Mathematics and Mechanics *of* Complex Systems

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**STATISTICALLY ISOTROPIC TENSOR RANDOM FIELDS:
CORRELATION STRUCTURES**

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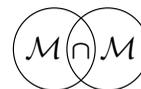
STATISTICALLY ISOTROPIC TENSOR RANDOM FIELDS: CORRELATION STRUCTURES

ANATOLIY MALYARENKO AND MARTIN OSTOJA-STARZEWSKI

Let V be a real finite-dimensional vector space. We introduce some physical problems that may be described by V-valued homogeneous and isotropic random fields on $\mathbb{R}^3$. We propose a general method for calculation of expectations and two-point correlation functions of such fields. Our results are equivalent to classical results by Robertson, when $V = \mathbb{R}^3$, and those by Lomakin, when V is the space of symmetric second-rank tensors over $\mathbb{R}^3$. Our solution involves an analogue of the classical Clebsch–Gordan coefficients.

1. Introduction

The entire field of continuum physics involves tensor fields. Overwhelmingly, most of the existing models and theories are deterministic and their stochastic generalizations necessitate construction of tensor-valued random fields (RF). While the literature on scalar RFs is vast (for example, [Cressie 1993; Christakos 2005; Marinucci and Peccati 2011; Leonenko and Sakhno 2012; Porcu et al. 2012]), that on vector RFs is largely limited to statistical turbulence [Monin and Yaglom 1965], and the case of higher tensor rank (second, fourth) RFs poses challenges. In this paper we focus on wide-sense stationary and statistically isotropic RFs of tensors of the first and second ranks. We present a new method of derivation of representations of their correlation functions, which in the case of first-rank tensors gives the same result as in [Robertson 1940], while in the case of second-rank tensors is equivalent to the result of [Lomakin 1964].

These representations have applications to tensor random fields (TRFs) governed by the field equations of continuum physics as well as those representing some spatially inhomogeneous constitutive properties of random media. The former type of TRFs is used in [Ostoja-Starzewski et al. 2013], where correlation functions are subject to constraints such as the equilibrium equation or strain-displacement relation. The basic properties of TRFs of a wide-sense homogeneous and isotropic kind with generally anisotropic realizations have been determined

in three continuum physics theories: thermal conduction, classical elasticity, and micropolar elasticity. The field equations (such as the linear and angular momentum balances and strain-displacement relations), all in a quasistatic setting, lead to consequences for the respective dependent fields involved. In effect, these consequences are restrictions on the admissible forms of the correlation functions describing the TRFs.

The latter type of TRFs provides models of random media described by the second-rank TRF. The typical example here is the thermal conductivity tensor and its mathematical analogies such as the antiplane stiffness tensor. Once the general representation of this TRF is established and the conditions of positive definiteness are imposed, one can turn to modeling and simulation of the entire range of statistical constitutive behaviors of all heat-conducting media or, say, elastic materials subjected to antiplane loading, for example, [Sena et al. 2013].

In particular, let V be a finite-dimensional real Hilbert space with norm $\|\cdot\|$. Let $T(x)$, $x \in \mathbb{R}^3$, be a random field taking values in (a subset of) V. Suppose that $\mathsf{E}[\|T(x)\|^2] < \infty$ and that $T(x)$ is mean-square continuous, that is, for any $x_0 \in \mathbb{R}^3$ we have

$$\lim_{\|x-x_0\| \to 0} \mathsf{E}[\|T(x) - T(x_0)\|^2] = 0.$$

Let $E(x) = \mathsf{E}[T(x)]$ be the expectation of the field, and let $B(x, y) = \mathsf{E}[T(\mathbf{x}) \otimes T(y)]$ be the two-point correlation function of the random field $T(x)$. The group $\mathbb{R}^3$ acts on itself by translations. Assume that the above functions are invariant with respect to this action, that is, for all $x, y, z \in \mathbb{R}^3$,

$$E(x + z) = E(x),$$
$$B(x + z, y + z) = B(x, y).$$

It follows that $E(x) = \boldsymbol{E} \in V$ is constant, while $B(x, y) \in V \otimes V$ depends only on the difference $x - y$.

Let $K = \mathrm{SO}(3)$ be the group of rotations in $\mathbb{R}^3$, and let (V, γ) be an orthogonal representation of K. Suppose that for all $k \in K$ and all $x \in \mathbb{R}^3$ we have

$$
\begin{aligned}
E(kx) &= \gamma(k)E(x), \\
B(kx) &= \gamma(k)B(x)\gamma^{-1}(k).
\end{aligned}
\tag{1-1}
$$

We would like to find a general form for the expectation and two-point correlation function of such a field.

In Section 2, we consider mathematical preliminaries. We use the book [Adams 1969], in which Adams considers both real and complex representations at the same time.

In Section 3 we consider two particular cases of the above problem:

(1) V has dimension 3 and $\gamma(k) = k$.

(2) V is the space of all second-rank tensors over $\mathbb{R}^3$, $\boldsymbol{T}(\boldsymbol{x})$ takes values in the set of all symmetric tensors, and the representation is $\gamma(k)\boldsymbol{T} = k\boldsymbol{T}k^{-1}$.

In the first case, the answer has been known since the classic paper [Robertson 1940]. In Theorem 3.1, we prove that our method of solution gives the same answer. Our new result is Theorem 3.2. Section 4 concludes. Proofs of our results are collected in the Appendix.

2. Mathematical preliminaries

Let $\mathbb{K}$ be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, and let K be a topological group with the identity element e. A *representation* of the group K over $\mathbb{K}$ is a pair (V, γ), where V is a finite-dimensional vector space over $\mathbb{K}$, and γ is a continuous homomorphism from K to the group Aut V of the invertible linear operators in V. In other words, for each $k \in K$ and for each $\boldsymbol{v} \in V$ there is a vector $\gamma(k)\boldsymbol{v} \in V$, and the following conditions hold true:

(1) $\gamma(e)\boldsymbol{v} = \boldsymbol{v}$ and $\gamma(k)(\gamma(k')\boldsymbol{v}) = \gamma(kk')\boldsymbol{v}$.

(2) $\gamma(k)\boldsymbol{v}$ is a $\mathbb{K}$-linear function of $\boldsymbol{v}$.

(3) $\gamma(k)\boldsymbol{v}$ is a continuous function of k and $\boldsymbol{v}$.

Let (V, γ) and (W, δ) be two representations. A map $G : V \to W$ is called a *K-map* if

$$G(\gamma(k)\boldsymbol{v}) = \delta(k)(G\boldsymbol{v}).$$

A $\mathbb{K}$-linear K-map is called an *intertwining operator*. The set of all intertwining operators is a vector space over $\mathbb{K}$. The representations (V, γ) and (W, δ) are called *equivalent* if the above space contains an invertible operator.

Let (V, γ) be a real representation of the group K. Build a complex representation (V', γ') as follows. Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$. Put $V' = \mathbb{C} \otimes_{\mathbb{R}} V$. The space V' is a complex vector space, where multiplication by a complex number z is defined as $z(z' \otimes \boldsymbol{v}) = zz' \otimes \boldsymbol{v}$. The representation γ' is

$$\gamma'(k)(z \otimes \boldsymbol{v}) = z \otimes \gamma(k)\boldsymbol{v}.$$

Define a map $j : V' \to V'$ by $j(z \otimes \boldsymbol{v}) = \bar{z} \otimes \boldsymbol{v}$. Then j is a *structural map*, that is, a K-map with

$$j(z\boldsymbol{v}) = \bar{z}j(\boldsymbol{v}), \quad j^2 = 1.$$

Conversely, let (V', γ') be a complex representation of K that admits a structural map j. Then V' is a direct sum of two eigenspaces V_+ and V_- of the map j that correspond to the eigenvalues $+1$ and -1. These spaces carry two equivalent real

representations. Multiplication by i is an invertible intertwining operator between the above representations.

The *direct sum* of two representations (V, γ) and (W, δ) is the representation $(V \oplus W, \gamma \oplus \delta)$, where

$$\gamma \oplus \delta(k)(\boldsymbol{v} \oplus \boldsymbol{w}) = (\gamma(k)\boldsymbol{v}) \oplus (\delta(k)\boldsymbol{w}).$$

The *tensor product* of two representations (V, γ) and (W, δ) is the representation $(V \otimes W, \gamma \otimes \delta)$, where

$$\gamma \otimes \delta(k)(\boldsymbol{v} \otimes \boldsymbol{w}) = (\gamma(k)\boldsymbol{v}) \otimes (\delta(k)\boldsymbol{w}).$$

A representation (V, γ) with $V \neq \{\boldsymbol{0}\}$ is called *reducible* if there exists a proper subspace W of V with $\gamma(k)\boldsymbol{w} \in W$ for all $\boldsymbol{w} \in W$ and $k \in K$ and *irreducible* otherwise. If K is a compact group, then any representation (V, γ) of K is a direct sum of irreducible representations. Moreover, the decomposition onto irreducible representations is unique in the following sense. If $m_i V_i$ denotes the direct sum of m_i copies of the representation V_i, and the representations $\bigoplus m_i V_i$ and $\bigoplus n_i V_i$ are equivalent, then $m_i = n_i$ for all i.

Let (V, γ) be a complex representation of a *compact* topological group K. By [Adams 1969, Proposition 3.16], there exists a K-invariant inner product $(\cdot, \cdot)$ on V. Moreover, if (V, γ) admits a structural map j, one can choose the above inner product in such a way that $(j\boldsymbol{v}, j\boldsymbol{w}) = \overline{(\boldsymbol{v}, \boldsymbol{w})}$, and the restriction of the inner product to either space V_+ or V_- is again an inner product.

