

NISSUNA UMANA INVESTIGAZIONE SI PUO DIMANDARE VERA SCIENZA
S'ESSA NON PASSA PER LE MATEMATICHE DIMOSTRAZIONI
LEONARDO DA VINCI

vol. 1

no. 1

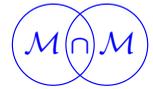
2013

MATHEMATICS AND MECHANICS
of
Complex Systems

XU WANG AND PETER SCHIAVONE

**DISLOCATIONS, IMPERFECT INTERFACES
AND INTERFACE CRACKS IN ANISOTROPIC ELASTICITY
FOR QUASICRYSTALS**





DISLOCATIONS, IMPERFECT INTERFACES AND INTERFACE CRACKS IN ANISOTROPIC ELASTICITY FOR QUASICRYSTALS

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We derive the detailed structures of the 6×6 matrices N_i and $N_i^{(-1)}$ ($i = 1, 2, 3$) in the Stroh formalism of anisotropic elasticity for quasicrystals. All six matrices are expressed explicitly in terms of the sixty-six reduced elastic compliances. The Green's functions for bi-quasicrystals are also obtained. Next, we derive compliant and stiff interface models in anisotropic quasicrystalline bimetals. It is observed that the phonon normal traction is always continuous across the stiff interface. Finally we present the asymptotic fields associated with a traction-free, semi-infinite interface crack in anisotropic quasicrystalline bimetals and solve the collinear interface crack problem. The interface crack-tip field consists of three two-dimensional oscillatory singularities which are evaluated via the introduction of three complex stress intensity factors.

1. Introduction

The theory of anisotropic elasticity for crystals has been actively investigated for more than half a century; see, for example, [Lekhnitskii 1950; Eshelby et al. 1953; Stroh 1958; Willis 1964; Willis 1970; Willis 1971; Clements 1971; Barnett and Lothe 1973; Suo 1990; Suo et al. 1992; Gao et al. 1992; Ting 1986; Ting 1988; Ting 1996; Ru 2001; Cheng and Reddy 2002; Ting and Schiavone 2010; Wang and Pan 2010]. As pointed out in [Ting 1996], the Stroh formalism [1958], which is based on Stroh eigenvalues and eigenvectors, allows for an elegant and powerful analysis of two-dimensional deformations of anisotropic crystalline solids where as many as fifteen elastic constants are involved. The beauty of the Stroh formalism has indeed been observed by various researchers; see, for example, [Suo 1990; Ru 2001; Cheng and Reddy 2002; Lazar and Kirchner 2005; Wang and Pan 2010].

Quasicrystalline structures were first reported in [Shechtman et al. 1984]. The generalized anisotropic elasticity for quasicrystals developed in [Hu et al. 2000] requires that anisotropic quasicrystals have as many as one hundred and twenty elastic constants. Even for the study of two-dimensional deformations of quasicrystals,

Communicated by David Steigmann.

MSC2010: primary 74B05, 74E10; secondary 74E15.

Keywords: quasicrystal, anisotropic elasticity, Stroh formalism, dislocation, interface crack.

sixty-six pertinent elastic constants remain. [Ding et al. \[1995\]](#) extended the original six-dimensional Stroh formalism for crystals by developing a twelve-dimensional version in order to study dislocation problems in quasicrystals.

Outline of paper. In [Section 2](#) we present the Stroh formalism for two-dimensional deformations in anisotropic quasicrystals. In [Section 3](#), we derive the detailed structures of N_i and $N_i^{(-1)}$ ($i = 1, 2, 3$) appearing in the Stroh formalism for quasicrystals through the introduction of sixty-six reduced elastic compliances $S_{ij} = S_{ji}$ ($i, j = 1 \dots 15$ and $i, j \neq 3, 9, 10, 14$). In [Section 4](#), the Green's functions for a line force and dislocation located

- (1) in a homogeneous quasicrystal,
- (2) along the interface of bi-quasicrystals, and
- (3) in one of two bonded quasicrystalline half-planes

are investigated. Based on the structures obtained for N_i ($i = 1, 2, 3$), we then develop in [Section 5](#) two imperfect interface models in anisotropic quasicrystalline bimetals. Finally, in [Section 6](#), we derive the near-tip field of an interface crack in anisotropic bi-quasicrystals consisting of three two-dimensional coupled oscillatory singularities. Three complex intensity factors K_1 , K_2 and K_3 are introduced to quantify the near-tip field. Also studied in [Section 6](#) is the collinear interface crack problems in anisotropic bi-quasicrystals following the decoupling method proposed in [\[Suo 1990\]](#) and [\[Suo et al. 1992\]](#).

2. The Stroh formalism

In a fixed rectangular coordinate system x_i ($i = 1, 2, 3$), let u_i and w_i be the phonon and phason displacements, σ_{ij} ($\sigma_{ij} = \sigma_{ji}$) and H_{ij} ($H_{ij} \neq H_{ji}$) be the phonon and phason stresses in an anisotropic quasicrystalline material. The stress-strain law and the equations of equilibrium are [\[Hu et al. 2000\]](#):

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}u_{k,l} + R_{ijkl}w_{k,l}, & \sigma_{ij,j} &= 0, \\ H_{ij} &= R_{klij}u_{k,l} + K_{ijkl}w_{k,l}, & H_{ij,j} &= 0, \end{aligned} \quad (1)$$

where the comma denotes differentiation, C_{ijkl} are the elastic constants in the phonon field, K_{ijkl} are the elastic constants in the phason field and R_{ijkl} are the phonon-phason coupling constants. In addition C_{ijkl} , R_{ijkl} and K_{ijkl} possess the following symmetry:

$$C_{ijkl} = C_{jikl} = C_{klij} = C_{ijlk}, \quad R_{ijkl} = R_{jikl}, \quad K_{ijkl} = K_{klij}. \quad (2)$$

For two-dimensional deformations in which u_i and w_i depend only on x_1 and x_2 , the general solutions can be expressed as

$$\begin{aligned} \mathbf{u} &= [u_1 \ u_2 \ u_3 \ w_1 \ w_2 \ w_3]^T = \mathbf{A}\mathbf{f}(z) + \overline{\mathbf{A}}\overline{\mathbf{f}(\bar{z})}, \\ \Phi &= [\Phi_1 \ \Phi_2 \ \Phi_3 \ \Psi_1 \ \Psi_2 \ \Psi_3]^T = \mathbf{B}\mathbf{f}(z) + \overline{\mathbf{B}}\overline{\mathbf{f}(\bar{z})}, \end{aligned} \quad (3)$$

where

$$\begin{aligned} \mathbf{A} &= [a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6], \quad \mathbf{B} = [b_1 \ b_2 \ b_3 \ b_4 \ b_5 \ b_6], \\ \mathbf{f}(z) &= [f_1(z_1) \ f_2(z_2) \ f_3(z_3) \ f_4(z_4) \ f_5(z_5) \ f_6(z_6)]^T, \\ z_i &= x_1 + p_i x_2, \quad \text{Im}\{p_i\} > 0 \quad (i = 1, \dots, 6), \end{aligned} \quad (4)$$

with

$$\begin{bmatrix} N_1 & N_2 \\ N_3 & N_1^T \end{bmatrix} \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix} = p_i \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix} \quad (i = 1, \dots, 6), \quad (5)$$

$$N_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad N_2 = \mathbf{T}^{-1}, \quad N_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}, \quad (6)$$

