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POSITIVITY, COMPLEX FIOS, AND TOEPLITZ OPERATORS

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We establish a characterization of complex linear canonical transformations that are positive with respect to a pair of strictly plurisubharmonic quadratic weights. As an application, we show that the boundedness of a class of Toeplitz operators on the Bargmann space is implied by the boundedness of their Weyl symbols.

1. Introduction and statement of results

The notion of a positive complex Lagrangian manifold, introduced in [Hörmander 1971], has long played an important role in microlocal analysis and spectral theory. Restricting the attention to the linear case, relevant for this work, let us recall that a complex Lagrangian plane $\Lambda \subset \mathbb{C}^{2n}$ is said to be positive if we have

$$\frac{1}{i}\sigma(\rho, \mathcal{C}(\rho)) \geq 0, \quad \rho \in \Lambda. \tag{1-1}$$

Here σ is the complex symplectic form on $\mathbb{C}^{2n}$ and $\mathcal{C} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is the antilinear map of complex conjugation. Let us mention here several familiar problems, where considerations of positive Lagrangian manifolds are essential. These include the spectral analysis and resolvent estimates for elliptic quadratic differential operators [Sjöstrand 1974; Hitrik et al. 2013], the study of spectral instability and pseudospectra for semiclassical nonnormal operators [Hörmander 1960; Dencker et al. 2004], as well as the construction of Gaussian beam quasimodes for semiclassical self-adjoint operators of principal type, associated with closed elliptic trajectories [Ralston 1976; Babich and Buldyrev 1991].

In [Sjöstrand 1982], one of us introduced and developed the notion of positivity of a complex Lagrangian space relative to a strictly plurisubharmonic quadratic weight, which is the starting point for the present work. To recall this notion, we let Φ_0 be a real-valued strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let us introduce the real linear subspace

$$\Lambda_{\Phi_0} = \left\{ \left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x) \right) : x \in \mathbb{C}^n \right\} \subset \mathbb{C}^{2n}. \tag{1-2}$$

We can view Λ_{Φ_0} as the image of the real phase space $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$ under a suitable complex linear canonical transformation on $\mathbb{C}^{2n}$, and in particular we notice that Λ_{Φ_0} is maximally totally real. In

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analogy with the discussion above, we say that a complex linear Lagrangian space $\Lambda \subset \mathbb{C}^{2n}$ is positive relative to Λ_{Φ_0} provided that the natural analog of (1-1) holds,

$$\frac{1}{i}\sigma(\rho, \iota_{\Phi_0}(\rho)) \geq 0, \quad \rho \in \Lambda. \tag{1-3}$$

Here the map of complex conjugation $\mathcal{C}$ has been replaced by the unique antilinear involution $\iota_{\Phi_0} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ such that $\iota_{\Phi_0}|_{\Lambda_{\Phi_0}} = 1$. A result of [Sjöstrand 1982] establishes a complete characterization of complex Lagrangians that are positive relative to Λ_{Φ_0} — see also Theorem 2.1 below.

In this work, we shall be mainly concerned with positive complex canonical transformations. Indeed, the main goal of the present work is to provide a characterization of positive complex linear canonical transformations relative to plurisubharmonic weights, and to consider Fourier integral operators (FIOs) in the complex domain associated to positive canonical transformations, establishing a link between such operators and Toeplitz operators. In particular, it seems that the point of view of complex FIOs allows us to shed some new light on some basic questions in the theory of Toeplitz operators. We would like to emphasize here that the original motivation for attempting to establish a link between FIOs in the complex domain and Toeplitz operators came from a talk delivered by Coburn at the conference "Complex and functional analysis and their interactions with harmonic analysis" at the Mathematical Research and Conference Center, Będlewo, June 2017.

We shall now proceed to define the notion of a complex linear canonical transformation which is positive relative to a strictly plurisubharmonic quadratic weight, and to state our main results. In fact, proceeding in the spirit of the discussion above, it will be more transparent to introduce the notion of positivity relative to a pair of strictly plurisubharmonic quadratic forms rather than relative to a single one. Thus, let Φ_1, Φ_2 be two strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$ with the corresponding antilinear involutions ι_{Φ_1}, ι_{Φ_2}. Let $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a complex linear canonical transformation, $\kappa^*\sigma = \sigma$. We say that κ is positive relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$ provided that

$$\frac{1}{i}\big(\sigma(\kappa(\rho), \iota_{\Phi_1}\kappa(\rho)) - \sigma(\rho, \iota_{\Phi_2}(\rho))\big) \geq 0, \quad \rho \in \mathbb{C}^{2n}. \tag{1-4}$$

The positivity of κ relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$ is said to be strict provided that the inequality in (1-4) is strict for all $0 \neq \rho \in \mathbb{C}^{2n}$. Let us remark that in the case when the positivity is taken relative to the real phase space $\mathbb{R}^{2n}$, see (1-1), such canonical transformations were studied in [Hörmander 1983, 1995]; see also the recent works [Pravda-Starov et al. 2018; Aleman and Viola 2018].

We can now state the first main result of this work.

Theorem 1.1. *Let $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a complex linear canonical transformation and let Φ_1, Φ_2 be strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$. The canonical transformation κ is positive relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$ precisely when we have*

$$\kappa(\Lambda_{\Phi_2}) = \Lambda_{\Phi}, \tag{1-5}$$

where Φ is a strictly plurisubharmonic quadratic form such that $\Phi \leq \Phi_1$.

Remark. The definition (1-4) of a positive canonical transformation is a direct adaptation of the corresponding notion of positivity due to Hörmander [1983; 1995] to the weighted setting. One advantage of the consideration of the general case of a pair of weights Φ_1, Φ_2 is that we can let κ be the identity in (1-4) and get an invariant notion of the positivity of one plurisubharmonic weight compared to another, in view of Theorem 1.1.

Our second main result is concerned with applications of Theorem 1.1 to the study of Toeplitz operators in the Bargmann space

$$H_{\Phi_0}(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)) \cap \mathrm{Hol}(\mathbb{C}^n),$$

where Φ_0 is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and $L(dx)$ is the Lebesgue measure on $\mathbb{C}^n$. See also (A-1). Specifically, we shall be concerned with the continuity properties of (in general unbounded) Toeplitz operators of the form

$$\mathrm{Top}(e^{2q}) = \Pi_{\Phi_0} \circ e^{2q} \circ \Pi_{\Phi_0} : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n), \tag{1-6}$$

where q is a complex-valued quadratic form on $\mathbb{C}^n$ and

$$\Pi_{\Phi_0} : L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)) \to H_{\Phi_0}(\mathbb{C}^n)$$

is the orthogonal projection. Sufficient conditions for the boundedness of $\mathrm{Top}(e^{2q})$ are provided in the following result.

Theorem 1.2. *Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let q be a quadratic form on $\mathbb{C}^n$ such that*

$$2\,\mathrm{Re}\, q(x) < \Phi_{\mathrm{herm}}(x) := \tfrac{1}{2}(\Phi_0(x) + \Phi_0(ix)), \quad x \neq 0, \tag{1-7}$$

$$\partial_x \partial_{\bar{x}}(\Phi_0 - q) \neq 0. \tag{1-8}$$

Let $a \in C^\infty(\Lambda_{\Phi_0})$ be the Weyl symbol of the Toeplitz operator $\mathrm{Top}(e^{2q})$*. Assume that $a \in L^\infty(\Lambda_{\Phi_0})$. Then the Toeplitz operator*

$$\mathrm{Top}(e^{2q}) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

is bounded.

Remark. Let us remark that Theorem 1.2 is closely related to the conjecture of [Berger and Coburn 1994; Coburn 2019] stating that a Toeplitz operator is bounded on $H_{\Phi_0}(\mathbb{C}^n)$ precisely when its Weyl symbol is bounded on Λ_{Φ_0}. Theorem 1.2 can therefore be regarded as establishing the sufficiency part of the conjecture in the special case when the Toeplitz symbol is of the form $\exp(2q)$, where q is a complex-valued quadratic form on $\mathbb{C}^n$, satisfying (1-7), (1-8).

Remark. As we shall see in Section 4, the strict inequality in condition (1-7) guarantees that the operator $\mathrm{Top}(e^{2q})$ is densely defined, and it seems difficult to weaken. Notice also that the Hermitian form Φ_{herm} in (1-7) is positive definite on $\mathbb{C}^n$, thanks to the strict plurisubharmonicity of Φ_0.

The plan of the paper is as follows. In Section 2, we establish the necessity part of Theorem 1.1, by means of direct geometric arguments, relying on some general results of [Sjöstrand 1982]; see also

[Caliceti et al. 2012; Hitrik and Sjöstrand 2018]. The proof of Theorem 1.1 is completed in Section 3, where we have found it convenient to introduce explicitly a Fourier integral operator in the complex domain quantizing the canonical transformation κ satisfying (1-5), when verifying the positivity of κ. Applications to Toeplitz operators are given in Section 4, where Theorem 1.2 is established. Appendix A is devoted to some elementary remarks concerning integral representations for linear continuous maps between weighted spaces of holomorphic functions, which can be regarded as a version of the Schwartz kernel theorem in this setting. These representations are to be applied in the main text when deriving a Bergman-type representation for our complex FIOs. Finally, Appendix B, for use in Section 4, characterizes boundedness properties of operators given as Weyl quantizations of symbols of the form $e^{iF(x,\xi)}$, where F is a holomorphic quadratic form on $\mathbb{C}^{2n}$.

2. Positive Lagrangian planes and positive canonical transformations in the H_Φ-setting

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. Associated to Φ_0 is the I-Lagrangian R-symplectic linear manifold Λ_{Φ_0}, given by

$$\Lambda_{\Phi_0} = \left\{ \left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x) \right) : x \in \mathbb{C}^n \right\} \subset \mathbb{C}^{2n}. \tag{2-1}$$

The linear manifold Λ_{Φ_0} is maximally totally real, and we let ι_{Φ_0} be the unique antilinear involution

$$\iota_{\Phi_0} : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \tag{2-2}$$

such that the restriction of ι_{Φ_0} to Λ_{Φ_0} is the identity. For future reference, we may recall the explicit description of the involution ι_{Φ_0} given in [Hitrik and Sjöstrand 2018],

$$\left(y, \frac{2}{i}(\Phi''_{0,xx} y + \Phi''_{0,x\bar{x}} \bar{x}) \right) \mapsto \left(x, \frac{2}{i}(\Phi''_{0,xx} x + \Phi''_{0,x\bar{x}} \bar{y}) \right). \tag{2-3}$$

We also have

$$\iota_{\Phi_0} : \left(y, \frac{2}{i}\overline{\partial_y \Psi_0(x, \bar{y})} \right) \mapsto \left(x, \frac{2}{i}\partial_x \Psi_0(x, \bar{y}) \right), \tag{2-4}$$

where $\Psi_0(x, y)$ is the polarization of Φ_0, i.e., the unique holomorphic quadratic form on $\mathbb{C}^n_x \times \mathbb{C}^n_y$ such that $\Psi_0(x, \bar{x}) = \Phi_0(x)$.

Let $\Lambda \subset \mathbb{C}^{2n}$ be a $\mathbb{C}$-Lagrangian space, i.e., a complex linear subspace such that $\dim_{\mathbb{C}} \Lambda = n$ and $\sigma|_\Lambda = 0$. Here σ is the standard symplectic form on $\mathbb{C}^{2n}$. Let us consider the Hermitian form

$$b(\nu, \mu) = \frac{1}{i}\sigma(\nu, \iota_{\Phi_0}(\mu)), \quad \nu, \mu \in \mathbb{C}^{2n}. \tag{2-5}$$

We say that Λ is positive relative to Λ_{Φ_0} if the Hermitian form (2-5) is positive semidefinite when restricted to Λ,

$$b(\mu, \mu) \geq 0, \quad \mu \in \Lambda. \tag{2-6}$$

The positivity is said to be strict if the form b in (2-5) is positive definite along Λ. As remarked in the introduction, this notion is a direct adaptation of the corresponding notion of positivity due to Hörmander

[1971], where in place of $(\Lambda_{\Phi_0}, \iota_{\Phi_0})$ we have $(\mathbb{R}^{2n}, \mathcal{C})$, with $\mathcal{C}$ being the antilinear map of complex conjugation.