Choose an orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ in V. Then, the complex representation γ takes values in the unitary group $U(n)$ and is called a *unitary representation*. A real representation takes values in the orthogonal group $O(n)$ and is called an *orthogonal representation*.

Realize $\mathbb{R}^3$ as the space of traceless Hermitian matrices in $\mathbb{C}^2$. Such a matrix has the form

$$A = \begin{pmatrix} x_0 & x_1 + x_{-1}\mathrm{i} \\ x_1 - x_{-1}\mathrm{i} & -x_0 \end{pmatrix}, \quad x_{-1}, x_0, x_1 \in \mathbb{R}.$$

The map $A \mapsto k^{-1}Ak$, where k is an element of the group SU(2) of unitary 2×2 matrices with unit determinant, is a rotation, that is, an element of the group SO(3). The matrices k and $-k$ determine the same rotation. Conversely, each rotation in SO(3) corresponds to a pair of matrices k and $-k$.

Let (V, γ) be an irreducible unitary representation of the group SU(2). If $\gamma(k) = \gamma(-k)$, then (V, γ) is an irreducible unitary representation of the group SO(3), and all irreducible unitary representations of SO(3) may be obtained in this way.

Let

$$k = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1,$$

be an element of the group SU(2). Let V^ℓ be the space of homogeneous polynomials of degree 2ℓ in two complex variables ξ and η. The representation

$$\gamma_\ell(k) f(\xi, \eta) = f(\bar{\alpha}\xi - \beta\eta, \bar{\beta}\xi + \alpha\eta) \qquad (2\text{-}1)$$

is irreducible. Conversely, any irreducible representation of the group SU(2) is equivalent to the representation (2-1).

If ℓ is an integer, then $\gamma_\ell(k) = \gamma_\ell(-k)$, and the representation (2-1) is an irreducible representation of the group SO(3). Moreover, put

$$jf(\xi, \eta) = \bar{f}(-\eta, \xi),$$

where $\bar{f}$ is the polynomial with coefficients which are complex conjugate to those of f. Then j is a structural map. If we choose an orthonormal basis e_m^ℓ, $-\ell \leq m \leq \ell$, satisfying the condition

$$je_m^\ell = e_m^\ell, \qquad (2\text{-}2)$$

then the restriction $(V_+^\ell, \gamma_{\ell,+})$ of the representation (V^ℓ, γ_ℓ) to the real linear span V_+^ℓ of the above basis is an irreducible real representation, and the matrix entries of the operators $\gamma_\ell(k)$ are real-valued functions on the group SO(3). If the basis e_m^ℓ satisfies the condition

$$je_m^\ell = -e_m^\ell, \qquad (2\text{-}3)$$

then the restriction $(V_-^\ell, \gamma_{\ell,-})$ of the representation (V^ℓ, γ_ℓ) to the real linear span V_-^ℓ of the above basis is an irreducible real representation, equivalent to $(V_+^\ell, \gamma_{\ell,+})$, and multiplication by i is an orthogonal intertwining operator between two equivalent representations.

The usual orthonormal basis in the space V^ℓ is as follows:

$$f_m^\ell(\xi, \eta) = (-1)^{\ell+m} \sqrt{\frac{(2\ell+1)!}{(\ell+m)!(\ell-m)!}} \xi^{\ell+m} \eta^{\ell-m}. \qquad (2\text{-}4)$$

The matrix entries of operators $\gamma_\ell(k)$ in this basis are called the *Wigner D-functions* and are denoted by $D_{mn}^\ell(k)$. The basis (2-4) does not satisfy (2-2). Gordienko

[2002] proposed the basis satisfying (2-2) as follows:

$$\boldsymbol{h}^{\ell}_{-m}(\xi, \eta) = \frac{(-\mathrm{i})^{\ell-1}}{\sqrt{2}}[(-1)^m \boldsymbol{f}^{\ell}_m(\xi, \eta) - \boldsymbol{f}^{\ell}_{-m}(\xi, \eta)],$$

$$\boldsymbol{h}^{\ell}_0(\xi, \eta) = (-\mathrm{i})^{\ell}\boldsymbol{e}^{\ell}_0(\xi, \eta),$$

$$\boldsymbol{h}^{\ell}_m(\xi, \eta) = -\frac{(-\mathrm{i})^{\ell}}{\sqrt{2}}[(-1)^m \boldsymbol{f}^{\ell}_m(\xi, \eta) + \boldsymbol{f}^{\ell}_{-m}(\xi, \eta)],$$

where $m \geq 1$. From now on, we define by $U^{\ell}(k)$ the matrices of the representation $(V^{\ell}_+, \gamma_{\ell,+})$ in the Gordienko basis and omit $+$ and $-$ for simplicity of notation. Note that $\mathrm{i}\boldsymbol{h}^{\ell}_m(\xi, \eta)$ is the Gordienko basis of the space V_-. Its vectors satisfy (2-3).

Any rotation k may be performed by three successive rotations:

- rotation $k_0(\psi)$ about the x_0-axis through an angle ψ, $0 \leq \psi < 2\pi$,

- rotation $k_{-1}(\theta)$ about the x_{-1}-axis through an angle θ, $0 \leq \theta \leq \pi$, and

- rotation $k_0(\varphi)$ about the x_0-axis through an angle φ, $0 \leq \varphi < 2\pi$.

The angles ψ, θ, and φ are the *Euler angles*. The map which maps the product of the above rotations $k(\psi, \theta, \varphi)$ to the point $(\psi, \theta, \varphi) \in (0, 2\pi) \times (0, \pi) \times (0, 2\pi)$ is a chart of the group manifold SO(3), and the domain of this chart is an open dense subset of SO(3). Moreover, the map which maps the rotation $k(0, \theta, \varphi)(0, 1, 0)^{\top}$ to the point $(\theta, \varphi) \in (0, \pi) \times (0, 2\pi)$ is a chart of the unit sphere S^2 centered at the origin of the space $\mathbb{R}^3$. The coordinates of the point $k(0, \theta, \varphi)(0, r, 0)^{\top}$, $r > 0$, are the spherical coordinates,

$$x_{-1} = r \sin \varphi \sin \theta,$$

$$x_0 = r \cos \theta,$$

$$x_1 = r \cos \varphi \sin \theta,$$

Gordienko [2002] calculated the matrix entries of the matrices $U^{\ell}(k)$. His result is as follows. If $k = k(\psi, \theta, \varphi)$, then

$$U^{\ell}(k) = U^{\ell}(k_0(\varphi))U^{\ell}(k_{-1}(\theta))U^{\ell}(k_0(\psi)), \tag{2-5}$$

by the definition of a representation. Denote the matrix entries of the matrix $U^{\ell}(k_0(\varphi))$ by $\Omega^{\ell}_{0,m,n}(\varphi)$, where $-\ell \leq m, n \leq \ell$. The nonzero entries are

$$\Omega^{\ell}_{0,0,0}(\varphi) = 1, \quad \Omega^{\ell}_{0,m,m}(\varphi) = \cos(m\varphi), \quad \Omega^{\ell}_{0,-m,m}(\varphi) = \sin(m\varphi), \tag{2-6}$$

where $m = \pm 1, \pm 2, \ldots, \pm \ell$. Denote the matrix entries of the matrix $U^\ell(k_{-1}(\theta))$ by $\Omega^\ell_{-1,m,n}(\theta)$. The nonzero entries are

$$\Omega^\ell_{-1,\pm m,\pm n}(\theta) = \frac{(-1)^{\ell-n}}{2^\ell(1-\mu^2)^{n/2}}\sqrt{\frac{(\ell+n)!}{(\ell-n)!(\ell-m)!(\ell+m)!}}$$
$$\times \left\{\left(\frac{1-\mu}{1+\mu}\right)^{n/2}\frac{\mathrm{d}^{\ell-n}}{\mathrm{d}\mu^{\ell-n}}[(1+\mu)^{\ell+m}(1-\mu)^{\ell-m}]\right.$$
$$\left.\pm(-1)^m\left(\frac{1+\mu}{1-\mu}\right)^{n/2}\frac{\mathrm{d}^{\ell-n}}{\mathrm{d}\mu^{\ell-n}}[(1+\mu)^{\ell-m}(1-\mu)^{\ell+m}]\right\},$$

$$\Omega^\ell_{-1,0,0}(\theta) = \frac{(-1)^\ell}{2^\ell\ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}\mu^\ell}(1-\mu^2)^\ell,$$

$$\Omega^\ell_{-1,0,n}(\theta) = \frac{(-1)^{\ell-n}}{2^\ell\ell!}\sqrt{\frac{2(\ell+n)!}{(\ell-n)!}}\frac{1}{(1-\mu^2)^{n/2}}\frac{\mathrm{d}^{\ell-n}}{\mathrm{d}\mu^{\ell-n}}(1-\mu^2)^\ell,$$

$$\Omega^\ell_{-1,m,0}(\theta) = -\frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{2(\ell+m)!}{(\ell-m)!}}\frac{1}{(1-\mu^2)^{m/2}}\frac{\mathrm{d}^{\ell-m}}{\mathrm{d}\mu^{\ell-m}}(1-\mu^2)^\ell,$$

$$(2\text{-}7)$$

where $m \geq 1$, $n \geq 1$, and where $\mu = \cos\theta$.