$$\mathbf{Q} = \begin{bmatrix} C_{11} & C_{16} & C_{15} & R_{11} & R_{19} & R_{15} \\ C_{16} & C_{66} & C_{56} & R_{61} & R_{69} & R_{65} \\ C_{15} & C_{56} & C_{55} & R_{51} & R_{59} & R_{55} \\ R_{11} & R_{61} & R_{51} & K_{11} & K_{19} & K_{15} \\ R_{19} & R_{69} & R_{59} & K_{19} & K_{99} & K_{59} \\ R_{15} & R_{65} & R_{55} & K_{15} & K_{59} & K_{55} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} C_{16} & C_{12} & C_{14} & R_{16} & R_{12} & R_{17} \\ C_{66} & C_{26} & C_{46} & R_{66} & R_{62} & R_{67} \\ C_{56} & C_{25} & C_{45} & R_{56} & R_{52} & R_{57} \\ R_{61} & R_{21} & R_{41} & K_{16} & K_{12} & K_{17} \\ R_{69} & R_{29} & R_{49} & K_{69} & K_{29} & K_{79} \\ R_{65} & R_{25} & R_{45} & K_{56} & K_{25} & K_{57} \end{bmatrix}, \quad (7)$$

$$\mathbf{T} = \begin{bmatrix} C_{66} & C_{26} & C_{46} & R_{66} & R_{62} & R_{67} \\ C_{26} & C_{22} & C_{24} & R_{26} & R_{22} & R_{27} \\ C_{46} & C_{24} & C_{44} & R_{46} & R_{42} & R_{47} \\ R_{66} & R_{26} & R_{46} & K_{66} & K_{26} & K_{67} \\ R_{62} & R_{22} & R_{42} & K_{26} & K_{22} & K_{27} \\ R_{67} & R_{27} & R_{47} & K_{67} & K_{27} & K_{77} \end{bmatrix}.$$

The matrices \mathbf{Q} and \mathbf{T} are symmetric and positive definite.

In (7), we have adopted the contracted notation

$$11 \leftrightarrow 1, \quad 22 \leftrightarrow 2, \quad 33 \leftrightarrow 3, \quad 23 \leftrightarrow 4, \quad 31 \leftrightarrow 5, \quad 12 \leftrightarrow 6, \quad 32 \leftrightarrow 7, \quad 13 \leftrightarrow 8, \quad 21 \leftrightarrow 9.$$

In addition the stress function vector Φ is defined, in terms of the phonon and phason stresses, by

$$\sigma_{i1} = -\Phi_{i,2}, \quad \sigma_{i2} = \Phi_{i,1}, \quad H_{i1} = -\Psi_{i,2}, \quad H_{i2} = \Psi_{i,1} \quad (i = 1, 2, 3). \quad (8)$$

The 6×6 matrices \mathbf{A} and \mathbf{B} satisfy the normalized orthogonal relationship

$$\begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \bar{\mathbf{B}}^T & \bar{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = \mathbf{I}. \quad (9)$$

Therefore three real Barnett–Lothe tensors \mathbf{S} , \mathbf{H} and \mathbf{L} can be introduced:

$$\mathbf{S} = i(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \quad \mathbf{H} = 2i\mathbf{A}\mathbf{A}^T, \quad \mathbf{L} = -2i\mathbf{B}\mathbf{B}^T. \quad (10)$$

Here \mathbf{H} and \mathbf{L} are positive definite matrices. It can also be easily checked that

$$\begin{bmatrix} \mathbf{N}_1^{(-1)} & \mathbf{N}_2^{(-1)} \\ \mathbf{N}_3^{(-1)} & \mathbf{N}_1^{(-1)T} \end{bmatrix} \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix} = \frac{1}{p_i} \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix} \quad (i = 1, \dots, 6), \quad (11)$$

where

$$\mathbf{N}_1^{(-1)} = -\mathbf{Q}^{-1}\mathbf{R}, \quad \mathbf{N}_2^{(-1)} = -\mathbf{Q}^{-1}, \quad \mathbf{N}_3^{(-1)} = \mathbf{T} - \mathbf{R}^T\mathbf{Q}^{-1}\mathbf{R}. \quad (12)$$

In addition, both the 6×6 matrix \mathbf{M} and its inverse \mathbf{M}^{-1} , defined by

$$\mathbf{M} = -i\mathbf{B}\mathbf{A}^{-1} = \mathbf{H}^{-1} + i\mathbf{H}^{-1}\mathbf{S}, \quad \mathbf{M}^{-1} = i\mathbf{A}\mathbf{B}^{-1} = \mathbf{L}^{-1} - i\mathbf{S}\mathbf{L}^{-1}, \quad (13)$$

are positive definite Hermitian matrices. In this work we make use of a positive definite Hermitian matrix \mathbf{N} involving bimaterial elastic constants and defined as

$$\mathbf{N} = \mathbf{M}_1^{-1} + \bar{\mathbf{M}}_2^{-1} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1} + i(\mathbf{S}_2\mathbf{L}_2^{-1} - \mathbf{S}_1\mathbf{L}_1^{-1}). \quad (14)$$

It is clear from (7) that for two-dimensional deformations of quasicrystalline materials, there are in total sixty-six elastic constants: fifteen in the phonon field, twenty-one in the phason field and thirty phonon-phason coupling constants. In the next section we present the detailed structures of \mathbf{N}_i and $\mathbf{N}_i^{(-1)}$ ($i = 1, 2, 3$).

3. The structures of \mathbf{N}_i and $\mathbf{N}_i^{(-1)}$

Consider first the structure of \mathbf{N}_i ($i = 1, 2, 3$). Since the second column of \mathbf{Q} is identical to the first column of \mathbf{R} , and the second row of \mathbf{R} is identical to the first row of \mathbf{T} , we have

$$\mathbf{N}_1 = \begin{bmatrix} * & -1 & * & * & * & * \\ * & 0 & * & * & * & * \\ * & 0 & * & * & * & * \\ * & 0 & * & * & * & * \\ * & 0 & * & * & * & * \\ * & 0 & * & * & * & * \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} * & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & * & * \\ * & 0 & * & * & * & * \\ * & 0 & * & * & * & * \\ * & 0 & * & * & * & * \end{bmatrix}, \quad (15)$$

where $*$ denotes a possibly nonzero element.

Next, we introduce the reduced elastic compliances $S_{ij} = S_{ji}$ ($i, j = 1 \dots 15$ and $i, j \neq 3, 9, 10, 14$) such that