Remark. It is easy to see and is established in [Caliceti et al. 2012; Hitrik and Sjöstrand 2018] that the Hermitian form b is nondegenerate along Λ precisely when Λ and Λ_{Φ_0} are transversal.

Our starting point is the following well-known result; see [Sjöstrand 1982; Caliceti et al. 2012; Hitrik and Sjöstrand 2018].

Theorem 2.1. *A $\mathbb{C}$-Lagrangian space Λ is positive relative to Λ_{Φ_0} if and only if $\Lambda = \Lambda_{\Psi}$, where Ψ is a pluriharmonic quadratic form such that $\Psi \leq \Phi_0$.*

The proof of Theorem 2.1 given in [Sjöstrand 1982; Caliceti et al. 2012; Hitrik and Sjöstrand 2018] discusses the case of strictly positive Lagrangian planes only and depends on the general fact that the set of all $\mathbb{C}$-Lagrangian spaces which are strictly positive relative to Λ_{Φ_0} is a connected component in the set of all $\mathbb{C}$-Lagrangian spaces that are transversal to Λ_{Φ_0}. Here we shall give a more direct proof, using the explicit description of the involution ι_{Φ_0}, given in (2-3), (2-4). Let $\Lambda \subset \mathbb{C}^{2n}$ be $\mathbb{C}$-Lagrangian, positive relative to Λ_{Φ_0}. It follows from (2-3), as explained in [Sjöstrand 1982; Hitrik and Sjöstrand 2018], that the fiber $\{(0, \xi); \xi \in \mathbb{C}^n\}$ is strictly negative relative to Λ_{Φ_0}, in the sense that the Hermitian form b in (2-5) is negative definite along the fiber, and therefore Λ is necessarily of the form $\xi = \partial_x \varphi(x)$, where φ is a holomorphic quadratic form on $\mathbb{C}^n$. It follows that

$$\Lambda = \Lambda_{\Psi}, \tag{2-7}$$

where $\Psi = -\operatorname{Im} \varphi$ is pluriharmonic quadratic. We shall now see that $\Psi \leq \Phi_0$, and to this end, let us consider the decomposition

$$\Phi_0 = \Phi_{\text{herm}} + \Phi_{\text{plh}}, \tag{2-8}$$

where

$$\Phi_{\text{herm}}(x) = \Phi''_{0,\bar{x}x} x \cdot \bar{x} \tag{2-9}$$

is positive definite Hermitian and

$$\Phi_{\text{plh}}(x) = \operatorname{Re}(\Phi''_{0,xx} x \cdot x) \tag{2-10}$$

is pluriharmonic. Let

$$A = \frac{2}{i}(\Phi_{\text{plh}})''_{xx} = \frac{2}{i}(\Phi_0)''_{xx},$$

and let us consider the complex linear "vertical" canonical transformation

$$\kappa_A(y, \eta) = (y, \eta + Ay). \tag{2-11}$$

We have

$$\kappa_A(\Lambda_{\Phi_{\text{herm}}}) = \Lambda_{\Phi_0}, \tag{2-12}$$

and letting $\iota_{\Phi_{\text{herm}}}$ be the antilinear involution associated to $\Lambda_{\Phi_{\text{herm}}}$, it is then clear that

$$\iota_{\Phi_{\text{herm}}} = \kappa_A^{-1} \circ \iota_{\Phi_0} \circ \kappa_A. \tag{2-13}$$

It follows that Λ is positive relative to Λ_{Φ_0} precisely when

$$\kappa_A^{-1}(\Lambda) = \Lambda_{\Psi - \Phi_{\mathrm{plh}}}$$

is positive relative to $\Lambda_{\Phi_{\mathrm{herm}}}$, and when proving Theorem 2.1 we may assume therefore that the pluriharmonic part of Φ_0 vanishes. In this discussion, we are also allowed to perform complex linear changes of variables in $\mathbb{C}^n$, which correspond to canonical transformations of the form $\kappa_C : (y, \eta) \mapsto (C^{-1}y, C^t\eta)$, where C is an invertible complex $n \times n$ matrix. We have $\kappa_C(\Lambda_{\Phi_0}) = \Lambda_{\Phi_1}$, $\Phi_1(x) = \Phi_0(Cx)$, and it follows therefore that when establishing Theorem 2.1 it suffices to consider the model case when

$$\Phi_0(x) = \tfrac{1}{2}|x|^2. \tag{2-14}$$

An application of (2-3) shows that the involution ι_{Φ_0} is then given by

$$(y, \eta) \mapsto \left(\frac{1}{i}\bar{\eta}, \frac{1}{i}\bar{y}\right), \tag{2-15}$$

and therefore

$$b(\mu, \mu) = \frac{1}{i}\sigma(\mu, \iota_{\Phi_0}(\mu)) = |x|^2 - |\xi|^2, \quad \mu = (x, \xi) \in \mathbb{C}^{2n}. \tag{2-16}$$

When $\mu \in \Lambda = \Lambda_\Psi$, we write $\xi = (2/i)\partial_x\Psi(x) = \partial_x\varphi(x)$, $\Psi(x) = -\operatorname{Im}\varphi$, where φ is a quadratic holomorphic form, and therefore if Λ is positive relative to Λ_{Φ_0}, then (2-16) shows that

$$|\varphi''_{xx}x| \le |x|, \qquad x \in \mathbb{C}^n \iff \|\varphi''_{xx}\| \le 1. \tag{2-17}$$

We get

$$\Psi(x) = -\operatorname{Im}\varphi(x) \le \tfrac{1}{2}|\varphi''_{xx}x \cdot x| \le \tfrac{1}{2}|x|^2 = \Phi_0(x), \quad x \in \mathbb{C}^n. \tag{2-18}$$

Conversely, let Λ be $\mathbb{C}$-Lagrangian of the form $\Lambda = \Lambda_\Psi$, where Ψ is pluriharmonic quadratic such that $\Psi \le \Phi_0$. Let us write $\Psi = -\operatorname{Im}\varphi$, where φ is a holomorphic quadratic form. We shall now see that Λ_Ψ is positive relative to Λ_{Φ_0}, and it follows from the remarks above that it suffices to verify the positivity in the model case when Φ_0 is given by (2-14), so that we have

$$\Psi(x) = -\operatorname{Im}\varphi(x) \le \Phi_0(x) = \tfrac{1}{2}|x|^2. \tag{2-19}$$

Writing

$$-\operatorname{Im}\varphi''_{xx}x \cdot x \le |x|^2, \tag{2-20}$$

replacing x by $e^{i\theta}x$ and varying $\theta \in \mathbb{R}$, we get

$$|\varphi''_{xx}x \cdot x| \le |x|^2, \quad x \in \mathbb{C}^n. \tag{2-21}$$

Next, writing

$$\varphi''_{xx}x \cdot y = \tfrac{1}{4}\big(\varphi''_{xx}(x+y)\cdot(x+y) - \varphi''_{xx}(x-y)\cdot(x-y)\big),$$

we get, using (2-21),

$$|\varphi''_{xx}x \cdot y| \le \tfrac{1}{4}(|x+y|^2 + |x-y|^2) = \tfrac{1}{2}(|x|^2 + |y|^2). \tag{2-22}$$

Replacing $x \mapsto \lambda^{1/2}x,\ y \mapsto \lambda^{-1/2}y,\ \lambda > 0$, we get

$$|\varphi''_{xx}x \cdot y| \le \frac{1}{2}\Big(\lambda |x|^2 + \frac{1}{\lambda}|y|^2\Big), \tag{2-23}$$

and choosing $\lambda = |y|/|x|$, assuming for simplicity that $x \neq 0,\ y \neq 0$, we obtain

$$|\varphi''_{xx}x \cdot y| \le |x|\,|y|.$$

Hence, $\|\varphi''_{xx}\| \le 1$ and the positivity of Λ_Ψ relative to Λ_{Φ_0} follows from (2-16), (2-17). The proof of Theorem 2.1 is complete.

Remark. Closely related to the proof of Theorem 2.1 given above is the normal form for strictly plurisubharmonic quadratic forms, given in Lemma 5.1 of [Hörmander 1997]; see also [Harvey and Wells 1973].

Let Φ_1, Φ_2 be two strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$ and let $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a complex linear canonical transformation which is positive relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$, in the sense of (1-4). In the remainder of this section, we shall establish the necessity part of Theorem 1.1, while the sufficiency is discussed in Section 3. To this end, let us observe first that the linear I-Lagrangian R-symplectic manifold $\kappa(\Lambda_{\Phi_2})$ is transversal to the fiber $\{(0, \xi) : \xi \in \mathbb{C}^n\}$. Indeed, we have in view of (1-4),

$$\frac{1}{i}\sigma(\rho, \iota_{\Phi_1}(\rho)) \ge 0, \qquad \rho \in \kappa(\Lambda_{\Phi_2}), \tag{2-24}$$

while, as recalled above, we know from [Sjöstrand 1982; Hitrik and Sjöstrand 2018] that the fiber is strictly negative relative to Λ_{Φ_1}. It follows that $\kappa(\Lambda_{\Phi_2}) = \Lambda_\Phi$, where Φ is a real quadratic form such that the Levi form $\bar\partial\partial\Phi$ is nondegenerate. When verifying that Φ is (necessarily strictly) plurisubharmonic, we claim that it suffices to do so when the pluriharmonic part of Φ_2 vanishes. Indeed, introducing the decomposition (2-8), with the quadratic form Φ_2 in place of Φ_0 and considering the canonical transformation κ_A given in (2-11), we see, using also (2-13), that κ is positive relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$ precisely when $\kappa_A^{-1} \circ \kappa \circ \kappa_A$ is positive relative to $(\Lambda_{\Phi_1 - \Phi_{2,\mathrm{plh}}}, \Lambda_{\Phi_{2,\mathrm{herm}}})$. Here $\Phi_{2,\mathrm{plh}}$ and $\Phi_{2,\mathrm{herm}}$ are the pluriharmonic and the Hermitian parts of Φ_2, respectively. Here it is also helpful to notice that

$$\iota_{\Phi_1 - \Phi_{2,\mathrm{plh}}} = \kappa_A^{-1} \circ \iota_{\Phi_1} \circ \kappa_A.$$

To summarize, if we know that the generating function of the linear I-Lagrangian R-symplectic manifold

$$\kappa_A^{-1} \circ \kappa \circ \kappa_A(\Lambda_{\Phi_{2,\mathrm{herm}}})$$

is plurisubharmonic, then the same property is also enjoyed by the generating function of $\kappa(\Lambda_{\Phi_2})$. In what follows we shall assume therefore that

$$\Phi_{2,xx} = \Phi_{2,\bar{x}\bar{x}} = 0. \tag{2-25}$$

As above, in this discussion, we are also allowed to perform complex linear changes of variables in $\mathbb{C}^n$, which correspond to canonical transformations of the form $(y, \eta) \mapsto (C^{-1}y, C^t\eta)$, where C is an invertible complex $n \times n$ matrix. Such canonical transformations preserve the plurisubharmonicity of the generating functions, and similarly to the proof of Theorem 2.1, it suffices therefore to consider the case when

$$\Phi_2(x) = \tfrac{1}{2}|x|^2. \tag{2-26}$$

Theorem 2.1 then shows that the $\mathbb{C}$-Lagrangian plane given by $\{(x,\xi) \in \mathbb{C}^{2n} : \xi = 0\}$ is strictly positive relative to Λ_{Φ_2}, and therefore $\kappa(\{(x,\xi) \in \mathbb{C}^{2n} : \xi = 0\})$ is strictly positive relative to Λ_{Φ_1}, in view of the positivity of κ. Another application of Theorem 2.1 gives that

$$\kappa(\{(x,\xi) \in \mathbb{C}^{2n} : \xi = 0\}) = \Lambda_{\Psi}, \tag{2-27}$$

where the quadratic form Ψ is pluriharmonic, with $\Psi \le \Phi_1$.