Let (V^m, γ_m) and (V^p, γ_p) be two irreducible orthogonal representations of the group SO(3). Their tensor product $(V^m \otimes V^p, \gamma_m \otimes \gamma_p)$ is equivalent to the direct sum $(V^{|m-p|} \oplus V^{|m-p|+1} \oplus \cdots \oplus V^{m+p}, \gamma_{|m-p|} \oplus \gamma_{|m-p|+1} \oplus \cdots \oplus \gamma_{m+p})$. Let G be the orthogonal intertwining operator between the above equivalent representations. Then we have

$$G(\gamma_m(k) \otimes \gamma_p(k)) = (\gamma_{|m-p|(k)} \oplus \gamma_{|m-p|+1}(k) \oplus \cdots \oplus \gamma_{m+p}(k))G. \qquad (2\text{-}8)$$

In the usual basis (2-4), this equality takes the form

$$\sum_{i=-m}^{m}\sum_{j=-p}^{p} C^{s\ell}_{mipj}D^m_{ni}(k)D^p_{qj}(k) = \sum_{t=-s}^{s} U^s_{t\ell}(k)C^{st}_{mnpq},$$

which is [Varshalovich et al. 1975, Equation (5), §4.6]. The matrix entries of the operator G in the basis (2-4), $C^{s\ell}_{mipj}$, are called the *Clebsch–Gordan coefficients*.

In the Gordienko basis, the same equality takes the form

$$\sum_{i=-m}^{m}\sum_{j=-p}^{p} g^{\ell[i,j]}_{s[m,p]}U^m_{ni}(k)U^p_{qj}(k) = \sum_{t=-s}^{s} U^s_{t\ell}(k)g^{t[n,q]}_{s[m,p]}.$$

We call the matrix entries of the operator G in the Gordienko basis, $g^{\ell[i,j]}_{s[m,p]}$, the

Godunov–Gordienko coefficients. They were calculated in [Godunov and Gordienko 2004].

It follows in particular that the matrix G transforms the *uncoupled basis* h_t^s, $|m - p| \le s \le m + p$, $-s \le t \le s$, to the *coupled basis* $h_n^m \otimes h_q^p$:

$$h_n^m \otimes h_q^p = \sum_{r=|m-p|}^{m+p} \sum_{s=-r}^{r} g_{r[m,p]}^{s[n,q]} h_s^r. \tag{2-9}$$

We multiply both sides of (2-8) by G^{-1} from the left and write the result in the Gordienko basis. We obtain

$$U_{ni}^m(k) U_{qj}^p(k) = \sum_{s=|m-p|}^{m+p} \sum_{t=-s}^{s} \sum_{\ell=-s}^{s} g_{s[m,p]}^{t[n,q]} U_{t\ell}^s(k) g_{s[m,p]}^{\ell[i,j]}. \tag{2-10}$$

The same equality in the usual basis is [Varshalovich et al. 1975, Equation (1), §4.6]. It is called the *Clebsch–Gordan expansion*.

Lemma 2.1 [Malyarenko 2013]. *Let U be an irreducible representation of a topological group K in a Hilbert space H. Let $x \in H$ be a common eigenvector of all operators $U(k)$, $g \in K$. If U is not trivial, then $x = \mathbf{0}$.*

Lemma 2.2. *The second equation in* (1-1) *may be written in the Gordienko basis as follows*:

$$B(k\mathbf{x}) = (U \otimes U)(k) B(\mathbf{x}). \tag{2-11}$$

3. The results

Theorem 3.1 [Robertson 1940]. *Let $\mathbf{T}(\mathbf{x})$ be a V-valued random field on $\mathbb{R}^3$ satisfying* (1-1) *with $U(k) = k$. Then*

$$\mathsf{E}[\mathbf{T}(\mathbf{x})] = \mathbf{0}$$

and there exist two continuous functions K_0, $K_2 : [0, \infty) \to \mathbb{R}$ with $K_2(0) = 0$ such that

$$B_{ij}(\mathbf{x}) = \delta_{ij} K_0(\|\mathbf{x}\|) + x_i x_j K_2(\|\mathbf{x}\|).$$

Let $k_{\mathbf{x}}$, $\mathbf{x} \ne \mathbf{0}$, be the rotation with Euler angles $(0, \theta, \varphi)$, where θ and φ are angular spherical coordinates of the point $\mathbf{x}$. We introduce the following notation:

$$M_{ij\ell m}^1(\boldsymbol{x}) = \frac{1}{3}\delta_{ij}\delta_{\ell m},$$

$$M_{ij\ell m}^2(\boldsymbol{x}) = \frac{1}{\sqrt{5}} \sum_{n=-2}^{2} g_{2[1,1]}^{n[i,j]} g_{2[1,1]}^{n[\ell,m]},$$

$$M_{ij\ell m}^3(\boldsymbol{x}) = \frac{1}{\sqrt{6}}\left[\delta_{ij} \sum_{n=-2}^{2} g_{2[1,1]}^{n[\ell,m]} U_{n0}^2(k_{\boldsymbol{x}}) + \delta_{\ell m} \sum_{n=-2}^{2} g_{2[1,1]}^{n[i,j]} U_{n0}^2(k_{\boldsymbol{x}})\right], \quad (3\text{-}1)$$

$$M_{ij\ell m}^4(\boldsymbol{x}) = \sum_{n=-2}^{2} \sum_{p=-2}^{2} g_{2[1,1]}^{n[i,j]} g_{2[1,1]}^{p[\ell,m]} \sum_{q=-2}^{2} g_{2[2,2]}^{q[n,p]} U_{q0}^2(k_{\boldsymbol{x}}),$$

$$M_{ij\ell m}^5(\boldsymbol{x}) = \sum_{n=-2}^{2} \sum_{p=-2}^{2} g_{2[1,1]}^{n[i,j]} g_{2[1,1]}^{p[\ell,m]} \sum_{q=-4}^{4} g_{4[2,2]}^{q[n,p]} U_{q0}^4(k_{\boldsymbol{x}}).$$

Theorem 3.2. *Let V be the space of all symmetric second-rank tensors over $\mathbb{R}^3$, let $\boldsymbol{T}(\boldsymbol{x})$ be a V-valued random field on $\mathbb{R}^3$ satisfying (1-1) with $U(k)\boldsymbol{T} = k\boldsymbol{T}k^{-1}$. Then $E_{ij}(\boldsymbol{x}) = C\delta_{ij}$, $C \in \mathbb{R}$, and there exist five continuous functions $K_1, \ldots, K_5 : [0, \infty) \to \mathbb{R}$ with $K_3(0) = K_4(0) = K_5(0) = 0$ such that*

$$B_{ij\ell m}(\boldsymbol{x}) = \sum_{n=1}^{5} M_{ij\ell m}^n(\boldsymbol{x}) K_n(\|\boldsymbol{x}\|). \quad (3\text{-}2)$$

A formula similar to (3-2) has been obtained by Lomakin [1964]. For any fixed $\boldsymbol{x} \in \mathbb{R}^3$, the tensor in the left-hand side of (3-2) is a symmetric linear operator acting in the space of symmetric tensors of the second rank. Following Boehler et al. [1994], denote the space of all such tensors by T_4^e. Under the action of SO(3), the space T_4^e decomposes into the following direct sum:

$$T_4^e = V^0 \oplus V^0 \oplus V^2 \oplus V^2 \oplus V^4,$$

where V^i may be considered as the space of completely symmetric traceless tensors of the i-th rank linearly dependent on x_i. Using the general form of such a tensor given by invariant theory (see, for example, [Spencer 1971]), we obtain the result of [Lomakin 1964]:

$$B_{ij\ell m}(\boldsymbol{x}) = \sum_{n=1}^{5} L_{ij\ell m}^n(\boldsymbol{x}) K_n(\|\boldsymbol{x}\|),$$

where (compare with [Boehler et al. 1994, Lemma, pp. 98–99])

$$L^1_{ij\ell m}(\boldsymbol{x}) = \delta_{ij}\delta_{\ell m},$$

$$L^2_{ij\ell m}(\boldsymbol{x}) = \delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{jl},$$

$$L^3_{ij\ell m}(\boldsymbol{x}) = \frac{x_j x_\ell}{\|\boldsymbol{x}\|^2}\delta_{im} + \frac{x_i x_m}{\|\boldsymbol{x}\|^2}\delta_{j\ell} + \frac{x_i x_\ell}{\|\boldsymbol{x}\|^2}\delta_{jm} + \frac{x_j x_m}{\|\boldsymbol{x}\|^2}\delta_{i\ell},$$

$$L^4_{ij\ell m}(\boldsymbol{x}) = \frac{x_i x_j}{\|\boldsymbol{x}\|^2}\delta_{\ell m} + \frac{x_\ell x_m}{\|\boldsymbol{x}\|^2}\delta_{ij},$$

$$L^5_{ij\ell m}(\boldsymbol{x}) = \frac{x_i x_j x_\ell x_m}{\|\boldsymbol{x}\|^4}.$$

$$(3\text{-}3)$$

We prove that Theorem 3.2 is equivalent to the result of [Lomakin 1964]. Indeed, we have

$$M^1_{ij\ell m}(\boldsymbol{x}) = \frac{1}{3}L^1_{ij\ell m}(\boldsymbol{x}),$$

$$M^2_{ij\ell m}(\boldsymbol{x}) = -\frac{1}{3\sqrt{5}}L^1_{ij\ell m}(\boldsymbol{x}) + \frac{1}{2\sqrt{5}}L^2_{ij\ell m}(\boldsymbol{x}),$$

$$M^3_{ij\ell m}(\boldsymbol{x}) = -\frac{1}{3}L^1_{ij\ell m}(\boldsymbol{x}) + \frac{1}{2}L^4_{ij\ell m}(\boldsymbol{x}),$$

$$M^4_{ij\ell m}(\boldsymbol{x}) = \frac{2\sqrt{2}}{3\sqrt{7}}L^1_{ij\ell m}(\boldsymbol{x}) - \frac{1}{\sqrt{14}}L^2_{ij\ell m}(\boldsymbol{x}) + \frac{3}{2\sqrt{14}}L^3_{ij\ell m}(\boldsymbol{x}) - \frac{\sqrt{2}}{\sqrt{7}}L^4_{ij\ell m}(\boldsymbol{x}),$$

$$M^5_{ij\ell m}(\boldsymbol{x}) = \frac{1}{2\sqrt{70}}L^1_{ij\ell m}(\boldsymbol{x}) + \frac{1}{2\sqrt{70}}L^2_{ij\ell m}(\boldsymbol{x}) - \frac{\sqrt{5}}{2\sqrt{14}}L^3_{ij\ell m}(\boldsymbol{x})$$

$$-\frac{\sqrt{5}}{2\sqrt{14}}L^4_{ij\ell m}(\boldsymbol{x}) + \frac{\sqrt{35}}{2\sqrt{2}}L^5_{ij\ell m}(\boldsymbol{x}).$$

$$(3\text{-}4)$$

It is easy to check that the transition matrix between Lomakin's functions (3-3) and the functions (3-1) is invertible. A proof of (3-4) may be found in the Appendix.

