c) \neq \mathcal{J}(I^{c-\epsilon})$ for all $\epsilon > 0$. Every jumping number is in fact rational. The smallest strictly positive jumping number is called the log canonical threshold of I, denoted $\operatorname{lct}(I)$. It turns out that $\mathcal{J}(I^0) = (1)$ is the trivial ideal, so $\operatorname{lct}(I)$ is the supremum of c such that $\mathcal{J}(I^c) = (1)$; equivalently, $\operatorname{lct}(I)$ is the first value of c such that $\mathcal{J}(I^c) \neq (1)$.

The portion dealing with monomial ideals was written first and distributed as the package *MonomialMultiplierIdeals*. The portion dealing with monomial curves was written by C. Raicu, B. Snapp, and the author at the 2011 IMA Special Workshop on *Macaulay2*, and distributed as the package *SpaceCurvesMultiplierIdeals*. The portion dealing with hyperplane arrangements is based on code written by Denham and Smith for the *HyperplaneArrangements* package [2011]. These portions were all integrated into the present package, and computations with generic determinantal ideals added, at the 2012 *Macaulay2* Workshop at Wake Forest.

MONOMIAL IDEALS. For a monomial ideal $I \subset \Bbbk[x_1, \dots, x_n]$, let $\operatorname{monom}(I) \subset \mathbb{Z}^n_{\geq 0}$ be the set of exponent vectors of monomials in I. The Newton polyhedron $\operatorname{Newt}(I)$ is the convex hull of $\operatorname{monom}(I)$. Let $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$.

Multiplier ideals of monomial ideals are described by the following theorem of Howald:

Theorem 1 [Howald 2001; Blickle 2004]. *The multiplier ideal* $\mathcal{J}(I^c)$ *is the monomial ideal containing* x^v *if and only if* $v + \mathbf{1} \in \operatorname{Int}(c \cdot \operatorname{Newt}(I))$. *Here* Int *denotes the topological interior of* $c \cdot \operatorname{Newt}(I)$ *relative to the nonnegative orthant, that is, as a subset of* $(\mathbb{R}_{\geq 0})^n$.

In other words, the multiplier ideal is the quotient ideal

$$\mathcal{J}(I^c) = \big(x^v : v \in \operatorname{Int}(c \cdot \operatorname{Newt}(I))\big) : x^{\mathbf{1}}.$$

The Newton polyhedron $\operatorname{Newt}(I)$ is defined by a system of inequalities $Av \geq b$, where A is an $r \times n$ matrix, b is a vector, and $\geq$ is the partial order of entry-wise comparison, where $a \geq b$ if and only if $a_i \geq b_i$, $1 \leq i \leq r$. Then $c \cdot \operatorname{Newt}(I)$ is defined by $Av \geq cb$. The interior $\operatorname{Int}(c \cdot \operatorname{Newt}(I))$ is the solution of the system of inequalities given by

$$\begin{cases} A_i v > cb_i & \text{if } b_i \neq 0, \\ A_i v \geq cb_i = 0 & \text{if } b_i = 0. \end{cases}$$

Since $\operatorname{Newt}(I)$ is a rational polyhedron, we can (and do) take the A and b to have integer entries. Furthermore, since I is an ideal the entries of A and b are nonnegative. In practice, it is sufficient to compute $\mathcal{J}(I^c)$ for rational $c = p/q$, and this can be done as follows: To find the integer vectors v lying in the topological interior of the solution region to $Av \geq cb$ (equivalently, $qAv \geq pb$), we add 1 to the nonzero entries of pb, yielding a vector b' with entries $b'_i = pb_i + 1$ if $b_i \neq 0$, and $b'_i = pb_i = b_i = 0$ otherwise. Then the multiplier ideal $\mathcal{J}(I^c)$ is the quotient $(x^v : qAv \geq b') : x^{\mathbf{1}}$.