$$\begin{bmatrix} C_{11} & C_{12} & C_{14} & C_{15} & C_{16} & R_{11} & R_{12} & R_{15} & R_{16} & R_{17} & R_{19} \\ C_{12} & C_{22} & C_{24} & C_{25} & C_{26} & R_{21} & R_{22} & R_{25} & R_{26} & R_{27} & R_{29} \\ C_{14} & C_{24} & C_{44} & C_{45} & C_{46} & R_{41} & R_{42} & R_{45} & R_{46} & R_{47} & R_{49} \\ C_{15} & C_{25} & C_{45} & C_{55} & C_{56} & R_{51} & R_{52} & R_{55} & R_{56} & R_{57} & R_{59} \\ C_{16} & C_{26} & C_{46} & C_{56} & C_{66} & R_{61} & R_{62} & R_{65} & R_{66} & R_{67} & R_{69} \\ R_{11} & R_{21} & R_{41} & R_{51} & R_{61} & K_{11} & K_{12} & K_{15} & K_{16} & K_{17} & K_{19} \\ R_{12} & R_{22} & R_{42} & R_{52} & R_{62} & K_{12} & K_{22} & K_{25} & K_{26} & K_{27} & K_{29} \\ R_{15} & R_{25} & R_{45} & R_{55} & R_{63} & K_{15} & K_{25} & K_{55} & K_{56} & K_{57} & K_{59} \\ R_{16} & R_{26} & R_{46} & R_{56} & R_{66} & K_{16} & K_{26} & K_{56} & K_{66} & K_{67} & K_{69} \\ R_{17} & R_{27} & R_{47} & R_{57} & R_{67} & K_{17} & K_{27} & K_{57} & K_{67} & K_{77} & K_{79} \\ R_{19} & R_{29} & R_{49} & R_{59} & R_{69} & K_{19} & K_{29} & K_{59} & K_{69} & K_{79} & K_{99} \end{bmatrix} \\ \times \begin{bmatrix} S_{11} & S_{12} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} & S_{111} & S_{112} & S_{113} & S_{115} \\ S_{12} & S_{22} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} & S_{211} & S_{212} & S_{213} & S_{215} \\ S_{14} & S_{24} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} & S_{411} & S_{412} & S_{413} & S_{415} \\ S_{15} & S_{25} & S_{45} & S_{55} & S_{56} & S_{57} & S_{58} & S_{511} & S_{512} & S_{513} & S_{515} \\ S_{16} & S_{26} & S_{46} & S_{56} & S_{66} & S_{67} & S_{68} & S_{611} & S_{612} & S_{613} & S_{615} \\ S_{17} & S_{27} & S_{47} & S_{57} & S_{67} & S_{77} & S_{78} & S_{711} & S_{712} & S_{713} & S_{715} \\ S_{18} & S_{28} & S_{48} & S_{58} & S_{68} & S_{78} & S_{88} & S_{811} & S_{812} & S_{813} & S_{815} \\ S_{111} & S_{211} & S_{411} & S_{511} & S_{611} & S_{711} & S_{811} & S_{1111} & S_{1112} & S_{1113} & S_{1115} \\ S_{112} & S_{212} & S_{412} & S_{512} & S_{612} & S_{712} & S_{812} & S_{1112} & S_{1212} & S_{1213} & S_{1215} \\ S_{113} & S_{213} & S_{413} & S_{513} & S_{613} & S_{713} & S_{813} & S_{1113} & S_{1213} & S_{1313} & S_{1315} \\ S_{115} & S_{215} & S_{415} & S_{515} & S_{615} & S_{715} & S_{815} & S_{1115} & S_{1215} & S_{1315} & S_{1515} \end{bmatrix} = \mathbf{I}. \quad (16)$$

Remark. We adopt the convention that if three digits appear as subscripts of S_{ij} , the first digit is i and the remaining two form j . If four digits appear in the subscripts of S_{ij} , the first two digits are i and the remaining two will form j .

It can be easily deduced from (16) that

$$\begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{R}^T & \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{q}_2 & \mathbf{r}_2 \\ \mathbf{r}_2^T & \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{I} - \mathbf{I}_2 & \mathbf{I}_{12}^T \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad (17)$$

where

$$\mathbf{q}_2 = \begin{bmatrix} S_{11} & 0 & S_{15} & S_{17} & S_{115} & S_{111} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ S_{15} & 0 & S_{55} & S_{57} & S_{515} & S_{511} \\ S_{17} & 0 & S_{57} & S_{77} & S_{715} & S_{711} \\ S_{115} & 0 & S_{515} & S_{715} & S_{1515} & S_{1115} \\ S_{111} & 0 & S_{511} & S_{711} & S_{1115} & S_{1111} \end{bmatrix},$$

$$\begin{aligned}
\mathbf{r}_2 &= \begin{bmatrix} S_{16} & S_{12} & S_{14} & S_{112} & S_{18} & S_{113} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ S_{56} & S_{25} & S_{45} & S_{512} & S_{58} & S_{513} \\ S_{67} & S_{27} & S_{47} & S_{712} & S_{78} & S_{713} \\ S_{615} & S_{215} & S_{415} & S_{715} & S_{815} & S_{1315} \\ S_{611} & S_{211} & S_{411} & S_{711} & S_{811} & S_{1113} \end{bmatrix}, \\
\mathbf{t} &= \begin{bmatrix} S_{66} & S_{26} & S_{46} & S_{612} & S_{68} & S_{613} \\ S_{26} & S_{22} & S_{24} & S_{212} & S_{28} & S_{213} \\ S_{46} & S_{24} & S_{44} & S_{412} & S_{48} & S_{413} \\ S_{612} & S_{212} & S_{412} & S_{1212} & S_{812} & S_{1213} \\ S_{68} & S_{28} & S_{48} & S_{812} & S_{88} & S_{813} \\ S_{613} & S_{213} & S_{413} & S_{1213} & S_{813} & S_{1313} \end{bmatrix}, \\
\mathbf{I}_{12} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{I}_2 = \text{diag}[0 \ 1 \ 0 \ 0 \ 0 \ 0].
\end{aligned}$$

After some algebraic manipulations, we finally arrive at

$$\mathbf{N}_3 = -\mathbf{q}_2^{-1} = \frac{-1}{\Delta} \begin{bmatrix} \widehat{W}_{11} & 0 & \widehat{W}_{12} & \widehat{W}_{13} & \widehat{W}_{14} & \widehat{W}_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \widehat{W}_{12} & 0 & \widehat{W}_{22} & \widehat{W}_{23} & \widehat{W}_{24} & \widehat{W}_{25} \\ \widehat{W}_{13} & 0 & \widehat{W}_{23} & \widehat{W}_{33} & \widehat{W}_{34} & \widehat{W}_{35} \\ \widehat{W}_{14} & 0 & \widehat{W}_{24} & \widehat{W}_{34} & \widehat{W}_{44} & \widehat{W}_{45} \\ \widehat{W}_{15} & 0 & \widehat{W}_{25} & \widehat{W}_{35} & \widehat{W}_{45} & \widehat{W}_{55} \end{bmatrix}, \quad (18)$$

$$\mathbf{N}_1 = \mathbf{r}_2^T \mathbf{q}_2^{-1} - \mathbf{I}_{12} = \begin{bmatrix} r_6 & -1 & s_6 & t_6 & u_6 & v_6 \\ r_2 & 0 & s_2 & t_2 & u_2 & v_2 \\ r_4 & 0 & s_4 & t_4 & u_4 & v_4 \\ r_{12} & 0 & s_{12} & t_{12} & u_{12} & v_{12} \\ r_8 & 0 & s_8 & t_8 & u_8 & v_8 \\ r_{13} & 0 & s_{13} & t_{13} & u_{13} & v_{13} \end{bmatrix}, \quad (19)$$

$$\mathbf{N}_2 = \mathbf{t} - \mathbf{r}_2^T \mathbf{q}_2^{-1} \mathbf{r}_2 = \begin{bmatrix} \kappa_{66} & \kappa_{26} & \kappa_{46} & \kappa_{612} & \kappa_{68} & \kappa_{613} \\ \kappa_{26} & \kappa_{22} & \kappa_{24} & \kappa_{212} & \kappa_{28} & \kappa_{213} \\ \kappa_{46} & \kappa_{24} & \kappa_{44} & \kappa_{412} & \kappa_{48} & \kappa_{413} \\ \kappa_{612} & \kappa_{212} & \kappa_{412} & \kappa_{1212} & \kappa_{812} & \kappa_{1213} \\ \kappa_{68} & \kappa_{28} & \kappa_{48} & \kappa_{812} & \kappa_{88} & \kappa_{813} \\ \kappa_{613} & \kappa_{213} & \kappa_{413} & \kappa_{1213} & \kappa_{813} & \kappa_{1313} \end{bmatrix}, \quad (20)$$

where \mathbf{q}_2^{-1} is the pseudo-inverse of \mathbf{q}_2 , Δ is the determinant of

$$\mathbf{W} = \begin{bmatrix} S_{11} & S_{15} & S_{17} & S_{115} & S_{111} \\ S_{15} & S_{55} & S_{57} & S_{515} & S_{511} \\ S_{17} & S_{57} & S_{77} & S_{715} & S_{711} \\ S_{115} & S_{515} & S_{715} & S_{1515} & S_{1115} \\ S_{111} & S_{511} & S_{711} & S_{1115} & S_{1111} \end{bmatrix}, \quad (21)$$