Let $\phi(x, y, \theta)$ be a holomorphic quadratic form on $\mathbb{C}^n_x \times \mathbb{C}^n_y \times \mathbb{C}^N_\theta$, which is a nondegenerate phase function in the sense of Hörmander, generating the graph of κ. It follows from (2-27), as explained in [Caliceti et al. 2012], that the quadratic form

$$\mathbb{C}^n \times \mathbb{C}^N \ni (y,\theta) \mapsto -\operatorname{Im}\phi(0, y, \theta) \tag{2-28}$$

is nondegenerate, and since it is pluriharmonic, the signature is necessarily $(n+N, n+N)$. Recalling that

$$\kappa(\Lambda_{\Phi_2}) = \Lambda_{\Phi}, \tag{2-29}$$

we see, using [Caliceti et al. 2012], that the quadratic form

$$(y,\theta) \mapsto -\operatorname{Im}\phi(0, y, \theta) + \Phi_2(y) \tag{2-30}$$

is nondegenerate as well. We would like to conclude that the signature of the quadratic form in (2-30) is also $(n+N, n+N)$, and to that end, we follow [Sjöstrand 1982] and consider the continuous deformation

$$[0,1] \ni t \mapsto -\operatorname{Im}\phi(0, y, \theta) + t\Phi_2(y). \tag{2-31}$$

Using (2-16) we see that

$$\frac{1}{i}\sigma(\mu, \iota_{\Phi_2}(\mu)) \ge 0, \quad \mu \in \Lambda_{t\Phi_2}, \quad 0 \le t \le 1. \tag{2-32}$$

It follows as before that the I-Lagrangian manifold $\kappa(\Lambda_{t\Phi_2})$ is transversal to the fiber, $0 \le t \le 1$, and therefore we conclude that the nondegeneracy of the quadratic forms in (2-31) is maintained along the deformation $0 \le t \le 1$. Recalling that the set of nondegenerate quadratic forms of a fixed given signature is a connected component in the set of all nondegenerate quadratic forms, we conclude that the signature of the quadratic form in (2-30) is $(n+N, n+N)$. Now, as explained in [Caliceti et al. 2012], the quadratic form Φ in (2-29) is given by

$$\Phi(x) = \operatorname{vc}_{y,\theta}(-\operatorname{Im}\phi(x, y, \theta) + \Phi_2(y)), \tag{2-33}$$

where $\operatorname{vc}_{y,\theta}$ stands for the critical value with respect to y, θ, and we conclude by the fundamental lemma of [Sjöstrand 1982], see also [Hitrik and Sjöstrand 2018], that Φ is plurisubharmonic. (As already observed, the plurisubharmonicity of Φ is necessarily strict.)

We shall next see that $\Phi \leq \Phi_1$, and when doing so it will be convenient to discuss the following auxiliary result first, which may be of some independent interest.

Proposition 2.2. *Let $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a complex linear canonical transformation which is positive relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$. If Φ_2 is strictly convex then κ has a generating function $\varphi(x,\eta)$ which is a holomorphic quadratic form such that*

$$\kappa : (\varphi'_\eta(x,\eta), \eta) \mapsto (x, \varphi'_x(x,\eta)). \tag{2-34}$$

Proof. It suffices to show that the map

$$\pi : \operatorname{graph}(\kappa) \ni (x,\xi; y,\eta) \mapsto (x,\eta) \in \mathbb{C}^{2n}$$

is bijective, i.e., injective. Let $(0,\xi; y, 0) \in \operatorname{Ker}(\pi)$ so that $\kappa : (y,0) \mapsto (0,\xi)$. Let us consider the Hermitian forms

$$b_j(\nu,\mu) = \frac{1}{i}\sigma(\nu, \iota_{\Phi_j}(\mu)), \quad j = 1,2.$$

The strict convexity of Φ_2 together with Theorem 2.1 implies

$$b_2((y,0),(y,0)) \asymp |y|^2, \quad y \in \mathbb{C}^n, \tag{2-35}$$

and the strict negativity of the fiber with respect to Λ_{Φ_1} gives

$$b_1((0,\xi),(0,\xi)) \asymp -|\xi|^2, \quad \xi \in \mathbb{C}^n.$$

Hence by the positivity of κ, we get

$$0 \leq b_1((0,\xi),(0,\xi)) - b_2((y,0),(y,0)) \asymp -(|\xi|^2 + |y|^2).$$

It follows that $(y,\xi) = 0$ and we conclude that π is injective. □

Remark. Suppose that the assumptions of Proposition 2.2 hold. The holomorphic quadratic form $\varphi(x,\theta) - y\cdot\theta$ is then a nondegenerate phase function generating the graph of κ.

Let us now turn to the proof of the fact that

$$\Phi \leq \Phi_1. \tag{2-36}$$

It follows from the remarks above that it suffices to verify (2-36) when the pluriharmonic part of Φ_2 vanishes, and since we are again allowed to perform complex linear changes of variables in $\mathbb{C}^n$, as before, we conclude that it suffices to consider the case when Φ_2 is given by (2-26). Proposition 2.2 applies and there exists therefore a holomorphic quadratic form $\varphi(x,\theta)$ such that

$$\kappa : (\varphi'_\theta(x,\theta), \theta) \mapsto (x, \varphi'_x(x,\theta)). \tag{2-37}$$

We shall now express the positivity of κ relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$ in terms of the generating function φ. To this end, we shall first obtain an explicit expression for the Hermitian form

$$\frac{1}{i}\sigma((y,\eta), \iota_{\Phi_1}(y,\eta)), \quad (y,\eta) \in \mathbb{C}^{2n},$$

where we write

$$\Phi_1(x) = \tfrac{1}{2} L\bar{x}\cdot x + \mathrm{Re}(Ax\cdot x), \quad L = 2\Phi''_{1,x\bar{x}}, \quad A = \Phi''_{1,xx}. \tag{2-38}$$

Here L is Hermitian positive definite and performing a unitary transformation, we may assume, for simplicity, that L is diagonal, with real positive diagonal elements. A simple computation using (2-3) shows that

$$\frac{1}{i}\sigma((y,\eta), \iota_{\Phi_1}(y,\eta)) = L\bar{y}\cdot y + (2Ay - i\eta)\cdot x, \tag{2-39}$$

where

$$L\bar{x} = i\eta - 2Ay,$$

and therefore we get

$$\frac{1}{i}\sigma((y,\eta), \iota_{\Phi_1}(y,\eta)) = L\bar{y}\cdot y - L^{-1}(2iAy+\eta)\cdot \overline{(2iAy+\eta)}. \tag{2-40}$$

Using also (2-37), we conclude that κ is positive relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$ precisely when

$$L^{-1}(\varphi'_x + 2iAx)\cdot\overline{(\varphi'_x + 2iAx)} + |\varphi'_\theta(x,\theta)|^2 \le L\bar{x}\cdot x + |\theta|^2, \quad (x,\theta)\in\mathbb{C}^{2n}. \tag{2-41}$$

It is now easy to conclude the proof of the necessity part of Theorem 1.1, using (2-41). It follows from (2-33) that we can write

$$\Phi(x) = \mathrm{vc}_{y,\theta}\big(-\mathrm{Im}(\varphi(x,\theta) - y\cdot\theta) + \Phi_2(y)\big). \tag{2-42}$$

At the unique critical point $(y(x),\theta(x))$, we have

$$y = \varphi'_\theta(x,\theta), \tag{2-43}$$

$$\frac{2}{i}\frac{\partial\Phi_2}{\partial y}(y) = \theta \iff \theta = \frac{1}{i}\bar{y}. \tag{2-44}$$

Injecting (2-44) into (2-42), we get

$$\Phi(x) = -\mathrm{Im}\,\varphi(x,\theta) - \tfrac{1}{2}|\theta|^2, \quad \theta = \theta(x), \tag{2-45}$$

and in view of (2-38), it suffices therefore to establish the inequality

$$-2\,\mathrm{Im}\,\varphi(x,\theta) \le L\bar{x}\cdot x + |\theta|^2 + 2\,\mathrm{Re}(Ax\cdot x), \quad (x,\theta)\in\mathbb{C}^{2n}. \tag{2-46}$$

When verifying (2-46), we write, using the Euler homogeneity relation,

$$2\varphi(x,\theta) = \varphi'_x(x,\theta)\cdot x + \varphi'_\theta(x,\theta)\cdot\theta, \tag{2-47}$$

and therefore,

$$-2\,\mathrm{Im}\,\varphi(x,\theta) = -\mathrm{Im}\big((\varphi'_x(x,\theta) + 2iAx)\cdot x + \varphi'_\theta(x,\theta)\cdot\theta\big) + 2\,\mathrm{Re}(Ax\cdot x). \tag{2-48}$$

An application of the Cauchy–Schwarz inequality with respect to the positive definite Hermitian forms $(x,y)\mapsto L^{-1}x\cdot\bar{y}$, $(x,y)\mapsto x\cdot\bar{y}$ together with the inequality $ab\le\frac{1}{2}a^2+\frac{1}{2}b^2$ allows us to conclude that the first term in the right-hand side of (2-48) does not exceed

$$\tfrac{1}{2}\big(L^{-1}(\varphi_x' + 2iAx)\cdot\overline{(\varphi_x' + 2iAx)} + L\bar{x}\cdot x + |\varphi_\theta'(x,\theta)|^2 + |\theta|^2\big).$$

The inequality (2-46) follows, in view of (2-41). The proof of the necessity part of Theorem 1.1 is complete.

Remark. In the context of Theorem 1.1, assume that $\Phi_1 = \Phi_2 =: \Phi_0$ and let us write

$$\Phi_0(x) = \sup_{y\in\mathbb{R}^n} (-\operatorname{Im}\varphi(x,y)), \tag{2-49}$$

where $\varphi(x,y)$ is a holomorphic quadratic form on $\mathbb{C}^n_x \times \mathbb{C}^n_y$ such that $\det\varphi''_{xy} \neq 0$ and $\operatorname{Im}\varphi''_{yy} > 0$. In the special case when Φ_0 is given by (2-26), we can take

$$\varphi(x,y) = i\big(\tfrac{1}{2}x^2 + \sqrt{2}x\cdot y + \tfrac{1}{2}y^2\big).$$

The complex canonical transformation

$$\kappa_\varphi : \mathbb{C}^{2n} \ni (y, -\varphi_y'(x,y)) \mapsto (x, \varphi_x'(x,y)) \in \mathbb{C}^{2n} \tag{2-50}$$

maps $\mathbb{R}^{2n}$ bijectively onto Λ_{Φ_0}, see [Hitrik and Sjöstrand 2018], and it exchanges the complex conjugation map $\mathcal{C}$ and the involution ι_{Φ_0}. Setting

$$\tilde{\kappa} = \kappa_\varphi^{-1} \circ \kappa \circ \kappa_\varphi, \tag{2-51}$$

we see that the complex linear canonical transformation $\tilde{\kappa}$ is positive in the sense of [Hörmander 1995],

$$\frac{1}{i}\big(\sigma(\tilde{\kappa}(\rho), \mathcal{C}\tilde{\kappa}(\rho)) - \sigma(\rho, \mathcal{C}(\rho))\big) \geq 0, \quad \rho \in \mathbb{C}^{2n}. \tag{2-52}$$

An application of Proposition 5.10 of [Hörmander 1995] allows us to conclude therefore that the map $\tilde{\kappa}$ enjoys the factorization

$$\tilde{\kappa} = \tilde{\kappa}_1 \circ \tilde{\kappa}_2 \circ \tilde{\kappa}_3, \tag{2-53}$$

where $\tilde{\kappa}_1$ and $\tilde{\kappa}_3$ are real linear canonical maps and the map $\tilde{\kappa}_2$ is of the form

$$\tilde{\kappa}_2 = \exp(-iH_{\tilde{q}}), \tag{2-54}$$

where $\tilde{q}$ is a quadratic form with $\operatorname{Re}\tilde{q} \geq 0$ on $\mathbb{R}^{2n}$ — see also the discussion in the proof of Proposition 5.12 of [Hörmander 1995]. We obtain the factorization

$$\kappa = \kappa_1 \circ \kappa_2 \circ \kappa_3, \tag{2-55}$$

where we have

$$\kappa_j : \Lambda_{\Phi_0} \to \Lambda_{\Phi_0}, \quad j = 1, 3, \tag{2-56}$$

and

$$\kappa_2 = \exp(-iH_q), \tag{2-57}$$

where q is a holomorphic quadratic form on $\mathbb{C}^{2n}$ such that $\operatorname{Re} q \geq 0$ along Λ_{Φ_0}. The representation (2-55) can be used to give an alternative proof of the basic inequality $\Phi \leq \Phi_0$ in Theorem 1.1, in this special case.