Given that $\boldsymbol{T}$ has diagonal and off-diagonal components, there are five special cases of $B_{ij\ell m}$ that shed light on the physical meaning of the K_n:

(1) $\mathsf{E}[T_{ij}(\boldsymbol{0})T_{kl}(\boldsymbol{x})]|_{i=j=k=l}$; that is, auto-correlations of diagonal terms

$$\mathsf{E}[T_{11}(\boldsymbol{0})T_{11}(\boldsymbol{x})] = K_1 + 2K_2 + 2x_1^2 K_3 + 4x_1^2 K_4 + x_1^4 K_5$$

and then $\mathsf{E}[T_{22}(\boldsymbol{0})T_{22}(\boldsymbol{x})]$ and $\mathsf{E}[T_{33}(\boldsymbol{0})T_{33}(\boldsymbol{x})]$ by cyclic permutations $1 \to 2 \to 3$.

(2) $\mathsf{E}[T_{ij}(\boldsymbol{0})T_{kl}(\boldsymbol{x})]|_{i=j\neq k=l}$; that is, cross-correlations of diagonal terms

$$\mathsf{E}[T_{11}(\boldsymbol{0})T_{22}(\boldsymbol{x})] = K_1 + (x_2^2 + x_1^2)K_3 + x_2^2 x_1^2 K_5$$

and then $\mathsf{E}[T_{22}(\boldsymbol{0})T_{33}(\boldsymbol{x})]$ and $\mathsf{E}[T_{33}(\boldsymbol{0})T_{11}(\boldsymbol{x})]$ by cyclic permutations $1 \to 2 \to 3$.

(3) $\mathsf{E}[T_{ij}(\mathbf{0})T_{kl}(\mathbf{x})]|_{i=k\neq j=l}$; that is, auto-correlations of off-diagonal terms

$$\mathsf{E}[T_{12}(\mathbf{0})T_{12}(\mathbf{x})] = K_2 + (x_1^2 + x_2^2)K_4 + x_1^2 x_2^2 K_5$$

and then $\mathsf{E}[T_{23}(\mathbf{0})T_{23}(\mathbf{x})]$ and $\mathsf{E}[T_{31}(\mathbf{0})T_{31}(\mathbf{x})]$ by cyclic permutations $1 \rightarrow 2 \rightarrow 3$.

(4) $\mathsf{E}[T_{ij}(\mathbf{0})T_{kl}(\mathbf{x})]|_{j\neq i=k\neq l\neq j}$; that is, cross-correlations of off-diagonal terms

$$\mathsf{E}[T_{12}(\mathbf{0})T_{13}(\mathbf{x})] = x_2 x_3 K_4 + x_1^2 x_2 x_3 K_5$$

and then $\mathsf{E}[T_{13}(\mathbf{0})T_{32}(\mathbf{x})]$ and $\mathsf{E}[T_{32}(\mathbf{0})T_{12}(\mathbf{x})]$ by cyclic permutations $1 \rightarrow 2 \rightarrow 3$.

(5) $\mathsf{E}[T_{ij}(\mathbf{0})T_{kl}(\mathbf{x})]|_{i=j=k\neq l\neq j}$; that is, cross-correlations of diagonal terms with off-diagonal terms such as

$$\mathsf{E}[T_{11}(\mathbf{0})T_{12}(\mathbf{x})] = x_1 x_2 (K_3 + 2K_4) + x_1 x_2^3 K_5$$

and

$$\mathsf{E}[T_{12}(\mathbf{0})T_{13}(\mathbf{x})] = x_2 x_3 K_3 + x_1^2 x_2 x_3 K_5$$

and the others by cyclic permutations $1 \rightarrow 2 \rightarrow 3$.

In principle, we can determine these five correlations for a specific physical situation. For example, when $\mathbf{T}$ is the antiplane elasticity tensor for a given resolution (or mesoscale) [Ostoja-Starzewski 2008], we can use micromechanics or experiments and then determine the best fits of the K_n ($n = 1, \ldots, 5$) coefficients.

4. Concluding remarks

Remark 4.1. On the one hand, Lomakin's functions (3-3) are simpler than functions (3-1). On the other hand, the restrictions of the functions (3-1) to the unit sphere $S^2 \subset \mathbb{R}^3$ are orthogonal in the space of the square-integrable functions on S^2. Using this property, in a forthcoming paper we will obtain spectral expansions of tensor-valued homogeneous and isotropic random fields similar to those of Yaglom [1957].

Remark 4.2. The *spherical harmonics* $Y_{\ell m}(\theta, \varphi)$ are proportional to the Wigner D-functions by [Varshalovich et al. 1975, Equation (37), §5.2]:

$$Y_{\ell m}(\theta, \varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} D^\ell_{-m0}(0, \theta, \varphi).$$

The matrix entries $U^i_{\ell 0}(k_{\mathbf{x}})$ are proportional to the *zonal, sectorial, and tesseral harmonics* defined by [Varshalovich et al. 1975, Equation (14), §5.1].

Remark 4.3. The matrix entries $U^i_{j0}(k_x)$, the Godunov–Gordienko coefficients, and (3-4) were calculated and proved by hand. Afterwards, they were checked using MATLAB and Symbolic Math Toolbox™ [Mathworks 2013]. The problem of an algebraic proof of the second and fourth equations in (3-4) remains open.

Appendix: Proofs

Proof of Lemma 2.2. The two-point correlation function $B(\boldsymbol{x})$ is a linear operator in V. It is known that the space of linear operators in V is isomorphic to the tensor product $V^* \otimes V$, where V^* is the set of all $\mathbb{R}$-linear maps $\boldsymbol{v}^* : V \to \mathbb{R}$. We need to prove that for any $S \in V^* \otimes V$ the following equality holds true:

$$U(k)\,\mathrm{SU}^{-1}(k) = (U \otimes U)(k)S.$$

Note that the set of tensors satisfying the above equality form a linear space. Therefore, it is enough to prove this equality for tensors of the form $\boldsymbol{v}^* \otimes \boldsymbol{v}$, where $\boldsymbol{v} \in V$ and $\boldsymbol{v}^* \in V^*$. The linear operator $\boldsymbol{v}^* \otimes \boldsymbol{v}$ acts on V by

$$(\boldsymbol{v}^* \otimes \boldsymbol{v})\boldsymbol{w} = \boldsymbol{v}^*(\boldsymbol{w})\boldsymbol{v}, \quad \boldsymbol{w} \in V.$$

For this operator,

$$(U(k)\boldsymbol{v}^* \otimes \boldsymbol{v}U^{-1}(k))\boldsymbol{w} = U(k)\boldsymbol{v}^*(U^{-1}(k)\boldsymbol{w})\boldsymbol{v} = (U(k)\boldsymbol{v}^*)(\boldsymbol{w})U(k)\boldsymbol{v}$$
$$= (U(k)\boldsymbol{v}^*) \otimes (U(k)\boldsymbol{v})\boldsymbol{w} = (U \otimes U)(k)(\boldsymbol{v}^* \otimes \boldsymbol{v})\boldsymbol{w}.$$

By linearity, this equality follows for any $S \in V^* \otimes V$. □

Proof of Theorem 3.1. The first equation in (1-1) may be written in the Gordienko basis as

$$U^1(k)\boldsymbol{E} = \boldsymbol{E}.$$

The representation U^1 is irreducible. By Lemma 2.1, $\boldsymbol{E} = \boldsymbol{0}$.

We cannot apply Lemma 2.1 to (2-11), because the representation $U^1 \otimes U^1$ acting in $V^1 \otimes V^1$ is reducible.

By definition, a tensor field $B(\boldsymbol{x})$ is a function on $\mathbb{R}^3$ with values in the linear space of real-valued bilinear forms on $V^1 \times V^1$. The component $B_{ij}(\boldsymbol{x})$ is the value of the bilinear form $B(\boldsymbol{x})$ at the point $(\boldsymbol{h}^1_i, \boldsymbol{h}^1_j) \in V^1 \times V^1$:

$$B_{ij}(\boldsymbol{x}) = B(\boldsymbol{x}; \boldsymbol{h}^1_i, \boldsymbol{h}^1_j).$$

The last equality may be rewritten as

$$B_{ij}(\boldsymbol{x}) = B(\boldsymbol{x}; \boldsymbol{h}^1_i \otimes \boldsymbol{h}^1_j),$$

where the right-hand side is the value of the linear form $B(\boldsymbol{x})$ at the point $\boldsymbol{h}^1_i \otimes \boldsymbol{h}^1_j \in V^1 \otimes V^1$. In other words, tensor products simplify multilinear forms to linear ones.