$\widehat{\mathbf{W}} = [\widehat{W}_{ij}]$ is the cofactor matrix of \mathbf{W} , and

$$\begin{aligned} r_\alpha &= \frac{1}{\Delta} \begin{vmatrix} S_{1\alpha} & S_{5\alpha} & S_{7\alpha} & S_{15\alpha} & S_{11\alpha} \\ S_{15} & S_{55} & S_{57} & S_{515} & S_{511} \\ S_{17} & S_{57} & S_{77} & S_{715} & S_{711} \\ S_{115} & S_{515} & S_{715} & S_{1515} & S_{1115} \\ S_{111} & S_{511} & S_{711} & S_{1115} & S_{1111} \end{vmatrix}, & s_\alpha &= \frac{1}{\Delta} \begin{vmatrix} S_{11} & S_{15} & S_{17} & S_{115} & S_{111} \\ S_{1\alpha} & S_{5\alpha} & S_{7\alpha} & S_{15\alpha} & S_{11\alpha} \\ S_{17} & S_{57} & S_{77} & S_{715} & S_{711} \\ S_{115} & S_{515} & S_{715} & S_{1515} & S_{1115} \\ S_{111} & S_{511} & S_{711} & S_{1115} & S_{1111} \end{vmatrix}, \\ t_\alpha &= \frac{1}{\Delta} \begin{vmatrix} S_{11} & S_{15} & S_{17} & S_{115} & S_{111} \\ S_{15} & S_{55} & S_{57} & S_{515} & S_{511} \\ S_{1\alpha} & S_{5\alpha} & S_{7\alpha} & S_{15\alpha} & S_{11\alpha} \\ S_{115} & S_{515} & S_{715} & S_{1515} & S_{1115} \\ S_{111} & S_{511} & S_{711} & S_{1115} & S_{1111} \end{vmatrix}, & u_\alpha &= \frac{1}{\Delta} \begin{vmatrix} S_{11} & S_{15} & S_{17} & S_{115} & S_{111} \\ S_{15} & S_{55} & S_{57} & S_{515} & S_{511} \\ S_{17} & S_{57} & S_{77} & S_{715} & S_{711} \\ S_{1\alpha} & S_{5\alpha} & S_{7\alpha} & S_{15\alpha} & S_{11\alpha} \\ S_{111} & S_{511} & S_{711} & S_{1115} & S_{1111} \end{vmatrix}, \\ v_\alpha &= \frac{1}{\Delta} \begin{vmatrix} S_{11} & S_{15} & S_{17} & S_{115} & S_{111} \\ S_{15} & S_{55} & S_{57} & S_{515} & S_{511} \\ S_{17} & S_{57} & S_{77} & S_{715} & S_{711} \\ S_{115} & S_{515} & S_{715} & S_{1515} & S_{1115} \\ S_{1\alpha} & S_{5\alpha} & S_{7\alpha} & S_{15\alpha} & S_{11\alpha} \end{vmatrix} & & (\alpha = 6, 2, 4, 12, 8, 13), \\ \kappa_{\alpha\beta} &= \frac{1}{\Delta} \begin{vmatrix} S_{11} & S_{1\alpha} & S_{15} & S_{17} & S_{115} & S_{111} \\ S_{1\beta} & S_{\alpha\beta} & S_{5\beta} & S_{7\beta} & S_{15\beta} & S_{11\beta} \\ S_{15} & S_{5\alpha} & S_{55} & S_{57} & S_{515} & S_{511} \\ S_{17} & S_{7\alpha} & S_{57} & S_{77} & S_{715} & S_{711} \\ S_{115} & S_{15\alpha} & S_{515} & S_{715} & S_{1515} & S_{1115} \\ S_{111} & S_{11\alpha} & S_{511} & S_{711} & S_{1115} & S_{1111} \end{vmatrix} & & (\alpha, \beta = 6, 2, 4, 12, 8, 13). \end{aligned}$$

In view of (6), N_2 is positive definite, while (18) and the fact that \mathbf{W} defined by (21) is positive definite result in $-N_3$ being positive semidefinite.

We next derive the structure of $N_i^{(-1)}$ ($i = 1, 2, 3$). It is not hard to check that

$$N_1^{(-1)} = \begin{bmatrix} 0 & * & * & * & * & * \\ -1 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}, \quad N_3^{(-1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}, \quad (22)$$

In addition, we have the identity

$$\begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{R}^T & \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{q} & \mathbf{r}_1 \\ \mathbf{r}_1^T & \mathbf{t}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I}_{12} & \mathbf{I} - \mathbf{I}_1 \end{bmatrix}, \quad (23)$$

where

$$\mathbf{q} = \begin{bmatrix} S_{11} & S_{16} & S_{15} & S_{17} & S_{115} & S_{111} \\ S_{16} & S_{66} & S_{56} & S_{67} & S_{615} & S_{611} \\ S_{15} & S_{56} & S_{55} & S_{57} & S_{515} & S_{511} \\ S_{17} & S_{67} & S_{57} & S_{77} & S_{715} & S_{711} \\ S_{115} & S_{615} & S_{515} & S_{715} & S_{1515} & S_{1115} \\ S_{111} & S_{611} & S_{511} & S_{711} & S_{1115} & S_{1111} \end{bmatrix}, \quad \mathbf{r}_1 = \begin{bmatrix} 0 & S_{12} & S_{14} & S_{112} & S_{18} & S_{113} \\ 0 & S_{26} & S_{46} & S_{612} & S_{68} & S_{613} \\ 0 & S_{25} & S_{45} & S_{512} & S_{58} & S_{513} \\ 0 & S_{27} & S_{47} & S_{712} & S_{78} & S_{713} \\ 0 & S_{215} & S_{415} & S_{715} & S_{815} & S_{1315} \\ 0 & S_{211} & S_{411} & S_{711} & S_{811} & S_{1113} \end{bmatrix},$$

$$\mathbf{t}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & S_{22} & S_{24} & S_{212} & S_{28} & S_{213} \\ 0 & S_{24} & S_{44} & S_{412} & S_{48} & S_{413} \\ 0 & S_{212} & S_{412} & S_{1212} & S_{812} & S_{1213} \\ 0 & S_{28} & S_{48} & S_{812} & S_{88} & S_{813} \\ 0 & S_{213} & S_{413} & S_{1213} & S_{813} & S_{1313} \end{bmatrix}, \quad \mathbf{I}_1 = \text{diag}[1 \ 0 \ 0 \ 0 \ 0 \ 0].$$