3. Positivity and Fourier integral operators

The purpose of this section is to establish the sufficiency part of Theorem 1.1. To this end, let Φ_1, Φ_2 be two strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$ and let $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a complex linear canonical transformation. Assume that

$$\kappa(\Lambda_{\Phi_2}) = \Lambda_{\Phi}, \tag{3-1}$$

where Φ is a strictly plurisubharmonic quadratic form such that

$$\Phi \le \Phi_1. \tag{3-2}$$

We shall establish the positivity of κ relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$ by making a judicious choice of a nondegenerate phase function generating the graph of κ, and to this end, it will be convenient to consider a metaplectic Fourier integral operator associated to κ. Let therefore $\varphi(x, y, \theta)$ be a holomorphic quadratic form on $\mathbb{C}^n_x \times \mathbb{C}^n_y \times \mathbb{C}^N_\theta$, which is a nondegenerate phase function in the sense of Hörmander, generating the graph of κ. It follows from [Caliceti et al. 2012] that the plurisubharmonic quadratic form

$$\mathbb{C}^n \times \mathbb{C}^N \ni (y, \theta) \mapsto -\operatorname{Im} \varphi(0, y, \theta) + \Phi_2(y) \tag{3-3}$$

is nondegenerate of signature $(n + N, n + N)$. We conclude, following [Sjöstrand 1982; Caliceti et al. 2012] that the Fourier integral operator

$$Au(x) = \iint e^{i\varphi(x,y,\theta)} a u(y)\, dy\, d\theta, \quad a \in \mathbb{C}, \tag{3-4}$$

quantizing κ, can be realized by means of a good contour and we obtain a bounded linear map,

$$A : H_{\Phi_2}(\mathbb{C}^n) \to H_{\Phi}(\mathbb{C}^n). \tag{3-5}$$

Here

$$H_{\Phi_2}(\mathbb{C}^n) = \operatorname{Hol}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, e^{-2\Phi_2} L(dx)),$$

with $H_\Phi(\mathbb{C}^n)$ having an analogous definition.

We shall now discuss a Bergman-type representation of the bounded operator in (3-5); see also [Melin and Sjöstrand 2003] for a related discussion. To this end, let us recall from Theorem A.1 that we can write

$$Au(x) = \int K_A(x, \bar{y}) u(y)\, e^{-2\Phi_2(y)}\, L(dy) =: \tilde{A}u(x). \tag{3-6}$$

Here the kernel $K_A(x, z)$ is holomorphic on $\mathbb{C}^n_x \times \mathbb{C}^n_z$, with

$$y \mapsto \overline{K(x, \bar{y})} \in H_{\Phi_2}(\mathbb{C}^n),$$

uniquely determined by (3-6). If $u \in L^2_{\Phi_2}(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\Phi_2} L(dx))$ is orthogonal to $H_{\Phi_2}(\mathbb{C}^n)$, we see from (3-6) that $\tilde{A}u = 0$. Hence the operator $\tilde{A}$ in (3-6) is a well-defined linear continuous map

$$\tilde{A} : L^2_{\Phi_2}(\mathbb{C}^n) \to H_{\Phi_2}(\mathbb{C}^n).$$

Furthermore, $\tilde{A}$ extends to a map: $\mathcal{E}'(\mathbb{C}^n) \to \mathrm{Hol}(\mathbb{C}^n)$ and we have

$$K_A(x, \bar{y})e^{-2\Phi_2(y)} = (\tilde{A}\delta_y)(x), \tag{3-7}$$

where $\delta_y \in \mathcal{E}'(\mathbb{C}^n)$ is the delta function at y. Let next $\Pi_2 : L^2_{\Phi_2}(\mathbb{C}^n) \to H_{\Phi_2}(\mathbb{C})$ be the orthogonal projection and let us recall from [Hitrik and Sjöstrand 2018] that the operator Π_2 is given by

$$\Pi_2 u(x) = a_2 \int e^{2\Psi_2(x,\bar{y})-\Phi_2(y)} u(y)\, L(dy), \quad a_2 > 0. \tag{3-8}$$

Here Ψ_2 is the polarization of Φ_2, i.e., a holomorphic quadratic form on $\mathbb{C}^{2n}_{x,y}$ such that $\Psi_2(x, \bar{x}) = \Phi_2(x)$. We get $\tilde{A}\delta_y = \tilde{A}\Pi_2\delta_y = A\Pi_2\delta_y$, and it follows from (3-7) that

$$K_A(x, \bar{y}) = A(a_2 e^{2\Psi_2(\cdot,\bar{y})})(x). \tag{3-9}$$

From [Hitrik and Sjöstrand 2018], let us recall the basic property

$$2\,\mathrm{Re}\,\Psi_2(x, \bar{y}) - \Phi_2(x) - \Phi_2(y) \sim -|x - y|^2$$

on $\mathbb{C}^n_x \times \mathbb{C}^n_y$, and in particular we have

$$2\,\mathrm{Re}\,\Psi_2(x, \bar{y}) \le \Phi_2(x) + \Phi_2(y). \tag{3-10}$$

It follows that

$$-\mathrm{Im}\,\varphi(0, \tilde{y}, \theta) + 2\,\mathrm{Re}\,\Psi_2(\tilde{y}, 0) \le -\mathrm{Im}\,\varphi(0, \tilde{y}, \theta) + \Phi_2(\tilde{y}). \tag{3-11}$$

Here, as observed in (3-3), the right-hand side is a nondegenerate plurisubharmonic quadratic form of signature $(n + N, n + N)$, and since the left-hand side is pluriharmonic, we conclude that it is also nondegenerate of signature $(n + N, n + N)$. Writing

$$-\mathrm{Im}\,\varphi(0, \tilde{y}, \theta) + 2\,\mathrm{Re}\,\Psi_2(\tilde{y}, 0) = \mathrm{Re}(i\varphi(0, \tilde{y}, \theta) + 2\Psi_2(\tilde{y}, 0)),$$

we conclude that the holomorphic quadratic form

$$\mathbb{C}^n \times \mathbb{C}^N \ni (\tilde{y}, \theta) \mapsto i\varphi(0, \tilde{y}, \theta) + 2\Psi_2(\tilde{y}, 0)$$

is nondegenerate. It follows that the holomorphic function

$$\mathbb{C}^n \times \mathbb{C}^N \ni (\tilde{y}, \theta) \mapsto i\varphi(x, \tilde{y}, \theta) + 2\Psi_2(\tilde{y}, z)$$

has a unique critical point which is nondegenerate for each $(x, z) \in \mathbb{C}^n \times \mathbb{C}^n$. An application of exact (quadratic) stationary phase allows us therefore to conclude that

$$K_A(x, \bar{y}) = \hat{a} e^{2\Psi(x,\bar{y})}, \quad \hat{a} \in \mathbb{C}. \tag{3-12}$$

Here $\Psi(x, z)$ is a holomorphic quadratic form on $\mathbb{C}^{2n}$ given by

$$2\Psi(x, z) = \mathrm{vc}_{\tilde{y},\theta}(i\varphi(x, \tilde{y}, \theta) + 2\Psi_2(\tilde{y}, z)). \tag{3-13}$$

Let us now make the following basic observation.

Proposition 3.1. *The holomorphic quadratic form* $\Psi(x,z)$ *given in* (3-13) *satisfies*

$$2\,\mathrm{Re}\,\Psi(x,\bar{y}) \le \Phi(x) + \Phi_2(y), \quad (x,y) \in \mathbb{C}^n_x \times \mathbb{C}^n_y. \tag{3-14}$$

Proof. It will be more convenient to verify that

$$2\,\mathrm{Re}\,\Psi(x,y) \le \Phi(x) + \Phi_2^*(y), \quad (x,y) \in \mathbb{C}^n_x \times \mathbb{C}^n_y, \tag{3-15}$$

where $\Phi_2^*(y) = \Phi_2(\bar{y})$. A direct calculation shows that

$$\frac{2}{i}\partial_y \Phi_2^*(y) = -\overline{\frac{2}{i}(\partial_y \Phi_2)(\bar{y})},$$

or equivalently,

$$\frac{2}{i}\partial_y (\Phi_2^*)(\bar{y}) = -\overline{\frac{2}{i}(\partial_y \Phi_2)(y)}.$$

It follows that the antilinear involution

$$\Gamma : \mathbb{C}^{2n} \ni (y,\eta) \mapsto (\bar{y}, -\bar{\eta}) \in \mathbb{C}^{2n} \tag{3-16}$$

maps Λ_{Φ_2} bijectively onto $\Lambda_{\Phi_2^*}$. We conclude in view of (3-1) that

$$\kappa \circ \Gamma : \Lambda_{\Phi_2^*} \to \Lambda_{\Phi}, \tag{3-17}$$

and let us consider the graph of the map in (3-17), $\mathrm{Graph}(\kappa \circ \Gamma) \cap (\Lambda_\Phi \times \Lambda_{\Phi_2^*})$. Here $\Lambda_\Phi \times \Lambda_{\Phi_2^*} = \Lambda_{\Phi(x)+\Phi_2^*(y)}$ is I-Lagrangian and R-symplectic for the standard symplectic form

$$d\xi \wedge dx + d\eta \wedge dy \tag{3-18}$$

on $\mathbb{C}^{2n}_{x,\xi} \times \mathbb{C}^{2n}_{y,\eta}$ and we claim that $\mathrm{Graph}(\kappa \circ \Gamma) \cap (\Lambda_\Phi \times \Lambda_{\Phi_2^*})$ is Lagrangian for the symplectic form in (3-18), restricted to $\Lambda_\Phi \times \Lambda_{\Phi_2^*}$. This can be seen by a direct computation: when $(t,s) \in \Lambda_{\Phi_2^*} \times \Lambda_{\Phi_2^*}$ we have, writing σ for the standard symplectic form on $\mathbb{C}^{2n}$,

$$\sigma\big(\kappa(\Gamma(t)), \kappa(\Gamma(s))\big) + \sigma(t,s) = \sigma(\Gamma(t),\Gamma(s)) + \sigma(t,s) = -\overline{\sigma(t,s)} + \sigma(t,s) = 0,$$

since $\sigma(t,s)$ is real. Here we have also used that, by a straightforward computation,

$$\sigma(\Gamma t, \Gamma s) = -\overline{\sigma(t,s)}. \tag{3-19}$$

It is then well known that $\pi_{x,y}(\mathrm{Graph}(\kappa\circ\Gamma)\cap(\Lambda_\Phi \times \Lambda_{\Phi_2^*}))$, the projection of $\mathrm{Graph}(\kappa\circ\Gamma)\cap(\Lambda_\Phi\times\Lambda_{\Phi_2^*})$ in $\mathbb{C}^{2n}_{x,y}$, is maximally totally real; see [Melin and Sjöstrand 2003].