The software *Normaliz* can compute the defining inequalities $Av \geq b$ of $\operatorname{Newt}(I)$ and the solutions to the modified system $qAv \geq b'$; *Macaulay2* can compute the ideal quotient by $x^{\mathbf{1}}$, giving the multiplier ideal:

```
Macaulay2, version 1.6
i1 : needsPackage "MultiplierIdeals";
i2 : R = QQ[x,y,z,w];
i3 : I = monomialIdeal(x*y, x*z, y*z, y*w, z*w^2);
o3 : MonomialIdeal of R
i4 : logCanonicalThreshold(I)
o4 = 2
i5 : multiplierIdeal(I,7/3)
                            2
o5 = ideal (y, z*w, z , x*z)
i6 : toString jumpingNumbers(I)
o6 = {{2, 7/3, 5/2, 8/3, 3, 10/3, 7/2, 11/3, 4}, {ideal(z,y),
     ideal(y,z*w,z^2,x*z), ideal(z*w,y*w,y*z,x*z,y^2,x*y),
     ideal(y*w,y*z,x*z,y^2,x*y,z*w^2,z^2*w),
     ideal(y*z*w,y^2*w,y*z^2,x*z^2,y^2*z,x*y*z,x*y^2,z^2*w^2), ...
```

The `jumpingNumbers` command produces a list with two elements:

(1) A list of the jumping numbers of I in the interval $(0, k(I)]$, where $k(I)$ is the analytic spread of I. A different interval may be specified as an optional argument.

(2) A list of the multiplier ideals at the jumping numbers. (The list is truncated in the above example.)

Thus the output of the last command says that this ideal I has jumping numbers $2, 7/3, \dots$, and gives the corresponding multiplier ideals: $\mathscr{J}(I^2) = (z, y)$, $\mathscr{J}(I^{7/3}) = (y, zw, z^2, xz)$, and so on. Multiplier ideals and jumping numbers for $c > k(I)$ are given by Skoda's theorem [Lazarsfeld 2004, Theorem 9.6.21]. Namely, for $c > k(I)$, $\mathscr{J}(I^c) = I\mathscr{J}(I^{c-1})$; and $c > k(I)$ is a jumping number if and only if $c - 1$ is a jumping number.

In the above example, the log canonical threshold, single multiplier ideal $\mathscr{J}(I^{7/3})$, and list of nine jumping numbers and multiplier ideals were each computed in a fraction of a second on a 2012 MacBook with dual-core 64-bit 2.9 GHz CPU and 8 GB RAM. By way of comparison, the *Dmodules* package takes about 42 seconds to compute the log canonical threshold on the same machine, and about 84 seconds to compute $\mathscr{J}(I^{7/3})$. This comparison is only intended to illustrate the advantages of using special algorithms where available, and we remind the reader that the *DModules* package uses a general method.

For monomial ideals, extra information is available: for any monomial x^v, the package computes the threshold value $\min\{c : x^v \notin \mathscr{J}(I^c)\}$, and the list of facets of the Newton polyhedron that impose the nonmembership:

```
i7 : toString logCanonicalThreshold(I,z^2*w)
o7 = (3,matrix {{2, 2, 1, 1, -3}, {2, 2, 0, 1, -2}})
```

This output means that $z^2w \notin \mathscr{J}(I^3)$ but $z^2w \in \mathscr{J}(I^c)$ for $c < 3$. That is, for the exponent vector $v = (0, 0, 2, 1)$, $v + \mathbf{1}$ lies on the boundary of $3 \cdot \mathrm{Newt}(I)$; and furthermore it lies on the intersection of two facets, the ones scaled up from the facets of $\mathrm{Newt}(I)$ defined by $2x + 2y + z + w = 3$ and $2x + 2y + w = 2$.

The log canonical threshold of the ideal I itself is the threshold value for $1 = x^0$.

MONOMIAL CURVES. An affine monomial curve is one parametrized by $t \mapsto (t^{a_1}, \dots, t^{a_n})$. We can and do assume that $1 \le a_1 \le \cdots \le a_n$ and $\gcd(a_1, \dots, a_n) = 1$. For convenience we denote this curve by $C(a_1, \dots, a_n)$. It has a singularity at the origin when $a_1 \ge 2$. The defining ideal is the kernel of the map $\Bbbk[x_1, \dots, x_n] \to \Bbbk[t]$ given by $x_i \mapsto t^{a_i}$. This is a binomial ideal.