We can now arrive at

$$\mathbf{N}_3^{(-1)} = \mathbf{t}_1^{-1} = \frac{1}{\Delta'} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{W}'_{11} & \widehat{W}'_{12} & \widehat{W}'_{13} & \widehat{W}'_{14} & \widehat{W}'_{15} \\ 0 & \widehat{W}'_{12} & \widehat{W}'_{22} & \widehat{W}'_{23} & \widehat{W}'_{24} & \widehat{W}'_{25} \\ 0 & \widehat{W}'_{13} & \widehat{W}'_{23} & \widehat{W}'_{33} & \widehat{W}'_{34} & \widehat{W}'_{35} \\ 0 & \widehat{W}'_{14} & \widehat{W}'_{24} & \widehat{W}'_{34} & \widehat{W}'_{44} & \widehat{W}'_{45} \\ 0 & \widehat{W}'_{15} & \widehat{W}'_{25} & \widehat{W}'_{35} & \widehat{W}'_{45} & \widehat{W}'_{55} \end{bmatrix}, \quad (24)$$

$$\mathbf{N}_1^{(-1)} = \mathbf{r}_1 \mathbf{t}_1^{-1} - \mathbf{I}_{12}^T = \begin{bmatrix} 0 & r'_1 & s'_1 & t'_1 & u'_1 & v'_1 \\ -1 & r'_6 & s'_6 & t'_6 & u'_6 & v'_6 \\ 0 & r'_5 & s'_5 & t'_5 & u'_5 & v'_5 \\ 0 & r'_7 & s'_7 & t'_7 & u'_7 & v'_7 \\ 0 & r'_{15} & s'_{15} & t'_{15} & u'_{15} & v'_{15} \\ 0 & r'_{11} & s'_{11} & t'_{11} & u'_{11} & v'_{11} \end{bmatrix}, \quad (25)$$

$$\mathbf{N}_2^{(-1)} = -\mathbf{q} + \mathbf{r}_1 \mathbf{t}_1^{-1} \mathbf{r}_1^T = - \begin{bmatrix} \kappa'_{11} & \kappa'_{16} & \kappa'_{15} & \kappa'_{17} & \kappa'_{115} & \kappa'_{111} \\ \kappa'_{16} & \kappa'_{66} & \kappa'_{56} & \kappa'_{67} & \kappa'_{615} & \kappa'_{611} \\ \kappa'_{15} & \kappa'_{56} & \kappa'_{55} & \kappa'_{57} & \kappa'_{515} & \kappa'_{511} \\ \kappa'_{17} & \kappa'_{67} & \kappa'_{57} & \kappa'_{77} & \kappa'_{715} & \kappa'_{711} \\ \kappa'_{115} & \kappa'_{615} & \kappa'_{515} & \kappa'_{715} & \kappa'_{1515} & \kappa'_{1115} \\ \kappa'_{111} & \kappa'_{611} & \kappa'_{511} & \kappa'_{711} & \kappa'_{1115} & \kappa'_{1111} \end{bmatrix}, \quad (26)$$

where Δ' is the determinant of

$$\mathbf{W}' = \begin{bmatrix} S_{22} & S_{24} & S_{212} & S_{28} & S_{213} \\ S_{24} & S_{44} & S_{412} & S_{48} & S_{413} \\ S_{212} & S_{412} & S_{1212} & S_{812} & S_{1213} \\ S_{28} & S_{48} & S_{812} & S_{88} & S_{813} \\ S_{213} & S_{413} & S_{1213} & S_{813} & S_{1313} \end{bmatrix}, \quad (27)$$

$\widehat{\mathbf{W}}' = [\widehat{W}'_{ij}]$ is the cofactor of \mathbf{W}' , and

$$\begin{aligned} r'_\alpha &= \frac{1}{\Delta'} \begin{vmatrix} S_{2\alpha} & S_{4\alpha} & S_{12\alpha} & S_{8\alpha} & S_{13\alpha} \\ S_{24} & S_{44} & S_{412} & S_{48} & S_{413} \\ S_{212} & S_{412} & S_{1212} & S_{812} & S_{1213} \\ S_{28} & S_{48} & S_{812} & S_{88} & S_{813} \\ S_{213} & S_{413} & S_{1213} & S_{813} & S_{1313} \end{vmatrix}, & s'_\alpha &= \frac{1}{\Delta'} \begin{vmatrix} S_{22} & S_{24} & S_{212} & S_{28} & S_{213} \\ S_{2\alpha} & S_{4\alpha} & S_{12\alpha} & S_{8\alpha} & S_{13\alpha} \\ S_{212} & S_{412} & S_{1212} & S_{812} & S_{1213} \\ S_{28} & S_{48} & S_{812} & S_{88} & S_{813} \\ S_{213} & S_{413} & S_{1213} & S_{813} & S_{1313} \end{vmatrix}, \\ t'_\alpha &= \frac{1}{\Delta'} \begin{vmatrix} S_{22} & S_{24} & S_{212} & S_{28} & S_{213} \\ S_{24} & S_{44} & S_{412} & S_{48} & S_{413} \\ S_{2\alpha} & S_{4\alpha} & S_{12\alpha} & S_{8\alpha} & S_{13\alpha} \\ S_{28} & S_{48} & S_{812} & S_{88} & S_{813} \\ S_{213} & S_{413} & S_{1213} & S_{813} & S_{1313} \end{vmatrix}, & u'_\alpha &= \frac{1}{\Delta'} \begin{vmatrix} S_{22} & S_{24} & S_{212} & S_{28} & S_{213} \\ S_{24} & S_{44} & S_{412} & S_{48} & S_{413} \\ S_{212} & S_{412} & S_{1212} & S_{812} & S_{1213} \\ S_{2\alpha} & S_{4\alpha} & S_{12\alpha} & S_{8\alpha} & S_{13\alpha} \\ S_{213} & S_{413} & S_{1213} & S_{813} & S_{1313} \end{vmatrix}, \\ v'_\alpha &= \frac{1}{\Delta'} \begin{vmatrix} S_{22} & S_{24} & S_{212} & S_{28} & S_{213} \\ S_{24} & S_{44} & S_{412} & S_{48} & S_{413} \\ S_{212} & S_{412} & S_{1212} & S_{812} & S_{1213} \\ S_{28} & S_{48} & S_{812} & S_{88} & S_{813} \\ S_{2\alpha} & S_{4\alpha} & S_{12\alpha} & S_{8\alpha} & S_{13\alpha} \end{vmatrix} & & (\alpha = 1, 6, 5, 7, 15, 11), \\ \kappa'_{\alpha\beta} &= \frac{1}{\Delta'} \begin{vmatrix} S_{\alpha\beta} & S_{2\beta} & S_{4\beta} & S_{12\beta} & S_{8\beta} & S_{13\beta} \\ S_{2\alpha} & S_{22} & S_{24} & S_{212} & S_{28} & S_{213} \\ S_{4\alpha} & S_{24} & S_{44} & S_{412} & S_{48} & S_{413} \\ S_{12\alpha} & S_{212} & S_{412} & S_{1212} & S_{812} & S_{1213} \\ S_{8\alpha} & S_{28} & S_{48} & S_{812} & S_{88} & S_{813} \\ S_{13\alpha} & S_{213} & S_{413} & S_{1213} & S_{813} & S_{1313} \end{vmatrix} & & (\alpha, \beta = 1, 6, 5, 7, 15, 11). \end{aligned}$$

In view of (12), the matrix $-N_2^{(-1)}$ is positive definite, while, by (24) and the fact that \mathbf{W}' defined by (27) is positive definite, we also have that $N_3^{(-1)}$ is positive semidefinite.