We now come to check (3-15). To this end, we observe that (3-13) gives

$$2\partial_x \Psi(x,y) = i\,\partial_x \varphi(x,\tilde{y},\theta), \tag{3-20}$$

$$2\partial_y \Psi(x,y) = 2\partial_y \Psi_2(\tilde{y},y), \tag{3-21}$$

where

$$\partial_\theta \varphi(x,\tilde{y},\theta) = 0, \quad \partial_{\tilde{y}} \varphi(x,\tilde{y},\theta) + \frac{2}{i}\partial_{\tilde{y}} \Psi_2(\tilde{y},y) = 0. \tag{3-22}$$

We shall consider (3-20), (3-21) at the points $(x, y) \in \pi_{x,y}(\mathrm{Graph}(\kappa \circ \Gamma) \cap \Lambda_\Phi \times \Lambda_{\Phi_2^*})$, which corresponds to $\tilde{y} = \bar{y}$ in (3-22). Using (3-22) together with the fact that

$$\partial_{\tilde{y}} \Psi_2(\tilde{y}, \bar{\tilde{y}}) = \partial_{\tilde{y}} \Phi_2(\tilde{y}),$$

and (3-21) together with the fact that

$$(\partial_y \Psi_2)(\bar{y}, y) = \partial_y \Phi_2^*(y),$$

we conclude that at the points

$$(x, y) \in \pi_{x,y}(\mathrm{Graph}(\kappa \circ \Gamma) \cap \Lambda_\Phi \times \Lambda_{\Phi_2^*}),$$

the following equalities hold:

$$\partial_x \Psi(x, y) = \partial_x \Phi(x), \quad \partial_y \Psi(x, y) = \partial_y \Phi_2^*(y). \tag{3-23}$$

In other words,

$$\partial_x (\Phi(x) - 2\,\mathrm{Re}\,\Psi(x, y)) = \partial_y (\Phi_2^*(y) - 2\,\mathrm{Re}\,\Psi(x, y)) = 0,$$

along $\pi_{x,y}(\mathrm{Graph}(\kappa \circ \Gamma) \cap \Lambda_\Phi \times \Lambda_{\Phi_2^*})$, and thus the gradient of the real-valued function

$$F(x, y) = \Phi(x) + \Phi_2^*(y) - 2\,\mathrm{Re}\,\Psi(x, y) \tag{3-24}$$

vanishes on $\pi_{x,y}(\mathrm{Graph}(\kappa \circ \Gamma) \cap \Lambda_\Phi \times \Lambda_{\Phi_2^*})$. It follows that the strictly plurisubharmonic quadratic form $F(x, y)$ vanishes to the second order along

$$\pi_{x,y}(\mathrm{Graph}(\kappa \circ \Gamma) \cap \Lambda_\Phi \times \Lambda_{\Phi_2^*}), \tag{3-25}$$

and since the latter is maximally totally real, we get $F \geq 0$, thus implying (3-15). □

Remark. The strictly plurisubharmonic quadratic form $F(x, y)$ in (3-24) vanishes to the second order along the maximally totally real subspace (3-25), and therefore the conclusion that $F \geq 0$ can be strengthened to

$$F(x, y) \asymp \mathrm{dist}\big((x, y), \pi_{x,y}(\mathrm{Graph}(\kappa \circ \Gamma) \cap \Lambda_\Phi \times \Lambda_{\Phi_2^*})\big)^2.$$

Let us now return to the Bergman-type representation of the Fourier integral operator A in (3-4) quantizing κ. Combining (3-6) and (3-12), we get

$$Au(x) = \iint \check{a} e^{2(\Psi(x,\bar{y}) - \Phi_2(y))} u(y)\, dy\, d\bar{y} \tag{3-26}$$

for some $\check{a} \in \mathbb{C}$. This can be viewed as a Fourier integral operator

$$Au(x) = \iint \check{a} e^{2(\Psi(x,\theta) - \Psi_2(y,\theta))} u(y)\, dy\, d\theta, \tag{3-27}$$

where we take the integration contour $\theta = \bar{y}$ in (3-27).

Since $\partial_y \partial_\theta \Psi_2(y, \theta)$ is nondegenerate, the phase function

$$\phi(x, y, \theta) = \frac{2}{i}(\Psi(x, \theta) - \Psi_2(y, \theta)) \tag{3-28}$$

is nondegenerate in the sense of Hörmander, and the canonical transformation κ takes the form

$$\kappa : \Big(y, \frac{2}{i}\partial_y \Psi_2(y,\theta)\Big) \mapsto \Big(x, \frac{2}{i}\partial_x \Psi(x,\theta)\Big), \quad \text{with } \partial_\theta \Psi(x,\theta) = \partial_\theta \Psi_2(y,\theta). \tag{3-29}$$

We may also notice here that if we define

$$\kappa_\Psi : \Big(\theta, -\frac{2}{i}\partial_\theta \Psi(y,\theta)\Big) \mapsto \Big(y, \frac{2}{i}\partial_y \Psi(y,\theta)\Big)$$

and κ_{Ψ_2} similarly, then $\kappa = \kappa_\Psi \circ \kappa_{\Psi_2}^{-1}$.

The discussion so far shows that the canonical transformation κ enjoying the mapping properties (3-1), (3-2), admits a nondegenerate phase function of the form (3-28), where the quadratic form Ψ satisfies

$$2\,\mathrm{Re}\,\Psi(x,\bar{y}) \le \Phi_1(x) + \Phi_2(y), \quad (x,y) \in \mathbb{C}^n_x \times \mathbb{C}^n_y. \tag{3-30}$$

The positivity of κ relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$ is then implied by the following general result.

Proposition 3.2. *Let κ be a canonical transformation satisfying* (3-1) *and let us consider a metaplectic Fourier integral operator of the form* (3-26), *or equivalently* (3-27), *associated to κ. Then the following conditions are equivalent*:

(i) *κ is positive relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$ in the sense of* (1-4):

$$\frac{1}{i}\sigma(t_1, \iota_{\Phi_1} t_1) - \frac{1}{i}\sigma(t_2, \iota_{\Phi_2} t_2) \ge 0, \quad \textit{whenever } t_1 = \kappa(t_2),\ t_2 \in \mathbb{C}^{2n}. \tag{3-31}$$

(ii) *$\Lambda_{2\,\mathrm{Re}\,\Psi(x,\bar{y})}$ is positive relative to $\Lambda_{\Phi_1(x)+\Phi_2(y)}$.*

(iii) *$2\,\mathrm{Re}\,\Psi(x,\bar{y}) - \Phi_1(x) - \Phi_2(y) \le 0$ on $\mathbb{C}^n_x \times \mathbb{C}^n_y$.*

Proof. The equivalence (ii)⇔(iii) follows from Theorem 2.1, so it suffices to show the equivalence (i)⇔(ii).

Clearly, (iii) is equivalent to

$$2\,\mathrm{Re}\,\Psi(x,y) - \Phi_1(x) - \Phi_2^*(y) \le 0 \quad \text{on } \mathbb{C}^{2n}_{x,y}, \tag{3-32}$$

where $\Phi_2^*(y) = \Phi_2(\bar{y}) (= \overline{\Phi_2(\bar{y})})$, and by Theorem 2.1(ii) is equivalent to

$$\Lambda_{2\,\mathrm{Re}\,\Psi(x,y)} \text{ is positive relative to } \Lambda_{\Phi_1(x)+\Phi_2^*(y)}. \tag{3-33}$$

We have

$$\begin{aligned}\Lambda_{2\,\mathrm{Re}\,\Psi} &= \Big\{\Big(x, \frac{2}{i}\partial_x 2\,\mathrm{Re}\,\Psi(x,y); y, \frac{2}{i}\partial_y 2\,\mathrm{Re}\,\Psi(x,y)\Big)\Big\} \\ &= \Big\{\Big(x, \frac{2}{i}\partial_x \Psi(x,y); y, \frac{2}{i}\partial_y \Psi(x,y)\Big)\Big\},\end{aligned} \tag{3-34}$$

and (3-33) means that

$$\frac{1}{i}\sigma(t_1, \iota_{\Phi_1} t_1) + \frac{1}{i}\sigma(t_2, \iota_{\Phi_2^*} t_2) \ge 0 \quad \text{for all } (t_1,t_2) \in \Lambda_{2\,\mathrm{Re}\,\Psi}. \tag{3-35}$$

Here, we shall relate the involutions $\iota_{\Phi_2^*}$ and ι_{Φ_2}. From (2-4) let us recall that ι_{Φ_2} is given by

$$\iota_{\Phi_2} : \Big(y, \frac{2}{i}\overline{\partial_y \Psi_2(x,\bar{y})}\Big) \mapsto \Big(x, \frac{2}{i}\partial_x \Psi_2(x,\bar{y})\Big). \tag{3-36}$$

We also know that the antilinear involution Γ, given in (3-16), maps Λ_{Φ_2} bijectively onto $\Lambda_{\Phi_2^*}$, and since $\iota_{\Phi_2}, \iota_{\Phi_2^*}$ are the unique antilinear maps equal to the identity on Λ_{Φ_2} and $\Lambda_{\Phi_2^*}$ respectively, it follows that

$$\iota_{\Phi_2^*} = \Gamma \iota_{\Phi_2} \Gamma. \tag{3-37}$$

From (3-19), let us recall that

$$\frac{1}{i}\sigma(\Gamma t, \Gamma s) = \overline{\frac{1}{i}\sigma(t,s)},$$

so using (3-37), we find that the second term in (3-35) is equal to

$$\frac{1}{i}\sigma(t_2, \Gamma\iota_{\Phi_2}\Gamma t_2) = \overline{\frac{1}{i}\sigma(\Gamma t_2, \iota_{\Phi_2}\Gamma t_2)} = \frac{1}{i}\sigma(\Gamma t_2, \iota_{\Phi_2}\Gamma t_2) = -\frac{1}{i}\sigma(\iota_{\Phi_2}\Gamma t_2, \Gamma t_2),$$

where we also used the fact that $(1/i)\sigma(t, \iota_{\Phi_2} t)$ is real. Hence (3-33) is equivalent, via (3-35), to

$$\frac{1}{i}\sigma(t_1, \iota_{\Phi_1} t_1) - \frac{1}{i}\sigma(\iota_{\Phi_2}\Gamma t_2, \Gamma t_2) \geq 0 \quad \text{for all } (t_1, t_2) \in \Lambda_{2\operatorname{Re}\Psi}. \tag{3-38}$$

From (3-36), we get

$$\iota_{\Phi_2}\Gamma : \Big(\bar{y}, -\overline{\frac{2}{i}\overline{\partial_y \Psi_2(x,\bar{y})}}\Big) \mapsto \Big(x, \frac{2}{i}\partial_x \Psi_2(x,\bar{y})\Big);$$

i.e.,

$$\iota_{\Phi_2}\Gamma : \Big(\theta, \frac{2}{i}\partial_\theta \Psi_2(y,\theta)\Big) \mapsto \Big(y, \frac{2}{i}\partial_y \Psi_2(y,\theta)\Big), \tag{3-39}$$

where we changed the notation slightly for convenience.

Write

$$\Lambda_{2\operatorname{Re}\Psi} \ni (t_1, t_2) = \Big(x, \frac{2}{i}\partial_x \Psi(x,\theta); \theta, \frac{2}{i}\partial_\theta \Psi(x,\theta)\Big),$$

and put $t_3 = \iota_{\Phi_2}\Gamma t_2$, so that by (3-39)

$$t_3 = \Big(y, \frac{2}{i}\partial_y \Psi_2(y,\theta)\Big),$$

where

$$\Big(\theta, \frac{2}{i}\partial_\theta \Psi(x,\theta)\Big) = \Big(\theta, \frac{2}{i}\partial_\theta \Psi_2(y,\theta)\Big).$$

Comparing with (3-29), we see that $t_1 = \kappa(t_3)$. Since $\Gamma t_2 = \iota_{\Phi_2}^2 \Gamma t_2 = \iota_{\Phi_2} t_3$, we see that (3-38) is equivalent to

$$\frac{1}{i}\sigma(t_1, \iota_{\Phi_1} t_1) - \frac{1}{i}\sigma(t_3, \iota_{\Phi_2} t_3) \geq 0, \quad \text{when } t_1 = \kappa(t_3), \tag{3-40}$$

which is precisely (3-31) up to a change of notation. This completes the proof of the equivalence (i)$\Leftrightarrow$(ii) and of the proposition. □

Combining Propositions 3.1 and 3.2, we see that the proof of the sufficiency part of Theorem 1.1 is now complete.

Remark. Let $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a complex linear canonical transformation such that (3-1) holds, where Φ_2, Φ are strictly plurisubharmonic. It follows from (3-23) that the holomorphic quadratic form $\Psi(x, y)$

depends only on κ and on the weights Φ_2, Φ, but not on the choice of a nondegenerate phase function $\varphi(x, y, \theta)$, $(x, y, \theta) \in \mathbb{C}^n_x \times \mathbb{C}^n_y \times \mathbb{C}^N_\theta$ such that

$$\Lambda'_\varphi = \mathrm{Graph}(\kappa),$$

where

$$\Lambda'_\varphi = \{(x, \varphi'_x(x, y, \theta); y, -\varphi'_y(x, y, \theta)) : \varphi'_\theta(x, y, \theta) = 0\}.$$

It follows that if $\psi(x, y, w)$, $(x, y, w) \in \mathbb{C}^n_x \times \mathbb{C}^n_y \times \mathbb{C}^{N'}_w$, is a second nondegenerate phase function such that

$$\Lambda'_\varphi = \Lambda'_\psi = \mathrm{Graph}(\kappa),$$

then both φ and ψ give rise to the same Fourier integral operators, realized as bounded linear maps: $H_{\Phi_2}(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n)$.