Using (2-9) with $m = p = 1$, $n = 1$, and $q = j$, we obtain

$$
\begin{aligned}
B_{ij}(\boldsymbol{x}) = B(\boldsymbol{x}; \boldsymbol{h}_i^1 \otimes \boldsymbol{h}_j^1) &= B\left(\boldsymbol{x}; \sum_{m=0}^{2} \sum_{\ell=-m}^{m} g_{m[1,1]}^{\ell[i,j]} \boldsymbol{h}_\ell^m\right) \\
&= \sum_{m=0}^{2} \sum_{\ell=-m}^{m} g_{m[1,1]}^{\ell[i,j]} B(\boldsymbol{x}; \boldsymbol{h}_\ell^m).
\end{aligned}
\tag{A.1}
$$

Let $\boldsymbol{B}_m(\boldsymbol{x}) \in V^m$ be the vector field with components $B(\boldsymbol{x}; \boldsymbol{h}_\ell^m)$, $-m \le \ell \le m$. For this field, we have

$$
\boldsymbol{B}_m(k\boldsymbol{x}) = U^m(k)\boldsymbol{B}_m(\boldsymbol{x}), \quad k \in \mathrm{SO}(3).
\tag{A.2}
$$

By Lemma 2.1, $\boldsymbol{B}_m(\boldsymbol{0}) = \boldsymbol{0}$ for $m \ge 1$, while $\boldsymbol{B}_0(\boldsymbol{0})$ may take any real value.

Let $\mathrm{SO}(2)$ be the subgroup of rotations around x_0 axis. The restriction of the representation U^m to $\mathrm{SO}(2)$ is the direct sum of the trivial representation W^0 of $\mathrm{SO}(2)$ acting in the one-dimensional space spanned by the vector $\boldsymbol{h}_0^m$ and of the irreducible representations

$$
W^\ell(\varphi) = \begin{pmatrix} \cos(\ell\varphi) & \sin(\ell\varphi) \\ -\sin(\ell\varphi) & \cos(\ell\varphi) \end{pmatrix}, \quad \varphi \in \mathrm{SO}(2),
$$

acting in the two-dimensional spaces spanned by the vectors $\boldsymbol{h}_\ell^m$ and $\boldsymbol{h}_{-\ell}^m$, $1 \le \ell \le m$. If $\boldsymbol{x} = (0, \|\boldsymbol{x}\|, 0)^\top \ne \boldsymbol{0}$, then $k\boldsymbol{x} = \boldsymbol{x}$, $k \in \mathrm{SO}(2)$, and

$$
(B(\boldsymbol{x}; \boldsymbol{h}_{-\ell}^m), B(\boldsymbol{x}; \boldsymbol{e}_\ell^m))^\top = W^\ell(\varphi)(B(\boldsymbol{x}; \boldsymbol{h}_{-\ell}^m), B(\boldsymbol{x}; \boldsymbol{h}_\ell^m))^\top, \quad 1 \le \ell \le m.
$$

By Lemma 2.1, $B(\boldsymbol{x}; \boldsymbol{h}_{-\ell}^m) = B(\boldsymbol{x}; \boldsymbol{h}_\ell^m) = 0$, if $1 \le \ell \le m$, while $B(\boldsymbol{x}; \boldsymbol{h}_0^m)$ is an arbitrary continuous real-valued function with $B(\boldsymbol{0}; \boldsymbol{h}_0^m) = 0$ for $m \ge 1$. In what follows, we denote this function by $B_m(\|\boldsymbol{x}\|)$.

By (A.2),

$$
B(\boldsymbol{x}; \boldsymbol{h}_\ell^m) = U_{\ell 0}^m(k_{\boldsymbol{x}}) B_m(\|\mathbf{x}\|).
\tag{A.3}
$$

Substitute (A.3) into (A.1). We obtain

$$
B_{ij}(\boldsymbol{x}) = \sum_{m=0}^{2} \sum_{\ell=-m}^{m} g_{m[1,1]}^{\ell[i,j]} U_{\ell 0}^m(k_{\boldsymbol{x}}) B_m(\|\boldsymbol{x}\|).
$$

In other words, $B_{ij}(\boldsymbol{x})$ is the sum of three matrix-valued functions

$$
B_{ij}(\boldsymbol{x}) = \sum_{m=0}^{2} B_{ij}^m(\boldsymbol{x}),
$$

with

$$
B_{ij}^m(\boldsymbol{x}) = \sum_{\ell=-m}^{m} g_{m[1,1]}^{\ell[i,j]} U_{\ell 0}^m(k_{\boldsymbol{x}}) B_m(\|\boldsymbol{x}\|).
$$

Let $m = 0$. Then

$$U_{00}^0(k_{\boldsymbol{x}}) = 1, \tag{A.4}$$

the matrix entry of the 1×1 identity matrix. To calculate the Godunov–Gordienko coefficients $g_{0[1,1]}^{0[i,j]}$, use the following property [Godunov and Gordienko 2004]:

$$g_{N[N_1,N_2]}^{n[n_1,n_2]} = (-1)^{N+N_1+N_2} \sqrt{\frac{2N+1}{2N_1+1}} \, g_{N_1[N,N_2]}^{n_1[n,n_2]}, \tag{A.5}$$

with $N = n = 0$, $N_1 = N_2 = 1$, $n_1 = i$, and $n_2 = j$. We have

$$g_{0[1,1]}^{0[i,j]} = \sqrt{1/3} \, g_{1[0,1]}^{i[0,j]}.$$

The coefficients in the right-hand side can be calculated using (2-9):

$$\boldsymbol{h}_0^0 \otimes \boldsymbol{h}_j^1 = \sum_{i=-1}^{1} g_{1[0,1]}^{i[0,j]} \boldsymbol{h}_i^1.$$

The left-hand side is clearly equal to $\boldsymbol{h}_j^1$. Therefore, $g_{1[0,1]}^{i[0,j]} = \delta_{ij}$,

$$g_{0[1,1]}^{0[i,j]} = \sqrt{1/3} \, \delta_{ij}, \tag{A.6}$$

and the first matrix is

$$B_{ij}^0(\boldsymbol{x}) = \sqrt{1/3} \, \delta_{ij} B_0(\|\boldsymbol{x}\|).$$

Let $m = 1$. Using (2-5)–(2-7), we obtain

$$\begin{pmatrix} U_{-10}^1(\varphi, \theta) \\ U_{00}^1(\varphi, \theta) \\ U_{10}^1(\varphi, \theta) \end{pmatrix} = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \sin\varphi \sin\theta \\ \cos\theta \\ \cos\varphi \sin\theta \end{pmatrix},$$

or

$$U_{\ell 0}^1(k_{\boldsymbol{x}}) = \frac{x_\ell}{r}.$$

The Godunov–Gordienko coefficients $g_{1[1,1]}^{\ell[i,j]}$ are calculated by [Godunov and Gordienko 2004, Formulae (1.26)–(1.29)]. The nonzero coefficients are

$$g_{1[1,1]}^{-1[1,0]} = g_{1[1,1]}^{0[-1,1]} = g_{1[1,1]}^{1[0,-1]} = \sqrt{1/2},$$

$$g_{1[1,1]}^{-1[0,1]} = g_{1[1,1]}^{0[1,-1]} = g_{1[1,1]}^{1[-1,0]} = -\sqrt{1/2}.$$

The second matrix becomes

$$\frac{1}{r\sqrt{2}} \begin{pmatrix} 0 & -x_1 & x_0 \\ x_1 & 0 & -x_{-1} \\ -x_0 & x_{-1} & 0 \end{pmatrix} B_1(\|\boldsymbol{x}\|).$$

This matrix is not symmetric but skew-symmetric. The matrix $B_{ij}(\boldsymbol{x})$ is symmetric if and only if $B_1(\|\boldsymbol{x}\|) = 0$.

The matrix entries $U^2_{\ell 0}(k_{\boldsymbol{x}})$ can be calculated using (2-5)–(2-7) as

$$U^2_{-20}(k_{\boldsymbol{x}}) = -\frac{\sqrt{3}x_{-1}x_1}{r^2}, \quad U^2_{-10}(k_{\boldsymbol{x}}) = \frac{\sqrt{3}x_{-1}x_0}{r^2}, \quad U^2_{00}(k_{\boldsymbol{x}}) = \frac{3x_0^2}{2r^2} - \frac{1}{2},$$

$$U^2_{10}(k_{\boldsymbol{x}}) = \frac{\sqrt{3}x_0 x_1}{r^2}, \qquad U^2_{20}(k_{\boldsymbol{x}}) = \frac{\sqrt{3}(x_{-1}^2 - x_1^2)}{2r^2}.$$

To calculate the Godunov–Gordienko coefficients $g^{\ell[i,j]}_{2[1,1]}$, use (A.5) with $N = 2$, $N_1 = N_2 = 1$, $n = \ell$, $n_1 = i$, and $n_2 = j$. We have

$$g^{\ell[i,j]}_{2[1,1]} = \sqrt{5/3}\, g^{i[\ell,j]}_{1[2,1]}.$$

The coefficients $g^{i[\ell,j]}_{1[2,1]}$ are calculated by [Godunov and Gordienko 2004, Formulae (1.30)–(1.34)]. The nonzero Godunov–Gordienko coefficients are

$$g^{-2[-1,1]}_{2[1,1]} = g^{-2[1,-1]}_{2[1,1]} = g^{2[1,1]}_{2[1,1]} = -\sqrt{1/2},$$

$$g^{0[-1,-1]}_{2[1,1]} = g^{0[1,1]}_{2[1,1]} = -\sqrt{1/6},$$

$$g^{-1[-1,0]}_{2[1,1]} = g^{-1[0,-1]}_{2[1,1]} = g^{1[0,1]}_{2[1,1]} = g^{1[1,0]}_{2[1,1]} = g^{2[-1,-1]}_{2[1,1]} = \sqrt{1/2},$$

$$g^{0[0,0]}_{2[1,1]} = \sqrt{2/3}.$$

In (2-10) put $k = j = 0$, $m = p = 1$, $n = i$, and $q = j$. We have

$$U^1_{i0}(k)U^1_{j0}(k) = \sum_{s=0}^{2}\sum_{t=-s}^{s}\sum_{\ell=-s}^{s} g^{t[i,j]}_{s[1,1]}U^s_{t\ell}(k)g^{\ell[0,0]}_{s[1,1]}.$$