4. Line force and line dislocation

4.1. Line force and dislocation in a homogeneous quasicrystal. Let a line of uniformly distributed force $\mathbf{p} = [p_1 \ p_2 \ p_3 \ q_1 \ q_2 \ q_3]^T$ per unit length be applied on the x_3 -axis which also includes a line dislocation with Burgers vector

$\mathbf{b} = [b_1 \ b_2 \ b_3 \ d_1 \ d_2 \ d_3]^T$. The analytic vector function $\mathbf{f}(z)$ due to the line force and dislocation is given by

$$\mathbf{f}(z) = \langle \ln z_\alpha \rangle \frac{\mathbf{A}^T \mathbf{p} + \mathbf{B}^T \mathbf{b}}{2\pi i}, \quad (28)$$

where $\langle * \rangle$ is a 6×6 diagonal matrix in which each component varies with the index α (from 1 to 6). The elastic energy for the annular region $0 < r_0 \leq r \leq R_0$ per unit length of the line force and dislocation is

$$U = \frac{1}{4\pi} \ln \left(\frac{R_0}{r_0} \right) (\mathbf{p}^T \mathbf{H} \mathbf{p} + \mathbf{b}^T \mathbf{L} \mathbf{b}) > 0, \quad (29)$$

which also provides an indirect proof that \mathbf{H} and \mathbf{L} must be positive definite if the elastic energy is to remain positive.

4.2. Interfacial Green's function. We consider a bimaterial made of two bonded dissimilar anisotropic quasicrystalline half-planes, denoted by #1 ($x_2 > 0$) and #2 ($x_2 < 0$). The bimaterial is subjected to a line force \mathbf{p} and line dislocation with Burgers vector \mathbf{b} at the origin. In fact the solution can be found from [Ting 1996]. The elastic energy for the annular region $0 < r_0 \leq r \leq R_0$ of the quasicrystalline bimaterial is

$$U = \frac{1}{2\pi} \ln \left(\frac{R_0}{r_0} \right) (\mathbf{p}^T \tilde{\mathbf{H}} \mathbf{p} + \mathbf{b}^T \tilde{\mathbf{L}} \mathbf{b}) > 0, \quad (30)$$

where $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{L}}$ are two 6×6 positive definite real symmetric matrices given by

$$\tilde{\mathbf{H}} = \text{Re}\{(\mathbf{M}_1 + \bar{\mathbf{M}}_2)^{-1}\}, \quad \tilde{\mathbf{L}} = \text{Re}\{(\mathbf{M}_1^{-1} + \bar{\mathbf{M}}_2^{-1})^{-1}\}. \quad (31)$$

4.3. Green's function for quasicrystalline bimaterials. We consider a bimaterial made of two perfectly bonded dissimilar anisotropic quasicrystalline half-planes again denoted by #1 ($x_2 > 0$) and #2 ($x_2 < 0$). A line force \mathbf{p} and line dislocation with Burgers vector \mathbf{b} are applied at $(x_1, x_2) = (0, \delta)$ ($\delta > 0$) in material #1. The structure of the solution is similar to that in [Suo 1990] and [Ting 1996]. The image force tending to move a pure dislocation ($\mathbf{p} = \mathbf{0}$) away from the interface is described by

$$\mathbf{F} = \frac{1}{4\pi\delta} \mathbf{b}^T (2\tilde{\mathbf{L}} - \mathbf{L}_1) \mathbf{b}, \quad (32)$$

where $\tilde{\mathbf{L}}$ is given by (31). For a sliding interface on which $\sigma_{12} = \sigma_{32} = 0$, the image force acting on the line dislocation is now characterized by

$$\mathbf{F} = \frac{1}{4\pi\delta} (2\mathbf{b}_0^T \hat{\mathbf{L}} \mathbf{b}_0 - \mathbf{b}^T \mathbf{L}_1 \mathbf{b}), \quad (33)$$

where

$$\mathbf{b}_0 = [b_2 \ d_1 \ d_2 \ d_3]^T, \quad \hat{\mathbf{L}} = \text{Re}\{\hat{\mathbf{N}}^{-1}\}, \quad \hat{\mathbf{N}} = \begin{bmatrix} N_{22} & N_{24} & N_{25} & N_{26} \\ \bar{N}_{24} & N_{44} & N_{45} & N_{46} \\ \bar{N}_{25} & \bar{N}_{45} & N_{55} & N_{56} \\ \bar{N}_{26} & \bar{N}_{46} & \bar{N}_{56} & N_{66} \end{bmatrix}, \quad (34)$$

N_{ij} being the components of the 6×6 Hermitian matrix N defined by (14). Both $\hat{\mathbf{N}}$ and $\hat{\mathbf{L}}$ are positive definite.

5. Imperfect interface models

In [Benveniste 2006], the author rigorously derives imperfect interface models for a three-dimensional curved interphase between two anisotropic crystalline solids by making use of the idea of Taylor expansion of the corresponding fields in thin regions. In this section we will derive compliant and stiff interface models in anisotropic quasicrystalline bimetals using the Stroh formalism and the structures of N_i presented in Sections 2 and 3. To simplify the analysis, we consider here a straight imperfect interface. The stress-strain law for an interphase of constant thickness h between the upper and lower anisotropic quasicrystalline materials 1 and 2 is described by

$$\boldsymbol{\sigma}_1 = \mathbf{Q}_c \mathbf{u}_{,1} + \mathbf{R}_c \mathbf{u}_{,2}, \quad \boldsymbol{\sigma}_2 = \mathbf{R}_c^T \mathbf{u}_{,1} + \mathbf{T}_c \mathbf{u}_{,2}, \quad (35)$$

where \mathbf{Q} , \mathbf{R} , \mathbf{T} are defined in (7) with the subscript c being used to identify the quantities associated with the intermediate interphase, and

$$\boldsymbol{\sigma}_1 = [\sigma_{11} \ \sigma_{21} \ \sigma_{31} \ H_{11} \ H_{21} \ H_{31}]^T, \quad \boldsymbol{\sigma}_2 = [\sigma_{12} \ \sigma_{22} \ \sigma_{32} \ H_{12} \ H_{22} \ H_{32}]^T. \quad (36)$$

- (1) If we assume that $C_{ijkl}^{(c)} \ll C_{ijkl}^{(1)}, C_{ijkl}^{(2)}$; $R_{ijkl}^{(c)} \ll R_{ijkl}^{(1)}, R_{ijkl}^{(2)}$ and $K_{ijkl}^{(c)} \ll K_{ijkl}^{(1)}, K_{ijkl}^{(2)}$ (the so-called compliant interphase) and that the interphase is also very thin, then it follows from (35)₂ that

$$\mathbf{u}_1 - \mathbf{u}_2 = h \mathbf{N}_2^{(c)} \boldsymbol{\sigma}_2^{(1)} = h \mathbf{N}_2^{(c)} \boldsymbol{\sigma}_2^{(2)} \text{ on the compliant interface.} \quad (37)$$

This is, in fact, the anisotropic spring-type interface.