We shall finish this section by making some remarks concerning metaplectic Fourier integral operators in the complex domain, associated to canonical transformations that are strictly positive relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$. Let

$$\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \tag{3-41}$$

be a complex linear canonical transformation which is strictly positive relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$. According to Theorem 1.1, we then have

$$\kappa(\Lambda_{\Phi_2}) = \Lambda_\Phi, \tag{3-42}$$

where Φ is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ such that

$$\Phi_1(x) - \Phi(x) \asymp |x|^2, \quad x \in \mathbb{C}^n. \tag{3-43}$$

Let

$$Tu(x) = \iint e^{i\phi(x,y,\theta)} au(y)\, dy\, d\theta, \quad a \in \mathbb{C},$$

be a Fourier integral operator associated to κ. As discussed above, it follows from [Caliceti et al. 2012; Sjöstrand 1982] that the operator T can be realized by means of a suitable good contour and we then obtain a bounded operator

$$T : H_{\Phi_2}(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n). \tag{3-44}$$

It follows from (3-43) that the inclusion map $H_\Phi(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C})$ is compact, and the operator $T : H_{\Phi_2}(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C}^n)$ is therefore compact. The following sharpening is essentially well known; see [Aleman and Viola 2018].

Proposition 3.3. *The operator*

$$T : H_{\Phi_2}(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C}^n)$$

is of trace class, with the singular values $s_j(T)$ satisfying

$$s_j(T) = \mathcal{O}(j^{-\infty}). \tag{3-45}$$

Proof. Let q be a holomorphic quadratic form on $\mathbb{C}^{2n}$ such that its restriction to Λ_{Φ_1} is real positive definite. Let us introduce the Weyl quantization of q, the operator $Q = q^w(x, D_x)$. The quadratic differential operator Q is self-adjoint on $H_{\Phi_1}(\mathbb{C}^n)$ with discrete spectrum, and let us consider the metaplectic Fourier integral operator e^{tQ}, $0 \leq t \leq t_0 \ll 1$, acting on the space $H_{\Phi}(\mathbb{C}^n)$. Using some well-known arguments, explained in detail in [Hérau et al. 2005; Hitrik and Pravda-Starov 2009; Hitrik et al. 2018], we see that, for $t \in [0, t_0]$ with $t_0 > 0$ small enough, the operator e^{tQ} is bounded,

$$e^{tQ} : H_{\Phi}(\mathbb{C}^n) \to H_{\Phi_t}(\mathbb{C}^n), \tag{3-46}$$

where Φ_t is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, depending smoothly on $t \geq 0$ small enough, such that

$$\Phi_t(x) = \Phi(x) + \mathcal{O}(t)|x|^2. \tag{3-47}$$

Combining this observation with (3-43) we conclude that there exists $\delta > 0$ small enough such that the operator

$$e^{\delta Q} T : H_{\Phi_2}(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C}^n) \tag{3-48}$$

is bounded. Writing

$$T = e^{-\delta Q} e^{\delta Q} T, \tag{3-49}$$

and applying the Ky Fan inequalities, we get

$$s_j(T) \leq s_j(e^{-\delta Q}) \| e^{\delta Q} T \|_{\mathcal{L}(H_{\Phi_2}, H_{\Phi_1})} = \mathcal{O}(j^{-\infty}).$$

Here we have also used the fact that the singular values of the compact positive self-adjoint operator $e^{-\delta Q}$ on $H_{\Phi_1}(\mathbb{C}^n)$ satisfy

$$s_j(e^{-\delta Q}) = \mathcal{O}(j^{-\infty}).$$

It follows that T is of trace class and the proof of the proposition is complete. ☐

4. Applications to Toeplitz operators

The purpose of this section is to apply the point of view of Fourier integral operators in the complex domain, developed in the previous sections, to the study of Toeplitz operators in the Bargmann space, establishing Theorem 1.2.

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let $p : \mathbb{C}^n \to \mathbb{C}$ be measurable. Associated to p is the Toeplitz operator

$$\mathrm{Top}(p) = \Pi_{\Phi_0} \circ p \circ \Pi_{\Phi_0} : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n). \tag{4-1}$$

Here

$$\Pi_{\Phi_0} : L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)) \to H_{\Phi_0}(\mathbb{C}^n)$$

is the orthogonal projection. We shall always assume that when equipped with the natural domain

$$\mathcal{D}(\mathrm{Top}(p)) = \{u \in H_{\Phi_0}(\mathbb{C}^n) : pu \in L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx))\}, \tag{4-2}$$

the operator $\mathrm{Top}(p)$ becomes densely defined.

For future reference, let us recall the link between the Toeplitz and Weyl quantizations on $\mathbb{C}^n$. Let $p \in L^\infty(\mathbb{C}^n)$, say. Then we have

$$\mathrm{Top}(p) = a^w(x, D_x), \tag{4-3}$$

where $a \in C^\infty(\Lambda_{\Phi_0})$ is given by

$$a\left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)\right) = \left(\exp\left(\tfrac{1}{4}(\Phi''_{0,x\bar{x}})^{-1}\partial_x \cdot \partial_{\bar{x}}\right)p\right)(x), \quad x \in \mathbb{C}^n. \tag{4-4}$$

See [Guillemin 1984; Sjöstrand 1996]. Here $-(\Phi''_{0,x\bar{x}})^{-1}\partial_x \cdot \partial_{\bar{x}}$ is a constant coefficient second-order differential operator on $\mathbb{C}^n$ whose symbol is the positive definite quadratic form

$$\tfrac{1}{4}(\Phi''_{0,x\bar{x}})^{-1}\bar{\xi} \cdot \xi > 0, \quad 0 \neq \xi \in \mathbb{C}^n \simeq \mathbb{R}^{2n},$$

and therefore the operator in (4-4) can be regarded as the forward heat flow acting on p.

In this section we shall be concerned with the question of when an operator of the form $\mathrm{Top}(p)$ is bounded,

$$\mathrm{Top}(p) \in \mathcal{L}(H_{\Phi_0}(\mathbb{C}^n), H_{\Phi_0}(\mathbb{C}^n)),$$

and following [Berger and Coburn 1994], in doing so we shall only consider Toeplitz symbols of the form

$$p = e^{2q}, \tag{4-5}$$

where q is a complex-valued quadratic form on $\mathbb{C}^n$. Let us first proceed to give an explicit criterion, guaranteeing that when equipped with the domain (4-2), the operator $\mathrm{Top}(e^{2q})$ is densely defined. Recalling the decomposition (2-8) and considering the unitary map

$$H_{\Phi_0}(\mathbb{C}^n) \ni u \mapsto ue^{-f} \in H_{\Phi_{\mathrm{herm}}}(\mathbb{C}^n), \quad f(x) = \Phi''_{0,xx}x \cdot x,$$

we may observe that the space $e^f\mathcal{P}(\mathbb{C}^n) = \{e^f p : p \in \mathcal{P}(\mathbb{C}^n)\}$ is dense in $H_{\Phi_0}(\mathbb{C}^n)$. Here $\mathcal{P}(\mathbb{C}^n)$ is the space of holomorphic polynomials on $\mathbb{C}^n$. It follows that

$$e^f\mathcal{P}(\mathbb{C}^n) \subset \mathcal{D}(\mathrm{Top}(e^{2q})),$$

so that $\mathrm{Top}(e^{2q})$ is densely defined, provided that

$$2\,\mathrm{Re}\, q(x) < \Phi_{\mathrm{herm}}(x), \tag{4-6}$$

in the sense of quadratic forms on $\mathbb{C}^n$.

Recalling (3-8), we may write

$$\mathrm{Top}(e^{2q})u(x) = C\int e^{2(\Psi_0(x,\bar{y})-\Phi_0(y))}e^{2q(y,\bar{y})}u(y)\,dy\,d\bar{y}, \quad u \in \mathcal{D}(\mathrm{Top}(e^{2q})). \tag{4-7}$$

Here $C > 0$ and Ψ_0 is the polarization of Φ_0. Similarly to (3-27), we get

$$\mathrm{Top}(e^{2q})u(x) = C\iint_\Gamma e^{2(\Psi_0(x,\theta)-\Psi_0(y,\theta)+q(y,\theta))}u(y)\,dy\,d\theta, \tag{4-8}$$

where Γ is the contour in $\mathbb{C}^{2n}$, given by $\theta = \bar{y}$. Here the holomorphic quadratic form

$$F(x, y, \theta) = \frac{2}{i}(\Psi_0(x, \theta) - \Psi_0(y, \theta) + q(y, \theta)) \tag{4-9}$$

is a nondegenerate phase function in the sense of Hörmander, in view of the fact that $\det \Psi''_{0,x\theta} \neq 0$, and therefore the operator $\mathrm{Top}(e^{2q})$ in (4-8) can be viewed as a metaplectic Fourier integral operator associated to a suitable canonical relation $\subset \mathbb{C}^{2n} \times \mathbb{C}^{2n}$. We have the formal factorization

$$\mathrm{Top}(e^{2q}) = AB,$$

where

$$Av(x) = \int e^{2\Psi_0(x,\theta)} v(\theta)\, d\theta, \quad Bu(\theta) = \int e^{-2\widetilde{\Psi}_0(y,\theta)} u(y)\, dy, \tag{4-10}$$

and where we have written $\widetilde{\Psi}_0(y, \theta) = \Psi_0(y, \theta) - q(y, \theta)$. Here the operator A, formally, is an elliptic Fourier integral operator associated to the canonical transformation

$$\Big(\theta, -\frac{2}{i}\partial_\theta \Psi_0(x, \theta)\Big) \mapsto \Big(x, \frac{2}{i}\partial_x \Psi_0(x, \theta)\Big).$$

It follows that the canonical relation associated to $\mathrm{Top}(e^{2q})$ is the graph of a canonical transformation if and only if this is the case for the Fourier integral operator B. We conclude that the operator $\mathrm{Top}(e^{2q})$ in (4-8) is associated to a canonical transformation precisely when

$$\partial_y \partial_\theta \widetilde{\Psi}_0 \neq 0. \tag{4-11}$$

The condition (4-11) is equivalent to the assumption (1-8) in Theorem 1.2. The canonical transformation is then given by

$$\kappa : (y, -\partial_y F(x, y, \theta)) \mapsto (x, \partial_x F(x, y, \theta)), \quad \partial_\theta F(x, y, \theta) = 0. \tag{4-12}$$

Example. In the following discussion, we shall revisit the family of examples discussed in Section 6 of [Berger and Coburn 1994] and show how the point of view of Fourier integral operators in the complex domain, developed above, allows one to recover the main findings of Section 6 of that paper, obtained there by means of a direct computation.

Let $\Phi_0(x) = \frac{1}{2}|x|^2$ and $q = \frac{1}{2}\lambda|y|^2$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda < \frac{1}{2}$. Here the restriction on $\mathrm{Re}\,\lambda$ implies that (4-6) holds, so that the operator $\mathrm{Top}(e^{2q})$ is densely defined in $H_{\Phi_0}(\mathbb{C}^n)$. We have

$$\Psi_0(x, y) = \tfrac{1}{2} x \cdot y,$$

and the phase function F in (4-9) is given by

$$F(x, y, \theta) = \frac{2}{i}\Big(\frac{1}{2}x \cdot \theta - \Big(\frac{1-\lambda}{2}\Big) y \cdot \theta\Big). \tag{4-13}$$

In particular, the condition (4-11) is satisfied and we may then compute the canonical transformation κ associated to the corresponding Fourier integral operator $\mathrm{Top}(e^{2q})$ in (4-8).