Using the above-calculated values of the matrix entries and Godunov–Gordienko coefficients, we may rewrite this formula as

$$\frac{x_i x_j}{r^2} = \frac{1}{3}\delta_{ij} + \sqrt{\frac{2}{3}}\sum_{t=-2}^{2} g^{t[i,j]}_{2[1,1]}U^2_{t0}(k_{\boldsymbol{x}})$$

or

$$\sum_{t=-2}^{2} g^{t[i,j]}_{2[1,1]}U^2_{t0}(k_{\boldsymbol{x}}) = \frac{\sqrt{3}}{\sqrt{2}r^2}\left(x_j x_j - \frac{r^2}{3}\delta_{ij}\right). \tag{A.7}$$

We introduce the following notation:

$$K_0(\|\boldsymbol{x}\|) = \frac{1}{\sqrt{3}}B_0(\|\boldsymbol{x}\|) - \frac{1}{\sqrt{6}}B_2(\|\boldsymbol{x}\|), \quad K_2(\|\boldsymbol{x}\|) = \frac{\sqrt{3}}{\sqrt{2}r^2}B_2(\|\boldsymbol{x}\|).$$

The theorem is proved. Note that the space V^0 consists of tensors proportional to the Kronecker delta, V^1 consists of skew-symmetric tensors, and V^2 consists of symmetric traceless tensors. □

Proof of Theorem 3.2. Use Lemma 2.2 to write the representation U as $U = U^1 \otimes U^1$. The first equation in (1-1) takes the form

$$E(k\boldsymbol{x}) = (U^1 \otimes U^1)(k)E(\boldsymbol{x}).$$

or

$$E = (U^1 \otimes U^1)(k)E, \quad k \in \mathrm{SO}(3).$$

In other words, E lies in the subspace of V where the trivial representation is realized, or

$$\mathsf{E}[T_{ij}(\boldsymbol{x})] = C\delta_{ij}, \quad C \in \mathbb{R}.$$

The second equation in (1-1) takes the form

$$B(k\boldsymbol{x}) = (U^1 \otimes U^1 \otimes U^1 \otimes U^1)(k)B(\boldsymbol{x}).$$

We write $B(\boldsymbol{x})$ as a bilinear form on pairs of tensors, $(V^1 \otimes V^1) \times (V^1 \otimes V^1)$:

$$B^{ijk\ell}(\boldsymbol{x}) = B(\boldsymbol{x}; \boldsymbol{h}_i^1 \otimes \boldsymbol{h}_j^1, \boldsymbol{h}_\ell^1 \otimes \boldsymbol{h}_m^1).$$

Note that the tensor field $B^{ij\ell m}(\boldsymbol{x})$ is symmetric in the following sense:

$$B^{\ell m ij}(\boldsymbol{x}) = B^{ij\ell m}(\boldsymbol{x}).$$

Recall that $V^1 \otimes V^1 = V^0 \oplus V^1 \oplus V^2$. Rewrite $B(\boldsymbol{x})$ as a bilinear form on $(V^0 \oplus V^1 \oplus V^2) \times (V^0 \oplus V^1 \oplus V^2)$:

$$B^{ij\ell m}(\boldsymbol{x}) = B\left(\boldsymbol{x}; \sum_{n=0}^{2}\sum_{p=-n}^{n} g_{n[1,1]}^{p[i,j]}\boldsymbol{h}_p^n, \sum_{q=0}^{2}\sum_{r=-q}^{q} g_{q[1,1]}^{r[\ell,m]}\boldsymbol{h}_r^q\right).$$

In fact, $B(\boldsymbol{x})$ is a bilinear form on pairs of symmetric tensors, which is to say, on $(V^0 \oplus V^2) \times (V^0 \oplus V^2)$:

$$B^{ij\ell m}(\boldsymbol{x}) = B\left(\boldsymbol{x}; \sum_{n\in\{0,2\}}\sum_{p=-n}^{n} g_{n[1,1]}^{p[i,j]}\boldsymbol{h}_p^n, \sum_{q\in\{0,2\}}\sum_{r=-q}^{q} g_{q[1,1]}^{r[\ell,m]}\boldsymbol{h}_r^q\right)$$

$$= \sum_{(n,q)\in\{0,2\}\times\{0,2\}}\sum_{p=-n}^{n}\sum_{r=-q}^{q} g_{n[1,1]}^{p[i,j]}g_{q[1,1]}^{r[\ell,m]} B(\boldsymbol{x}; \boldsymbol{h}_p^n, \boldsymbol{h}_r^q)$$

$$= \sum_{(n,q)\in\{0,2\}\times\{0,2\}}\sum_{p=-n}^{n}\sum_{r=-q}^{q} g_{n[1,1]}^{p[i,j]}g_{q[1,1]}^{r[\ell,m]} B(\boldsymbol{x}; \boldsymbol{h}_p^n \otimes \boldsymbol{h}_r^q)$$

$$= \sum_{(n,q)\in\{0,2\}\times\{0,2\}}\sum_{p=-n}^{n}\sum_{r=-q}^{q} g_{n[1,1]}^{p[i,j]}g_{q[1,1]}^{r[\ell,m]} B\left(\boldsymbol{x}; \sum_{s=|n-q|}^{n+q}\sum_{t=-s}^{s} g_{s[m,p]}^{t[n,q]}\boldsymbol{h}_t^s\right).$$

The possible values for s are $0 \le s \le 4$. Let A_s be the set of all pairs $(n,q) \in \{0,2\} \times \{0,2\}$ such that $V^s \subseteq V^n \otimes V^q$. We have

$$A_0 = \{(0,0), (2,2)\}, \quad A_1 = A_3 = A_4 = \{(2,2)\}, \quad A_2 = \{(0,2), (2,0), (2,2)\},$$

and

$$B^{ij\ell m}(\boldsymbol{x}) = \sum_{s=0}^{4} B_s^{ij\ell m}(\boldsymbol{x}),$$

where

$$B_s^{ij\ell m}(\boldsymbol{x}) = \sum_{(n,q)\in A_s} \sum_{p=-n}^{n} \sum_{r=-q}^{q} g_{n[1,1]}^{p[i,j]} g_{q[1,1]}^{r[\ell,m]} \sum_{t=-s}^{s} g_{s[m,p]}^{t[n,q]} B(\boldsymbol{x}; \boldsymbol{h}_t^s(n,q)),$$

and where the pairs $(n,q) \in A_s$ enumerate copies of the vector $\boldsymbol{h}_t^s$. By (A.3),

$$B(\boldsymbol{x}; \boldsymbol{h}_t^s(n,q)) = U_{t0}^s(k_{\boldsymbol{x}}) B_{s(n,q)}(\|\boldsymbol{x}\|),$$

where $B_{s(n,q)}(\|\boldsymbol{x}\|)$ are continuous real-valued functions with $B_{s(n,q)}(0) = 0$ for $s \geq 1$. We obtain

$$B_s^{ij\ell m}(\boldsymbol{x}) = \sum_{(n,q)\in A_s} C_{nqs}^{ij\ell m}(\boldsymbol{x}),$$

where

$$C_{nqs}^{ij\ell m}(\boldsymbol{x}) = \sum_{p=-n}^{n} \sum_{r=-q}^{q} g_{n[1,1]}^{p[i,j]} g_{q[1,1]}^{r[\ell,m]} \sum_{t=-s}^{s} g_{s[n,q]}^{t[p,r]} U_{t0}^s(k_{\boldsymbol{x}}) B_{s(n,q)}(\|\boldsymbol{x}\|).$$

We calculate the functions $C_{nqs}^{ij\ell m}(\boldsymbol{x})$.

Let $(n,q) = (0,0) \in A_0$. The corresponding term is

$$C_{000}^{ij\ell m}(\boldsymbol{x}) = \tfrac{1}{3} \delta_{ij} \delta_{\ell m} B_{0(0,0)}(\|\boldsymbol{x}\|) = M_{ij\ell m}^1(\boldsymbol{x}) K_1(\|\boldsymbol{x}\|),$$

by (A.4) and (A.6) with $K_1(\|\boldsymbol{x}\|) = B_{0(0,0)}(\|\boldsymbol{x}\|)$.