- (2) If we assume that $C_{ijkl}^{(c)} \gg C_{ijkl}^{(1)}, C_{ijkl}^{(2)}$ and $K_{ijkl}^{(c)} \gg K_{ijkl}^{(1)}, K_{ijkl}^{(2)}$ (the so-called stiff interphase) and that the interphase is also very thin, then it follows from (35) and the equilibrium equation $\boldsymbol{\sigma}_{1,1} + \boldsymbol{\sigma}_{2,2} = \mathbf{0}$ that

$$\mathbf{u}_1 = \mathbf{u}_2, \quad \boldsymbol{\sigma}_2^{(1)} - \boldsymbol{\sigma}_2^{(2)} = h \mathbf{N}_3^{(c)} \mathbf{u}_{1,1} = h \mathbf{N}_3^{(c)} \mathbf{u}_{2,1} \text{ on the stiff interface.} \quad (38)$$

This is an extension of the Young–Laplace model to anisotropic quasicrystalline materials.

In view of the structure of N_3 given by (18), the phonon normal traction component σ_{22} is *continuous* across the stiff interface.

6. Interface crack problems

First we derive the asymptotic fields associated with a traction-free semi-infinite crack which lies along the interface between the upper and lower anisotropic quasicrystalline half-planes #1 ($x_2 > 0$) and #2 ($x_2 < 0$). The portion $x_2 = 0$, $x_1 > 0$ of the interface is perfectly bonded, whereas the remaining portion $x_2 = 0$ and $x_1 < 0$ of the interface is fully debonded. In the following analysis, in view of the fact that $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = z$ on the interface $x_2 = 0$, we will replace the complex variables z_k ($k = 1, \dots, 6$) by the common complex variable $z = x_1 + ix_2$. After the analysis is completed, we can simply revert back to the corresponding complex variables.

We introduce an analytic vector function $\mathbf{h}(z)$ defined by

$$\mathbf{h}(z) = \mathbf{B}_1 \mathbf{f}'_1(z) = N^{-1} \bar{N} \mathbf{B}_2 \mathbf{f}'_2(z). \quad (39)$$

Consequently the traction and displacement jumps can be expressed in terms of $\mathbf{h}(z)$ as

$$\sigma_2(x_1) = \mathbf{h}^+(x_1) + \bar{N}^{-1} N \mathbf{h}^-(x_1), \quad \mathbf{id}'(x_1) = N[\mathbf{h}^+(x_1) - \mathbf{h}^-(x_1)], \quad (40)$$

We then arrive at the following homogeneous vector Riemann–Hilbert problem:

$$\begin{aligned} \mathbf{h}^+(z) - \mathbf{h}^-(z) &= \mathbf{0}, & z \notin C, \\ \bar{N} \mathbf{h}^+(z) + N \mathbf{h}^-(z) &= \mathbf{0}, & z \in C. \end{aligned} \quad (41)$$

Consider the eigenvalue problem

$$\bar{N} \mathbf{w} = e^{2\pi\epsilon} N \mathbf{w}. \quad (42)$$

It can be concluded that:

- (1) As a result of the positive definiteness of N , the eigenvalue $e^{2\pi\epsilon}$ is always positive; thus ϵ is real.
- (2) If (ϵ, \mathbf{w}) is an eigenpair, then $(-\epsilon, \bar{\mathbf{w}})$ is another eigenpair.

Three positive real numbers $\epsilon_1, \epsilon_2, \epsilon_3$ and three complex vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ form six eigenpairs:

$$(\epsilon_1, \mathbf{w}_1), (-\epsilon_1, \bar{\mathbf{w}}_1), (\epsilon_2, \mathbf{w}_2), (-\epsilon_2, \bar{\mathbf{w}}_2), (\epsilon_3, \mathbf{w}_3), (-\epsilon_3, \bar{\mathbf{w}}_3). \quad (43)$$

The following orthogonal relationships can also be established:

$$\mathbf{w}_i^T N \mathbf{w}_j = \bar{\mathbf{w}}_k^T N \mathbf{w}_l = \mathbf{w}_k^T N \bar{\mathbf{w}}_l = 0 \quad (i, j, k, l = 1, 2, 3 \text{ and } k \neq l), \quad (44)$$

The three positive real numbers (or oscillatory indices) $\epsilon_1, \epsilon_2, \epsilon_3$ are given by

$$\epsilon_j = \frac{1}{2\pi} \ln \frac{1 + \beta_j}{1 - \beta_j} \quad (j = 1, 2, 3), \quad (45)$$

where β_j ($j = 1, 2, 3$) are the three positive roots of the following cubic equation in β^2 :

$$\beta^6 + \frac{1}{2} \text{tr}(\check{S}^2) \beta^4 - \frac{1}{2} |\check{S}| \text{tr}(\check{S}^{-2}) \beta^2 - |\check{S}| = 0. \quad (46)$$

Here

$$\check{S} = (L_1^{-1} + L_2^{-1})^{-1} (S_1 L_1^{-1} - S_2 L_2^{-1}). \quad (47)$$

The oscillatory indexes $\epsilon_1, \epsilon_2, \epsilon_3$ are then explicitly determined.

Following Suo [1990], we can obtain for $\mathbf{h}(z)$ the expression

$$\mathbf{h}(z) = \sum_{j=1}^3 \frac{e^{\pi\epsilon_j} K_j z^{i\epsilon_j} \mathbf{w}_j + e^{-\pi\epsilon_j} \bar{K}_j z^{-i\epsilon_j} \bar{\mathbf{w}}_j}{2(2\pi z)^{\frac{1}{2}} \cosh \pi\epsilon_j}, \quad (48)$$

where K_1, K_2 and K_3 are three complex stress intensity factors. $\mathbf{f}'_1(z)$ and $\mathbf{f}'_2(z)$ in the two half-planes are then given by

$$\begin{aligned} \mathbf{B}_1 \mathbf{f}'_1(z) &= \sum_{j=1}^3 \frac{e^{\pi\epsilon_j} K_j z^{i\epsilon_j} \mathbf{w}_j + e^{-\pi\epsilon_j} \bar{K}_j z^{-i\epsilon_j} \bar{\mathbf{w}}_j}{2(2\pi z)^{\frac{1}{2}} \cosh \pi\epsilon_j} \quad (\text{Im}\{z\} > 0), \\ \mathbf{B}_2 \mathbf{f}'_2(z) &= \sum_{j=1}^3 \frac{e^{-\pi\epsilon_j} K_j z^{i\epsilon_j} \mathbf{w}_j + e^{\pi\epsilon_j} \bar{K}_j z^{-i\epsilon_j} \bar{\mathbf{w}}_j}{2(2\pi z)^{\frac{1}{2}} \cosh \pi\epsilon_j} \quad (\text{Im}\{z\} < 0). \end{aligned} \quad (49)$$

The traction at the bonded interface a distance r ahead of the crack tip is

$$\sigma_2(r) = \sum_{j=1}^3 [t_j(r) \mathbf{w}_j + \bar{t}_j(r) \bar{\mathbf{w}}_j], \quad (50)$$

where

$$t_j(r) = t_{2j} + it_{1j} = \frac{\bar{\mathbf{w}}_j^T \mathbf{N} \sigma_2(r)}{\bar{\mathbf{w}}_j^T \mathbf{N} \mathbf{w}_j} = \frac{K_j r^{i\epsilon_j}}{\sqrt{2\pi r}}.$$

Equation (50) states that the interface traction can be decomposed into three components, each in the plane spanned by $\text{Re}\{\mathbf{w}_j\}$ and $\text{Im}\{\mathbf{w}_j\}$ ($j = 1, 2, 3$).