The critical set C_F of the phase F is given by $\partial_\theta F = 0 \iff x = (1-\lambda)y$, and the corresponding canonical transformation κ is of the form

$$\kappa : (y, -\partial_y F(x, y, \theta)) \mapsto (x, \partial_x F(x, y, \theta)), \quad (x, y, \theta) \in C_F. \tag{4-14}$$

It follows that κ is given by

$$\kappa : (y, \eta) \mapsto \Big((1-\lambda)y, \frac{\eta}{1-\lambda}\Big). \tag{4-15}$$

We shall now determine when the canonical transformation κ is positive relative to Λ_{Φ_0}, which can be done by a direct computation: it follows from (2-4) that the involution ι_{Φ_0} is given by

$$\iota_{\Phi_0} : (y, \eta) \mapsto \Big(\frac{1}{i}\bar{\eta}, \frac{1}{i}\bar{y}\Big), \tag{4-16}$$

and therefore, we may compute,

$$\begin{aligned} \frac{1}{i}\sigma(\kappa(y,\eta), \iota_{\Phi_0}\kappa(y,\eta)) &= \frac{1}{i}\sigma\bigg(\Big((1-\lambda)y, \frac{\eta}{1-\lambda}\Big), \Big(\frac{1}{i}\frac{\bar{\eta}}{1-\bar{\lambda}}, \frac{1}{i}(1-\bar{\lambda})\bar{y}\Big)\bigg) \\ &= |1-\lambda|^2|y|^2 - \frac{|\eta|^2}{|1-\lambda|^2}. \end{aligned} \tag{4-17}$$

Similarly, we have

$$\frac{1}{i}\sigma((y,\eta), \iota_{\Phi_0}(y,\eta)) = |y|^2 - |\eta|^2. \tag{4-18}$$

Combining (4-17), (4-18) we see that the κ is positive relative to Λ_{Φ_0} if and only if

$$|1-\lambda| \geq 1. \tag{4-19}$$

This condition occurs in [Berger and Coburn 1994, pp. 581–582] (with the inessential difference that in the discussion in that paper one considers $\Phi_0(x) = \frac{1}{4}|x|^2$), where it is verified that the operator $\mathrm{Top}(e^{2q})$ is in $\mathcal{L}(H_{\Phi_0}(\mathbb{C}^n), H_{\Phi_0}(\mathbb{C}^n))$ precisely when (4-19) holds.

In the case when the strict inequality holds in (4-19), the canonical transformation κ in (4-15) is strictly positive relative to Λ_{Φ_0} and it follows from Proposition 3.3 that the Toeplitz operator $\mathrm{Top}(e^{2q})$ is of trace class on $H_{\Phi_0}(\mathbb{C}^n)$.

We shall now proceed to discuss the "boundary" case when

$$|1-\lambda| = 1. \tag{4-20}$$

In this case, using (4-15) we immediately see that $\kappa(\Lambda_{\Phi_0}) = \Lambda_{\Phi_0}$, and therefore we conclude, in view of [Caliceti et al. 2012; Sjöstrand 1982], that the operator

$$\mathrm{Top}(e^{2q}) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n) \tag{4-21}$$

is bounded, with a bounded two-sided inverse.

We claim next that the operator in (4-21) is in fact unitary when (4-20) holds, and when verifying the unitarity, it will be convenient to pass to the Weyl quantization, computing the Weyl symbol of $\mathrm{Top}(e^{2q})$.

It follows from (4-4) that

$$a\Big(x, \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x)\Big) = \Big(\exp\Big(\frac{\Delta}{8}\Big)e^{2q}\Big)(x) = \Big(\frac{2}{\pi}\Big)^n \int_{\mathbb{C}^n} e^{-2|x-y|^2} e^{\lambda|y|^2}\, L(dy). \tag{4-22}$$

Here Δ is the Laplacian on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Computing the Gaussian integral in (4-22) by the exact version of stationary phase, we get, see also [Berger and Coburn 1994],

$$a\Big(x, \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x)\Big) = \Big(\frac{2}{2-\lambda}\Big)^n \exp\Big(\frac{2\lambda}{2-\lambda}|x|^2\Big). \tag{4-23}$$

Here we may notice that

$$\mathrm{Re}\Big(\frac{2\lambda}{2-\lambda}\Big) = 0,$$

when (4-20) holds, reflecting the fact that the associated canonical transformation in (4-15) is "real" in this case. We conclude that the Weyl symbol of the Toeplitz operator $\mathrm{Top}(e^{2q})$ is given by

$$a(x,\xi) = \Big(\frac{2}{2-\lambda}\Big)^n \exp(iF(x,\xi)), \quad F(x,\xi) = \frac{2\lambda}{2-\lambda} x\cdot\xi, \tag{4-24}$$

so that

$$\mathrm{Top}(e^{2q}) = \Big(\frac{2}{2-\lambda}\Big)^n (\exp(iF))^w. \tag{4-25}$$

We have $(\mathrm{Im}\, F)|_{\Lambda_{\Phi_0}} = 0$ and an application of Proposition 5.11 of [Hörmander 1995] together with the metaplectic invariance of the Weyl quantization allows us to conclude that the operator

$$\sqrt{\det(I-\mathcal{F}/2)}(\exp(iF))^w : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n) \tag{4-26}$$

is unitary. Here $\mathcal{F}$ is the Hamilton map of F, i.e., the matrix of the (linear) Hamilton field H_F, and it remains therefore to check that

$$\sqrt{\det(I-\mathcal{F}/2)} = \Big(\frac{2}{2-\lambda}\Big)^n e^{i\theta}, \quad \theta\in\mathbb{R}. \tag{4-27}$$

To this end, we compute using (4-24),

$$\mathcal{F}/2 = \frac{\lambda}{2-\lambda}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad I-\mathcal{F}/2 = \frac{2}{2-\lambda}\begin{pmatrix}1-\lambda & 0\\ 0 & 1\end{pmatrix},$$

and (4-27) follows, thanks to (4-20). We conclude therefore that the Toeplitz operator $\mathrm{Top}(e^{2q})$ is unitary on $H_{\Phi_0}(\mathbb{C}^n)$, when $\mathrm{Re}\,\lambda < \frac{1}{2}$ and (4-20) holds. The unitarity property has also been observed in [Berger and Coburn 1994].

Remark. In the case when $\mathrm{Re}\,\lambda < \frac{1}{2}$, $|1-\lambda| > 1$, we observed that the operator $\mathrm{Top}(e^{2q})$ is of trace class on $H_{\Phi_0}(\mathbb{C}^n)$, and we get, using (4-24) and the metaplectic invariance of the Weyl quantization,

$$\mathrm{tr}\,\mathrm{Top}(e^{2q}) = \frac{1}{(2\pi)^n}\iint_{\Lambda_{\Phi_0}} a\frac{(\sigma|_{\Lambda_{\Phi_0}})^n}{n!},$$

where a is given in (4-24).

We are now ready to discuss the proof of Theorem 1.2. It follows from Theorem 1.1 and the discussion in this section that it suffices to check that the canonical transformation (4-12) associated to the operator $\mathrm{Top}(e^{2q})$ is positive relative to Λ_{Φ_0}. To this end, let us consider the Weyl symbol of $\mathrm{Top}(e^{2q})$, given by (4-4),

$$a(x,\xi) = \bigl(\exp\bigl(\tfrac{1}{4}(\Phi''_{0,x\bar{x}})^{-1}\partial_x \cdot \partial_{\bar{x}}\bigr)e^{2q}\bigr)(x), \quad (x,\xi) \in \Lambda_{\Phi_0}. \tag{4-28}$$

A simple computation of the inverse Fourier transform of a real Gaussian shows that

$$a(x,\xi) = C_{\Phi_0} \int_{\mathbb{C}^n} \exp(-4\Phi_{\mathrm{herm}}(x-y))e^{2q(y)}\, L(dy), \quad C_{\Phi_0} \neq 0. \tag{4-29}$$

Here the convergence of the integral in (4-29) is guaranteed by (4-6). In view of the exact version of stationary phase, it is therefore clear that

$$a(x,\xi) = C \exp(iF(x,\xi)), \quad (x,\xi) \in \Lambda_{\Phi_0}, \tag{4-30}$$

for some constant $C \neq 0$, where F is a holomorphic quadratic form on $\mathbb{C}^{2n}$. Proposition B.1 shows that the positivity of κ in (4-12) relative to Λ_{Φ_0} is equivalent to the fact that the Weyl symbol in (4-30) is such that $\mathrm{Im}\, F|_{\Lambda_{\Phi_0}} \geq 0 \Longleftrightarrow \exp(iF) \in L^\infty(\Lambda_{\Phi_0})$. The proof of Theorem 1.2 is complete.

Appendix A: Schwartz kernel theorem in the H_Φ-setting

In this appendix we shall make some elementary remarks concerning integral representations for linear continuous maps between weighted spaces of holomorphic functions. Such observations are essentially well known; see for instance [Peetre 1990].

Let $\Omega_j \subset \mathbb{C}^{n_j}$ be open, $j = 1, 2$, and let $\Phi_j \in C(\Omega_j; \mathbb{R})$. We introduce the weighted spaces

$$H_{\Phi_j}(\Omega_j) = \mathrm{Hol}(\Omega_j) \cap L^2(\Omega_j, e^{-2\Phi_j}\, L(dy_j)), \quad j = 1, 2, \tag{A-1}$$

where $L(dy_j)$ is the Lebesgue measure on $\mathbb{C}^{n_j}$. When viewed as closed subspaces of $L^2(\Omega_j, e^{-2\Phi_j} L(dy_j))$, the spaces $H_{\Phi_j}(\Omega_j)$ are separable complex Hilbert spaces and the natural embeddings $H_{\Phi_j}(\Omega_j) \to \mathrm{Hol}(\Omega_j)$ are continuous. Here the space $\mathrm{Hol}(\Omega_j)$ is equipped with its natural Fréchet space topology of locally uniform convergence. Let

$$T : H_{\Phi_1}(\Omega_1) \to H_{\Phi_2}(\Omega_2) \tag{A-2}$$

be a linear continuous map. Let us also write $\overline{\Omega}_1 = \{z \in \mathbb{C}^{n_1} : \bar{z} \in \Omega_1\}$.

Theorem A.1. *There exists a unique function $K(x,z) \in \mathrm{Hol}(\Omega_2 \times \overline{\Omega}_1)$ such that*

$$\Omega_1 \ni y \mapsto \overline{K(x,\bar{y})} \in H_{\Phi_1}(\Omega_1) \tag{A-3}$$

for each $x \in \Omega_2$, and

$$Tf(x) = \int_{\Omega_1} K(x,\bar{y}) f(y) e^{-2\Phi_1(y)}\, L(dy), \quad f \in H_{\Phi_1}(\Omega_1). \tag{A-4}$$

We also have

$$\Omega_2 \ni x \mapsto K(x,z) \in H_{\Phi_2}(\Omega_2) \tag{A-5}$$

for each $z \in \overline{\Omega}_1$.

When proving Theorem A.1, we observe that it follows from the remarks above that for each $x \in \Omega_2$, the linear form

$$H_{\Phi_1}(\Omega_1) \ni f \mapsto (Tf)(x) \in \mathbb{C} \tag{A-6}$$

is continuous, and there exists therefore a unique element $k_x \in H_{\Phi_1}(\Omega_1)$ such that for all $f \in H_{\Phi_1}(\Omega_1)$ we have

$$Tf(x) = (f, k_x)_{\Phi_1}, \quad x \in \Omega_2. \tag{A-7}$$

Here and in what follows $(\cdot, \cdot)_{\Phi_j}$ stands for the scalar product in the space $H_{\Phi_j}(\Omega_j)$, $j = 1, 2$.

Letting (e_j) be an orthonormal basis for $H_{\Phi_1}(\Omega_1)$, we may write with convergence in $H_{\Phi_1}(\Omega_1)$, for each $x \in \Omega_2$ fixed,

$$k_x = \sum_{j=1}^{\infty} (k_x, e_j)_{\Phi_1} e_j = \sum_{j=1}^{\infty} \overline{Te_j(x)} e_j. \tag{A-8}$$

By Parseval's formula we get

$$\|k_x\|_{\Phi_1}^2 = \sum_{j=1}^{\infty} |Te_j(x)|^2, \quad x \in \Omega_2. \tag{A-9}$$

Here we know that

$$\|k_x\|_{\Phi_1} = \sup_{\|f\|_{\Phi_1} \leq 1} |Tf(x)|, \tag{A-10}$$

and it follows that the function $\Omega_2 \ni x \mapsto \|k_x\|_{\Phi_1}$ is locally bounded. Let us now make the following elementary observation: Let $\Omega \subset \mathbb{C}^n$ be open and let $f_n \in \mathrm{Hol}(\Omega)$ be such that the series

$$\sum_{n=1}^{\infty} |f_n(z)|^2 \tag{A-11}$$

converges for each $z \in \Omega$, with the sum being locally integrable in Ω. Then the series converges locally uniformly in Ω. Indeed, let us write

$$\sum_{n=1}^{\infty} |f_n(z)|^2 =: F(z) \in L^1_{\mathrm{loc}}(\Omega).$$

Let $K \subset \Omega$ be compact and let ω be an open neighborhood of K such that $K \subset \omega \Subset \Omega$. Then by Cauchy's integral formula and the Cauchy–Schwarz inequality we have

$$\sup_K |f_n|^2 \leq \mathcal{O}_{K,\omega}(1) \|f_n\|^2_{L^2(\omega)}.$$

We get therefore the uniform bound

$$\sum_{n=1}^{N} \sup_K |f_n|^2 \leq \mathcal{O}_{K,\omega}(1) \|F\|_{L^1(\omega)}, \quad N = 1, 2, \ldots,$$

implying the locally uniform convergence of (A-11).