The multiplier ideals of affine monomial curves in dimension $n = 3$ have been found by Howard Thompson [2014],[1] using the combinatorial description of the

[1] This paper states the result over $\mathbb{C}$, but it holds over any field.

resolution of singularities of a binomial ideal given in [González Pérez and Teissier 2002]. This yields a combinatorial formula in terms of the vector (a, b, c) of exponents appearing in the parametrization $t \mapsto (t^a, t^b, t^c)$. Our software package implements Thompson's result, again calling on *Normaliz* to find generators for the semigroup of integer solutions to certain linear inequalities:

```
i8 : R = QQ[x,y,z]; S = QQ[t];
i10 : I = kernel map(S,R,{t^3,t^4,t^5}) -- ideal of C(3,4,5)
                 2          2      2   3
o10 = ideal (y  - x*z, x y - z , x  - y*z)
```

To compute the multiplier ideals and log canonical threshold of I, we input the list of exponents in the parametrization:

```
i11 : toString logCanonicalThreshold(R,{3,4,5})
o11 = 13/9
i12 : multiplierIdeal(R,{3,4,5},13/9)
o12 = ideal (z, y, x)
```

GENERIC DETERMINANTAL IDEALS. Let $X = (x_{i,j})_{1\le i\le m, 1\le j\le n}$ be an $m \times n$ generic matrix, meaning one whose entries are independent variables. Let $I_r(X)$ be the ideal generated by the $r \times r$ minors of X. The multiplier ideals of $I_r(X)$ have been found by Amanda Johnson [2003]:[2]

Theorem 2. *With X, m, n, and r as above, the multiplier ideals are given by the following intersection of symbolic powers of determinantal ideals*:

$$\mathcal{J}(I_r(X)^c) = \bigcap_{i=1}^{r} I_i(X)^{(\lfloor c(r+1-i)\rfloor + 1 - (n-i+1)(m-i+1))}.$$

Recall that symbolic powers of generic determinantal ideals may be expressed as

$$I_r(X)^{(a)} = \sum_{\kappa_1+\cdots+\kappa_s=a} \prod_{i=1}^{s} I_{r-1+\kappa_i}(X),$$

with the sum taken over partitions of a. See [Bruns and Vetter 1988, Theorem 10.4].

We may compute multiplier ideals of determinantal ideals in our software by giving the matrix X and the size of minors. Here we examine multiplier ideals of the size-2 and size-3 minors of a 4×5 generic matrix:

```
i13 : x = getSymbol"x"; R = QQ[x_1..x_20];
i15 : X = genericMatrix(R,4,5); -- a 4x5 generic matrix
i16 : logCanonicalThreshold(X,2) -- lct of the ideal of 2x2 minors
o16 = 10
i17 : multiplierIdeal(X,2,10) == minors(1,X) -- J(I^10) where I = 2x2 minors
o17 = true
```

[2] This dissertation states the result for algebraically closed fields, but it holds over any field.

```
i18 : multiplierIdeal(X,2,11) == (minors(1,X))^3 -- J(I^11)
o18 = true
```

HYPERPLANE ARRANGEMENTS. A formula for multiplier ideals of hyperplane arrangements was found by Mustaţă [2006] and simplified in [Teitler 2008].[3] The *HyperplaneArrangements* package [Denham and Smith 2011] uses these results to compute multiplier ideals and log canonical thresholds of hyperplane arrangements. To this we add the ability to compute jumping numbers and other minor modifications. I thank Graham Denham and Gregory G. Smith, the authors of *HyperplaneArrangements*, for their permission to copy and modify their package's source code.

The following is Example 6.3 of [Berkesch and Leykin 2010]:

```
i19 : R = QQ[x,y,z];
i20 : ff = toList factor ( (x^2-y^2)*(x^2-z^2)*(y^2-z^2)*z ) / first;
i21 : A = arrangement ff;
i22 : toString jumpingNumbers(A,IntervalType=>"ClosedOpen")
o22 = {{3/7, 4/7, 2/3, 6/7}, {ideal(z,y,x), ideal(z^2,y*z,x*z,y^2,x*y,x^2),
      ideal(y^2*z-z^3,x^2*z-z^3,x*y^2-x*z^2,x^2*y-y*z^2), ...
```

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ZACH TEITLER:

zteitler@boisestate.edu
Department of Mathematics, Boise State University, 1910 University Drive, Boise, ID 83725-1555, United States

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