The displacement jump a distance r behind the crack tip is

$$\mathbf{d}(r) = (\mathbf{N} + \bar{\mathbf{N}}) \left(\frac{r}{2\pi} \right)^{\frac{1}{2}} \sum_{j=1}^3 \left[\frac{K_j r^{i\epsilon_j} \mathbf{w}_j}{(1 + 2i\epsilon_j) \cosh \pi\epsilon_j} + \frac{\bar{K}_j r^{-i\epsilon_j} \bar{\mathbf{w}}_j}{(1 - 2i\epsilon_j) \cosh \pi\epsilon_j} \right]. \quad (51)$$

The energy release rate is therefore given by

$$G = \lim_{\delta \rightarrow 0} \frac{1}{2\delta} \int_0^\delta \boldsymbol{\sigma}_2^T (\delta - r) \mathbf{d}(r) dr = \sum_{j=1}^3 \frac{\bar{\mathbf{w}}_j^T (N + \bar{N}) \mathbf{w}_j}{4 \cosh^2 \pi \epsilon_j} |K_j|^2 > 0. \quad (52)$$

Next, we consider a set of collinear cracks between two dissimilar anisotropic quasicrystalline half-planes with prescribed traction $\mathbf{t}_0(x_1)$ on the crack lines C . Suppose there are n finite cracks in the intervals (a_j, b_j) ($j = 1, 2, \dots, n$) and two semi-infinite cracks $(-\infty, b_0)$ and $(a_0, +\infty)$. The prescribed traction $\mathbf{t}_0(x_1)$ on the crack lines C will result in the inhomogeneous Riemann–Hilbert vector problem:

$$\mathbf{h}^+(x_1) + \bar{N}^{-1} N \mathbf{h}^-(x_1) = \mathbf{t}_0(x_1), \quad x_1 \in C. \quad (53)$$

In order to solve this, we follow the method in [Suo 1990] and [Suo et al. 1992] and write $\mathbf{h}(z)$ and $\mathbf{t}_0(x_1)$ in terms of their components using the eigenvector representation

$$\mathbf{h}(z) = h_1(z) \mathbf{w}_1 + h_2(z) \bar{\mathbf{w}}_1 + h_3(z) \mathbf{w}_2 + h_4(z) \bar{\mathbf{w}}_2 + h_5(z) \mathbf{w}_3 + h_6(z) \bar{\mathbf{w}}_3,$$

$$\mathbf{t}_0(x_1) = t_{01}(x_1) \mathbf{w}_1 + \bar{t}_{01}(x_1) \bar{\mathbf{w}}_1 + t_{02}(x_1) \mathbf{w}_2 + \bar{t}_{02}(x_1) \bar{\mathbf{w}}_2 + t_{03}(x_1) \mathbf{w}_3 + \bar{t}_{03}(x_1) \bar{\mathbf{w}}_3.$$

As a result, (53) can be decoupled as follows:

$$\left. \begin{aligned} h_1^+(x_1) + e^{-2\pi\epsilon_1} h_1^-(x_1) &= t_{01}(x_1) \\ h_2^+(x_1) + e^{+2\pi\epsilon_1} h_2^-(x_1) &= \bar{t}_{01}(x_1) \\ h_3^+(x_1) + e^{-2\pi\epsilon_2} h_3^-(x_1) &= t_{02}(x_1) \\ h_4^+(x_1) + e^{+2\pi\epsilon_2} h_4^-(x_1) &= \bar{t}_{02}(x_1) \\ h_5^+(x_1) + e^{-2\pi\epsilon_3} h_5^-(x_1) &= t_{03}(x_1) \\ h_6^+(x_1) + e^{+2\pi\epsilon_3} h_6^-(x_1) &= \bar{t}_{03}(x_1) \end{aligned} \right\} \text{ for } x_1 \in C, \quad (54)$$

whose solution can be given simply by

$$h_1(z) = \frac{\chi_1(z)}{2\pi i} \int_C \frac{t_{01}(x_1) dx_1}{\chi_1^+(x_1)(x_1 - z)} + \chi_1(z) P_1(z),$$

$$h_2(z) = \frac{\bar{\chi}_1(z)}{2\pi i} \int_C \frac{\bar{t}_{01}(x_1) dx_1}{\bar{\chi}_1^+(x_1)(x_1 - z)} + \bar{\chi}_1(z) P_2(z),$$

$$h_3(z) = \frac{\chi_2(z)}{2\pi i} \int_C \frac{t_{02}(x_1) dx_1}{\chi_2^+(x_1)(x_1 - z)} + \chi_2(z) P_3(z),$$

$$h_4(z) = \frac{\bar{\chi}_2(z)}{2\pi i} \int_C \frac{\bar{t}_{02}(x_1) dx_1}{\bar{\chi}_2^+(x_1)(x_1 - z)} + \bar{\chi}_2(z) P_4(z),$$

$$h_5(z) = \frac{\chi_3(z)}{2\pi i} \int_C \frac{t_{03}(x_1) dx_1}{\chi_3^+(x_1)(x_1 - z)} + \chi_3(z) P_5(z),$$

$$h_6(z) = \frac{\bar{\chi}_3(z)}{2\pi i} \int_C \frac{\bar{t}_{03}(x_1) dx_1}{\bar{\chi}_3^+(x_1)(x_1 - z)} + \bar{\chi}_3(z) P_6(z),$$

where $\chi_1(z)$, $\chi_2(z)$ and $\chi_3(z)$ are defined as

$$\chi_j(z) = \prod_{k=0}^n (z - a_k)^{-\frac{1}{2} - i\epsilon_j} (z - b_k)^{-\frac{1}{2} + i\epsilon_j} \quad (j = 1, 2, 3), \quad (55)$$

and $P_i(z)$ ($i = 1, \dots, 6$) are polynomials in z of order less than $n + 1$ [Ting 1996].

7. Conclusions

In this paper all six 6×6 matrices N_i and $N_i^{(-1)}$ ($i = 1, 2, 3$) in the Stroh formalism of anisotropic elasticity for quasicrystals have been explicitly expressed in terms of the sixty-six reduced elastic compliances $S_{ij} = S_{ji}$ ($i, j = 1 \dots 15$ and $i, j \neq 3, 9, 10, 14$). It is found that N_2 and $-N_2^{(-1)}$ are positive definite, whilst $-N_3$ and $N_3^{(-1)}$ are positive semidefinite.

In the study of Green's functions, we present the elastic energy expressions (29) for a line force and dislocation in a homogeneous quasicrystal and (30) for a line force and dislocation lying on a bi-quasicrystal interface, and obtain the image force on a dislocation near a perfect or sliding interface between two anisotropic quasicrystalline half-planes.

We also derive compliant and stiff interface models using the Stroh formalism. Green's function solutions for quasicrystalline bimetals with imperfect interface can be further studied by using the method described in [Wang and Pan 2010].

Perhaps the most interesting conclusion from this research is that the interface crack-tip field consists of *three* two-dimensional oscillatory stress singularities $r^{-\frac{1}{2} \pm i\epsilon_j}$ ($j = 1, 2, 3$) characterized through the introduction of three complex stress intensity factors K_j ($j = 1, 2, 3$). We end by again noting the beauty and power of the Stroh formalism which is fully demonstrated here.

Acknowledgements

Wang W. was supported by Innovation Program of Shanghai Municipal Education Commission (No. 12ZZ058). Schiavone acknowledges the support of the Natural Sciences and Engineering Research Council of Canada.

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Received 23 Feb 2012. Accepted 5 Apr 2012.

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MEMOCS (ISSN 2325-3444 electronic, 2326-7186 printed) is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell’Aquila, Italy.

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