Let $(n,q) = (2,2) \in A_0$. The corresponding term is

$$C_{220}^{ij\ell m}(\boldsymbol{x}) = \sum_{r=-2}^{2} \sum_{s=-2}^{2} g_{2[1,1]}^{r[i,j]} g_{2[1,1]}^{s[\ell,m]} g_{0[2,2]}^{0[r,s]} U_{00}^0(k_{\boldsymbol{x}}) B_{0(2,2)}(\|\boldsymbol{x}\|).$$

The Godunov–Gordienko coefficients $g_{2[1,1]}^{r[i,j]}$ and $g_{2[1,1]}^{s[\ell,m]}$ are calculated by (A.6), while the coefficients $g_{0[2,2]}^{0[n,q]}$ are calculated using (A.5) with $n = N = 0$, $n_1 = r$, $n_2 = s$, $N_1 = N_2 = 2$ and (2-9):

$$g_{0[2,2]}^{0[r,s]} = \sqrt{1/5} g_{2[0,2]}^{r[0,s]} = \sqrt{1/5} \delta_{rs}.$$

Therefore, we obtain

$$C_{220}^{ij\ell m}(\boldsymbol{x}) = \sqrt{1/5} \sum_{r=-2}^{2} g_{2[1,1]}^{r[i,j]} g_{2[1,1]}^{r[\ell,m]} B_{0(2,2)}(\|\boldsymbol{x}\|) = M_{ij\ell m}^2(\boldsymbol{x}) K_2(\|\boldsymbol{x}\|)$$

with $K_2(\|\boldsymbol{x}\|) = B_{0(2,2)}(\|\boldsymbol{x}\|)$.

Let $(n, q) = (2, 2) \in A_1$. The corresponding term is

$$C_{221}^{ij\ell m}(\boldsymbol{x}) = \sum_{p=-2}^{2} \sum_{s=-2}^{2} g_{2[1,1]}^{p[i,j]} g_{2[1,1]}^{r[\ell,m]} \sum_{s=-1}^{1} g_{1[2,2]}^{s[p,r]} U_{s0}^{1}(k_{\boldsymbol{x}}) B_{1(2,2)}(\|\boldsymbol{x}\|).$$

The Godunov–Gordienko coefficients $g_{1[2,2]}^{s[p,r]}$ are calculated by [Godunov and Gordienko 2004, Formulae (1.26)–(1.29)]. The nonzero elements of the *Godunov–Gordienko matrix* $G_{1[2,2]}^{-1}$ are

$$g_{1[2,2]}^{-1[0,1]} = -\sqrt{3/10}, \quad g_{1[2,2]}^{-1[-2,-1]} = g_{1[2,2]}^{-1[2,1]} = -\sqrt{1/10},$$
$$g_{1[2,2]}^{-1[1,0]} = \sqrt{3/10}, \quad g_{1[2,2]}^{-1[-1,-2]} = g_{1[2,2]}^{-1[1,2]} = \sqrt{1/10},$$

and those of the matrix $G_{1[2,2]}^{0}$ are

$$g_{1[2,2]}^{0[2,-2]} = -\sqrt{2/5}, \quad g_{1[2,2]}^{0[1,-1]} = -\sqrt{1/10},$$
$$g_{1[2,2]}^{0[-2,2]} = \sqrt{2/5}, \quad g_{1[2,2]}^{0[-1,1]} = \sqrt{1/10}.$$

Finally, the nonzero elements of the Godunov–Gordienko matrix $G_{1[2,2]}^{-1}$ are

$$g_{1[2,2]}^{1[0,-1]} = \sqrt{3/10}, \quad g_{1[2,2]}^{1[1,-2]} = g_{1[2,2]}^{1[2,-1]} = -\sqrt{1/10},$$
$$g_{1[2,2]}^{1[-1,0]} = -\sqrt{3/10}, \quad g_{1[2,2]}^{1[-2,1]} = g_{1[2,2]}^{1[-1,2]} = \sqrt{1/10}.$$

We see that the Godunov–Gordienko matrices $g_{1[2,2]}^{s}$, $-1 \leq s \leq 1$, are skew-symmetric. We have $B_{1(2,2)}(\|\boldsymbol{x}\|) = 0$ by the same reasons as $B_1(\|\boldsymbol{x}\|) = 0$ in the proof of Theorem 3.1.

Let $(n, q) = (0, 2) \in A_2$. The corresponding term is

$$C_{022}^{ij\ell m}(\boldsymbol{x}) = g_{0[1,1]}^{0[i,j]} \sum_{p=-2}^{2} g_{2[1,1]}^{p[\ell,m]} \sum_{s=-2}^{2} g_{2[0,2]}^{s[0,p]} U_{s0}^{2}(k_{\boldsymbol{x}}) B_{2(0,2)}(\|\boldsymbol{x}\|).$$

Using (A.6), we obtain

$$C_{022}^{ij\ell m}(\boldsymbol{x}) = \sqrt{1/3}\delta_{ij} \sum_{p=-2}^{2} g_{2[1,1]}^{p[\ell,m]} \sum_{s=-2}^{2} g_{2[0,2]}^{s[0,r]} U_{s0}^{2}(k_{\boldsymbol{x}}) B_{2(0,2)}(\|\boldsymbol{x}\|).$$

Clearly $g_{2[0,2]}^{s[0,p]} = \delta_{ps}$. Therefore,

$$C_{022}^{ij\ell m}(\boldsymbol{x}) = \sqrt{1/3}\delta_{ij} \sum_{p=-2}^{2} g_{2[1,1]}^{p[\ell,m]} U_{p0}^{2}(k_{\boldsymbol{x}}) B_{2(0,2)}(\|\boldsymbol{x}\|).$$

Let $(n, q) = (2, 0) \in A_2$. The corresponding term is

$$C_{202}^{ij\ell m}(\boldsymbol{x}) = g_{0[1,1]}^{0[\ell,m]} \sum_{p=-2}^{2} g_{2[1,1]}^{p[i,j]} \sum_{s=-2}^{2} g_{2[2,0]}^{t[p,0]} U_{s0}^{2}(k_{\boldsymbol{x}}) B_{2(2,0)}(\|\boldsymbol{x}\|).$$

Using (A.6), we obtain

$$C_{202}^{ij\ell m}(\boldsymbol{x}) = \sqrt{1/3}\delta_{\ell m} \sum_{p=-2}^{2} g_{2[1,1]}^{p[i,j]} \sum_{s=-2}^{2} g_{2[2,0]}^{t[p,0]} U_{s0}^2(k_{\boldsymbol{x}}) B_{2(2,0)}(\|\boldsymbol{x}\|).$$

Clearly $g_{2[2,0]}^{s[p,0]} = \delta_{ps}$. Therefore,

$$C_{202}^{ij\ell m}(\boldsymbol{x}) = \sqrt{1/3}\delta_{\ell m} \sum_{p=-2}^{2} g_{2[1,1]}^{p[i,j]} U_{p0}^2(k_{\boldsymbol{x}}) B_{2(2,0)}(\|\boldsymbol{x}\|).$$

We see that the tensor field $B^{ij\ell m}(\boldsymbol{x})$ is symmetric if and only if $B_{2(0,2)}(\|\boldsymbol{x}\|) = B_{2(2,0)}(\|\boldsymbol{x}\|)$. Denote their common value by $K_3(\|\boldsymbol{x}\|)$ and put

$$C_2^{ij\ell m}(\boldsymbol{x}) = \frac{1}{\sqrt{2}}(C_{022}^{ij\ell m}(\boldsymbol{x}) + C_{202}^{ij\ell m}(\boldsymbol{x})).$$

We obtain

$$C_2^{ij\ell m}(\boldsymbol{x}) = M_{ij\ell m}^3(\boldsymbol{x}) K_3(\|\boldsymbol{x}\|).$$

Let $(n, q) = (2, 2) \in A_2$. The corresponding term is

$$C_{222}^{ij\ell m}(\boldsymbol{x}) = \sum_{p=-2}^{2} \sum_{r=-2}^{2} g_{2[1,1]}^{p[i,j]} g_{2[1,1]}^{r[\ell,m]} \sum_{s=-2}^{2} g_{2[2,2]}^{s[p,r]} U_{s0}^2(k_{\boldsymbol{x}}) B_{2(2,2)}(\|\boldsymbol{x}\|)$$

$$= M_{ij\ell m}^4(\boldsymbol{x}) K_4(\|\boldsymbol{x}\|),$$

with $K_4(\|\boldsymbol{x}\|) = B_{2(2,2)}(\|\boldsymbol{x}\|)$.

To calculate Godunov–Gordienko coefficients $g_{2[2,2]}^{s[p,r]}$, we use generating function from [Godunov and Gordienko 2004, Formula (2.6)]. The nonzero coefficients are:

$$g_{2[2,2]}^{-2[0,-2]} = g_{2[2,2]}^{-2[-2,0]} = g_{2[2,2]}^{0[-2,-2]} = g_{2[2,2]}^{0[2,2]} = g_{2[2,2]}^{2[2,0]} = g_{2[2,2]}^{2[0,2]} = -\sqrt{2/7},$$

$$g_{2[2,2]}^{-2[1,-1]} = g_{2[2,2]}^{-2[-1,1]} = g_{2[2,2]}^{-1[1,-2]} = g_{2[2,2]}^{-1[-2,1]} = g_{2[2,2]}^{1[-1,-2]} = g_{2[2,2]}^{1[-2,-1]} = g_{2[2,2]}^{1[2,1]}$$

$$= g_{2[2,2]}^{1[1,2]} = g_{2[2,2]}^{2[1,1]} = -\sqrt{3/14},$$

$$g_{2[2,2]}^{-1[0,-1]} = g_{2[2,2]}^{-1[-1,0]} = g_{2[2,2]}^{0[-1,-1]} = g_{2[2,2]}^{0[1,1]} = g_{2[2,2]}^{1[0,1]} = g_{2[2,2]}^{1[1,0]} = \sqrt{1/14},$$

$$g_{2[2,2]}^{-1[2,-1]} = g_{2[2,2]}^{-1[-1,2]} = g_{2[2,2]}^{2[-1,-1]} = \sqrt{3/14},$$

$$g_{2[2,2]}^{0[0,0]} = \sqrt{2/7}.$$

Let $(n, q) = (2, 2) \in A_3$. The corresponding term is

$$C_{223}^{ij\ell m}(\boldsymbol{x}) = \sum_{r=-2}^{2} \sum_{s=-2}^{2} g_{2[1,1]}^{r[i,j]} g_{2[1,1]}^{s[\ell,m]} \sum_{t=-3}^{3} g_{3[2,2]}^{t[r,s]} U_{t0}^3(k_{\boldsymbol{x}}) B_{3(2,2)}(\|\boldsymbol{x}\|).$$