It follows that (A-9) holds with locally uniform convergence in $x \in \Omega_2$, and in particular the function $\Omega_2 \ni x \mapsto \|k_x\|^2_{\Phi_1}$ is continuous plurisubharmonic. We may therefore conclude that the series in (A-8) converges locally uniformly in $\Omega_1 \times \Omega_2$. Letting

$$K(x,z) := \overline{k_x(\bar{z})} = \sum_{j=1}^{\infty} Te_j(x)\overline{e_j(\bar{z})}, \tag{A-12}$$

we conclude that $K \in \mathrm{Hol}(\Omega_2 \times \overline{\Omega}_1)$ is such that (A-3) and (A-4) hold, and these properties characterize the kernel K uniquely.

When verifying (A-5), we let $\tilde{k}_x \in H_{\Phi_2}(\Omega_2)$ be the reproducing kernel for $H_{\Phi_2}(\Omega_2)$. We may then write, when $f \in H_{\Phi_1}(\Omega_1),\ x \in \Omega_2$,

$$Tf(x) = (Tf, \tilde{k}_x)_{\Phi_2} = (f, T^*\tilde{k}_x)_{\Phi_1}, \tag{A-13}$$

and therefore,

$$k_x = T^*\tilde{k}_x. \tag{A-14}$$

Here

$$T^* : H_{\Phi_2}(\Omega_2) \to H_{\Phi_1}(\Omega_1)$$

is the adjoint of T. Letting (f_j) be an orthonormal basis for $H_{\Phi_2}(\Omega_2)$ and recalling that

$$\tilde{k}_x = \sum_{j=1}^{\infty} \overline{f_j(x)} f_j, \tag{A-15}$$

we get

$$k_x(y) = \sum_{j=1}^{\infty} \overline{f_j(x)} T^* f_j(y), \tag{A-16}$$

Therefore,

$$K(x,\bar{y}) = \sum_{j=1}^{\infty} f_j(x)\overline{T^* f_j(y)}.$$

and we see that (A-5) follows. We also get

$$\|K(\cdot,\bar{y})\|^2_{\Phi_2} = \sum_{j=1}^{\infty} |T^* f_j(y)|^2. \tag{A-17}$$

Remark. It follows from (A-9) that $T \in \mathcal{L}(H_{\Phi_1}(\Omega_1), H_{\Phi_2}(\Omega_2))$ is of Hilbert–Schmidt class precisely when

$$\iint_{\Omega_1\times\Omega_2} |K(x,\bar{y})|^2 e^{-2(\Phi_1(y)+\Phi_2(x))}\, L(dy)\, L(dx) < \infty.$$

Remark. An alternative proof of Theorem A.1 can be obtained by applying the Schwartz kernel theorem directly to the linear continuous map

$$\Pi_{\Phi_2} T \Pi_{\Phi_1} : L^2(\Omega_1, e^{-2\Phi_1} L(dy_1)) \to L^2(\Omega_2, e^{-2\Phi_2} L(dy_2)).$$

Here

$$\Pi_{\Phi_j} : L^2(\Omega_j, e^{-2\Phi_j} L(dy_j)) \to H_{\Phi_j}(\Omega_j)$$

is the orthogonal projection. Writing the Schwartz kernel of $\Pi_{\Phi_2} T \Pi_{\Phi_1}$ in the form $K(x, \bar{y}) e^{-2\Phi_1(y)}$, we see that K should satisfy $\partial_{\bar{x}} K(x, \bar{y}) = 0$. Now the distribution kernel of the adjoint $\Pi_{\Phi_1} T^* \Pi_{\Phi_2}$ is given by $\overline{K(y, \bar{x})} e^{-2\Phi_2(y)}$, and it follows that $\partial_{\bar{x}}(\overline{K(y, \bar{x})}) = 0$. We get $\partial_x(K(y, \bar{x})) = 0$, so that $(\partial_{\bar{y}} K)(y, \bar{x}) = 0 \iff \partial_{\bar{y}} K(x, y) = 0$. We conclude that $K(x, y)$ is holomorphic in (x, y).

Appendix B. Positivity and Weyl quantization

The purpose of this appendix is to characterize the boundedness of the Weyl quantization of a symbol of the form $\exp(iF(x, \xi))$, where F a complex quadratic form, in the H_Φ-setting. See also [Hörmander 1995] for a related discussion in the context of L^2-boundedness.

Let $F = F(x, \xi)$ be a complex-valued holomorphic quadratic form on $\mathbb{C}^{2n}$ and let us consider formally the Weyl quantization of $e^{iF(x,\xi)}$,

$$Au(x) = \mathrm{Op}^w(e^{iF}) u(x) = \frac{1}{(2\pi)^n} \iint e^{i((x-y)\cdot\theta + F((x+y)/2, \theta))} u(y)\, dy\, d\theta. \tag{B-1}$$

The holomorphic quadratic form $(x - y) \cdot \theta + F\big(\frac{1}{2}(x + y), \theta\big)$ is a nondegenerate phase function in the sense of Hörmander and generates a canonical relation

$$\kappa : (y, \eta) \mapsto (x, \xi), \tag{B-2}$$

given by

$$\begin{aligned} x &= \frac{x+y}{2} - \tfrac{1}{2} F'_\xi\Big(\frac{x+y}{2}, \theta\Big), \quad \xi = \theta + \tfrac{1}{2} F'_x\Big(\frac{x+y}{2}, \theta\Big), \\ y &= \frac{x+y}{2} + \tfrac{1}{2} F'_\xi\Big(\frac{x+y}{2}, \theta\Big), \quad \eta = \theta - \tfrac{1}{2} F'_x\Big(\frac{x+y}{2}, \theta\Big). \end{aligned} \tag{B-3}$$

The graph is parametrized by $\rho = \big(\frac{1}{2}(x + y), \theta\big) \in \mathbb{C}^{2n}$ and (B-2), (B-3) take the form

$$\kappa : \rho + \tfrac{1}{2} H_F(\rho) \mapsto \rho - \tfrac{1}{2} H_F(\rho), \tag{B-4}$$

where $H_F(\rho) = (F'_\xi(\rho), -F'_x(\rho))$ is the Hamilton field of F at ρ.

We shall now give a criterion for when κ in (B-4) is a canonical transformation. Recall that $H_F(\rho) = \mathcal{F}\rho$, where

$$\mathcal{F} = \begin{pmatrix} F''_{\xi x} & F''_{\xi\xi} \\ -F''_{xx} & -F''_{x\xi} \end{pmatrix}$$

is the fundamental matrix of F (usually appearing as the linearization of a Hamilton vector field, which in our case is already linear). We have

$$\mathcal{F} = J F'', \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad F'' = \begin{pmatrix} F''_{xx} & F''_{x\xi} \\ F''_{\xi x} & F''_{\xi\xi} \end{pmatrix},$$

and we notice that $J^2 = -1,\ J^\top = -J$. Then (B-4) is the relation

$$(1+\mathcal{F}/2)\rho \mapsto (1-\mathcal{F}/2)\rho. \tag{B-5}$$

Now $\mathcal{F}$ is antisymmetric with respect to the bilinear form $\sigma(\nu,\mu) = J\nu\cdot\mu$; hence $1-\mathcal{F}/2$ is bijective if and only if its transpose $1+\mathcal{F}/2$ with respect to σ is bijective. We conclude that the following three statements are equivalent:

(i) κ is a canonical transformation.

(ii) $1-\mathcal{F}/2$ is bijective.

(iii) $1+\mathcal{F}/2$ is bijective.

In what follows, we shall assume that (i)–(iii) hold.

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let $\iota_{\Phi_0} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the corresponding antilinear involution, i.e., the unique antilinear map which is equal to the identity on Λ_{Φ_0}. We shall now proceed to characterize the positivity of the canonical transformation κ in (B-4) relative to Λ_{Φ_0}. Let

$$[\mu,\nu] = \tfrac{1}{2} b(\mu,\nu), \tag{B-6}$$

where $b(\mu,\nu)$ has been defined in (2-5). It is a Hermitian form and κ is positive relative to Λ_{Φ_0} precisely when

$$[\nu,\nu] \geq [\mu,\mu] \quad \text{for all } \nu,\mu \text{ with } \nu = \kappa(\mu). \tag{B-7}$$

By (B-4) this is equivalent to

$$\bigl[\rho - \tfrac{1}{2} H_F(\rho), \rho - \tfrac{1}{2} H_F(\rho)\bigr] \geq \bigl[\rho + \tfrac{1}{2} H_F(\rho), \rho + \tfrac{1}{2} H_F(\rho)\bigr], \quad \rho \in \mathbb{C}^{2n},$$

or equivalently,

$$\operatorname{Re}[H_F(\rho),\rho] \leq 0, \quad \rho \in \mathbb{C}^{2n}. \tag{B-8}$$

To simplify the following discussion, we shall make use of the invariance (exact Egorov theorem) under conjugation of A in (B-1) with a unitary metaplectic Fourier integral operator $U : L^2(\mathbb{R}^n) \to H_{\Phi_0}(\mathbb{C}^n)$ with the associated canonical transformation κ_U, mapping $\mathbb{R}^{2n}$ onto Λ_{Φ_0}. The operator $B = U^{-1}AU$ is the Weyl quantization of e^{iG}, where $G = F \circ \kappa_U$. Also $\iota_{\Phi_0} = \kappa_U \mathcal{C} \kappa_U^{-1}$, where $\mathcal{C}$ is the involution associated to $\mathbb{R}^{2n}$, which is just the map of ordinary complex conjugation. By abuse of notation we write F also for the pull back $F \circ \kappa_U$ and we continue the discussion in the case when Λ_{Φ_0} has been replaced with $\mathbb{R}^{2n}$ and ι_{Φ_0} with $\mathcal{C}$, $\mathcal{C}(\rho) = \bar{\rho}$. In this setting, (B-8) becomes

$$\operatorname{Im} \sigma(F'_\xi(\rho), -F'_x(\rho); \bar{x}, \bar{\xi}) \leq 0 \quad \text{for all } \rho = (x,\xi) \in \mathbb{C}^{2n};$$

i.e.,

$$\operatorname{Im}(F'_x(x,\xi)\cdot\bar{x} + F'_\xi(x,\xi)\cdot\bar{\xi}) \geq 0, \quad (x,\xi) \in \mathbb{C}^{2n},$$

or even more simply,

$$\operatorname{Im}(F''_{\rho\rho}\rho\cdot\bar{\rho}) \geq 0.$$

Writing $\rho = \mu + i\nu,\ \mu, \nu \in \mathbb{R}^{2n}$, we see that the last inequality is equivalent to

$$\operatorname{Im} F''\mu \cdot \mu + \operatorname{Im} F''\nu \cdot \nu \geq 0;$$

i.e.,

$$\operatorname{Im} F'' \geq 0;$$

i.e.,

$$\operatorname{Im} F \geq 0 \quad \text{on } \mathbb{R}^{2n}.$$

By the metaplectic invariance it follows that the positivity condition (B-7) is equivalent to

$$\operatorname{Im} F \geq 0 \quad \text{on } \Lambda_{\Phi_0}, \tag{B-9}$$

now with the original F.

Remark. The condition (B-9) is quite natural since we know that for ordinary symbols instead of e^{iF}, the natural contour of integration in (B-1) should be

$$\theta = \frac{2}{i}\partial_x \Phi\left(\frac{x+y}{2}\right);$$

see [Sjöstrand 1996; Hitrik and Sjöstrand 2018].

We summarize the discussion in this section in the following result.

Proposition B.1. *Let F be a holomorphic quadratic form on $\mathbb{C}^{2n}$ such that the fundamental matrix of F does not have the eigenvalues ± 2. Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. The canonical transformation associated to the Fourier integral operator* $\operatorname{Op}^w(e^{iF})$ *is positive relative to Λ_{Φ_0} precisely when*

$$\operatorname{Im} F|_{\Lambda_{\Phi_0}} \geq 0. \tag{B-10}$$

In particular, if (B-10) *holds, then the operator*

$$\operatorname{Op}^w(e^{iF}) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

is bounded.

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