The nonzero elements of the Godunov–Gordienko matrix $g_{3[2,2]}^{-3}$ are

$$g_{3[2,2]}^{-3[-1,-2]} = g_{3[2,2]}^{-3[2,1]} = -1/2, \quad g_{3[2,2]}^{-3[-2,-1]} = g_{3[2,2]}^{-3[1,2]} = 1/2,$$

those of the matrix $g_{3[2,2]}^{-2}$ are

$$g_{3[2,2]}^{-2[2,0]} = -\sqrt{1/2}, \quad g_{3[2,2]}^{-2[0,2]} = \sqrt{1/2},$$

those of the matrix $g_{3[2,2]}^{-1}$ are

$$g_{3[2,2]}^{-1[0,1]} = -\sqrt{1/5}, \qquad\qquad g_{3[2,2]}^{-1[1,0]} = \sqrt{1/5},$$
$$g_{3[2,2]}^{-1[-1,-2]} = g_{3[2,2]}^{-1[1,2]} = -\sqrt{3/20}, \quad g_{3[2,2]}^{-1[-2,-1]} = g_{3[2,2]}^{-1[2,1]} = \sqrt{3/20},$$

those of the matrix $g_{3[2,2]}^{0}$ are

$$g_{3[2,2]}^{0[1,-1]} = -\sqrt{2/5}, \quad g_{3[2,2]}^{0[-1,1]} = \sqrt{2/5},$$
$$g_{3[2,2]}^{0[-2,2]} = -\sqrt{1/10}, \quad g_{3[2,2]}^{0[2,-2]} = \sqrt{1/10},$$

those of the matrix $g_{3[2,2]}^{1}$ are

$$g_{3[2,2]}^{1[-1,0]} = -\sqrt{1/5}, \qquad\qquad g_{3[2,2]}^{1[0,1]} = \sqrt{1/5},$$
$$g_{3[2,2]}^{1[-2,1]} = g_{3[2,2]}^{1[-1,2]} = -\sqrt{3/20}, \quad g_{3[2,2]}^{1[1,-2]} = g_{3[2,2]}^{1[2,-11]} = \sqrt{3/20},$$

and those of the matrix $g_{3[2,2]}^{2}$ are

$$g_{3[2,2]}^{2[-2,0]} = -\sqrt{1/2}, \quad g_{3[2,2]}^{2[0,-22]} = \sqrt{1/2}.$$

Finally, the nonzero elements of the Godunov–Gordienko matrix $g_{3[2,2]}^{3}$ are

$$g_{3[2,2]}^{3[-1,2]} = g_{3[2,2]}^{3[1,-2]} = -1/2, \quad g_{3[2,2]}^{3[-2,1]} = g_{3[2,2]}^{3[2,-1]} = 1/2.$$

We see that the Godunov–Gordienko matrices $g_{3[2,2]}^{t}$, $-3 \leq t \leq 3$, are skew-symmetric. We have $B_{2(2,2)}(\|\boldsymbol{x}\|) = 0$ by the same reasons as before.

Let $(n, q) = (2, 2) \in A_4$. The corresponding term is

$$C_{224}^{ij\ell m}(\boldsymbol{x}) = \sum_{r=-2}^{2} \sum_{s=-2}^{2} g_{2[1,1]}^{r[i,j]} g_{2[1,1]}^{s[\ell,m]} \sum_{t=-4}^{4} g_{4[2,2]}^{t[r,s]} U_{t0}^{4}(k_{\boldsymbol{x}}) B_{4(2,2)}(\|\boldsymbol{x}\|)$$
$$= M_{ij\ell m}^{5}(\boldsymbol{x}) K_5(\|\boldsymbol{x}\|)$$

with $K_5(\|\boldsymbol{x}\|) = B_{4(2,2)}(\|\boldsymbol{x}\|)$.

The nonzero elements of the Godunov–Gordienko matrices $g_{4[2,2]}^{\pm 4}$ are

$$g_{4[2,2]}^{-4[-2,2]} = g_{4[2,2]}^{-4[2,-2]} = g_{4[2,2]}^{4[2,2]} = -\sqrt{1/2}, \quad g_{4[2,2]}^{4[-2,-2]} = \sqrt{1/2},$$

those of the matrices $g_{4[2,2]}^{\pm 3}$ are

$$g_{4[2,2]}^{-3[-2,1]} = g_{4[2,2]}^{-3[1,-2]} = g_{4[2,2]}^{-3[-1,2]} = g_{4[2,2]}^{-3[2,-1]} = g_{4[2,2]}^{3[1,2]} = g_{4[2,2]}^{3[2,1]} = -1/2,$$
$$g_{4[2,2]}^{3[-2,-1]} = g_{4[2,2]}^{3[-1,-2]} = 1/2,$$

those of the matrices $g_{4[2,2]}^{\pm 2}$ are

$$g_{4[2,2]}^{-2[-1,1]} = g_{4[2,2]}^{-2[1,-1]} = g_{4[2,2]}^{2[1,1]} = -\sqrt{2/7},$$
$$g_{4[2,2]}^{-2[-2,0]} = g_{4[2,2]}^{-2[0,-2]} = g_{4[2,2]}^{2[0,2]} = g_{4[2,2]}^{2[2,0]} = \sqrt{3/14},$$
$$g_{4[2,2]}^{2[-1,-1]} = \sqrt{2/7},$$

and those of the matrices $g_{4[2,2]}^{\pm 1}$ are

$$g_{4[2,2]}^{-1[-1,2]} = g_{4[2,2]}^{-1[2,-1]} = -1/(2\sqrt{7}),$$
$$g_{4[2,2]}^{-1[-2,1]} = g_{4[2,2]}^{-1[1,-2]} = g_{4[2,2]}^{1[-2,-1]} = g_{4[2,2]}^{1[-1,-2]} = g_{4[2,2]}^{1[1,2]} = g_{4[2,2]}^{1[2,1]} = 1/(2\sqrt{7}),$$
$$g_{4[2,2]}^{-1[-1,0]} = g_{4[2,2]}^{-1[0,-1]} = g_{4[2,2]}^{1[0,1]} = g_{4[2,2]}^{1[1,0]} = \sqrt{3/7}.$$

Finally, the nonzero elements of the Godunov–Gordienko matrix $g_{4[2,2]}^{0}$ are

$$g_{4[2,2]}^{0[-1,-1]} = g_{4[2,2]}^{0[1,1]} = -\sqrt{8/35},$$
$$g_{4[2,2]}^{0[-2,-2]} = g_{4[2,2]}^{0[2,2]} = \sqrt{1/70},$$
$$g_{4[2,2]}^{0[0,0]} = 6/\sqrt{70}. \qquad \square$$

Proof of (3-4). The first equation is obvious. The second and fourth equations may be checked by brute force, using the values of the matrix entries and Godunov–Gordienko coefficients calculated above. To prove the third equation, substitute (A.7) into (3-1). For the fifth equation, put $i = j = 0$ and $m = p = 2$ in (2-10). We have

$$U_{r0}^2(k_x)U_{s0}^2(k_x) = \sum_{t=0}^{4} \sum_{v=-t}^{t} \sum_{w=-t}^{t} g_{t[2,2]}^{v[r,s]} U_{vw}^t(k_x) g_{t[2,2]}^{w[0,0]}.$$

Using the matrix entries calculated above and the Godunov–Gordienko coefficients, we may rewrite this formula as

$$U_{r0}^2(k_x)U_{s0}^2(k_x) = \frac{1}{5}\delta_{rs} + \sqrt{2/7} \sum_{t=-2}^{2} g_{2[2,2]}^{t[r,s]} U_{t0}^2(k_x) + \frac{3\sqrt{2}}{\sqrt{35}} \sum_{t=-4}^{4} g_{4[2,2]}^{t[r,s]} U_{t0}^4(k_x).$$

Move all terms but the last from the right-hand side to the left-hand side, and substitute the result into the fifth line in (3-1). We obtain

$$
M_{ij\ell m}^5(\boldsymbol{x}) = \frac{\sqrt{35}}{3\sqrt{2}} \sum_{n=-2}^{2} \sum_{p=-2}^{2} g_{2[1,1]}^{n[i,j]} g_{2[1,1]}^{p[\ell,m]} \bigg(U_{r0}^2(k_{\boldsymbol{x}}) U_{s0}^2(k_{\boldsymbol{x}}) - \tfrac{1}{5}\delta_{np}
$$
$$
- \sqrt{2/7} \sum_{t=-2}^{2} g_{2[2,2]}^{t[r,s]} U_{t0}^2(k_{\boldsymbol{x}}) \bigg).
$$

Using (A.7) and (3-1), we get

$$
M_{ij\ell m}^5(\boldsymbol{x}) = \frac{\sqrt{35}}{3\sqrt{2}} \big(\tfrac{3}{2} L_{ij\ell m}^5(\boldsymbol{x}) - \tfrac{1}{2} L_{ij\ell m}^4(\boldsymbol{x}) + \tfrac{1}{6} L_{ij\ell m}^1(\boldsymbol{x}) \big)
$$
$$
- \frac{\sqrt{7}}{3\sqrt{2}} M_{ij\ell m}^2(\boldsymbol{x}) - \frac{\sqrt{5}}{3} M_{ij\ell m}^4(\boldsymbol{x}).
$$

We finish the proof by using the second and fourth equations.   □

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