

Pacific Journal of Mathematics

RATIO TESTS FOR CONVERGENCE OF SERIES

RALPH PALMER AGNEW

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1. Introduction. The following theorem was proved and used by Jehlke [2] to obtain elegant improvements of the classic tests of Gauss and Weierstrass for convergence of series of real and of complex terms.

THEOREM 1. *If the terms of two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are such that*

$$(1) \quad \frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n} (1 + c_n) \quad (n = 0, 1, \dots),$$

where $\sum_{n=0}^{\infty} c_n$ is absolutely convergent, then the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent or both divergent.

It is the main object of this note to prove that Theorem 1 is a best possible theorem in that no hypothesis weaker than the hypothesis that $\sum_{n=0}^{\infty} |c_n| < \infty$ is sufficient to imply the conclusion of the theorem. The final result, Theorem 4, is obtained from two preliminary theorems, Theorems 2 and 3, which seem to have independent interest.

2. Preliminary theorems. We first establish the following result.

THEOREM 2. *Let $c_n \neq -1$, $n = 0, 1, 2, \dots$. In order that the sequence $\{c_n\}$ be such that $\sum_{n=0}^{\infty} b_n$ converges whenever (1) holds and $\sum_{n=0}^{\infty} a_n$ converges, it is necessary and sufficient that*

$$(2) \quad \sum_{n=1}^{\infty} |(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})c_n| < \infty.$$

Proof. To prove Theorem 2, let (1) hold. Then

$$(3) \quad \frac{b_{n+1}}{a_{n+1}} = \frac{b_n}{a_n} (1 + c_n) \quad (n = 0, 1, 2, \dots),$$

and hence

Received November 8, 1950.

Pacific J. Math. 1(1951), 1-3.

$$(4) \quad \frac{b_n}{a_n} = \frac{b_0}{a_0} (1 + c_0)(1 + c_1) \cdots (1 + c_{n-1}) \quad (n = 1, 2, \dots).$$

Let

$$(5) \quad p_n = \frac{b_0}{a_0} (1 + c_0)(1 + c_1) \cdots (1 + c_{n-1}) \quad (n = 1, 2, \dots).$$

Then $b_n = p_n a_n$. But by a well-known theorem of Hadamard [1], $\sum_{n=0}^{\infty} p_n a_n$ converges whenever $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} |p_{n+1} - p_n| < \infty$. But (5) implies that

$$(6) \quad p_{n+1} - p_n = \frac{b_0}{a_0} (1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})c_n,$$

and the conclusion of Theorem 2 follows.

THEOREM 3. *Let $c_n \neq -1$, $n = 0, 1, 2, \dots$. In order that the sequence $\{c_n\}$ be such that $\sum_{n=0}^{\infty} a_n$ converges whenever (1) holds and $\sum_{n=0}^{\infty} b_n$ converges, it is necessary and sufficient that*

$$(7) \quad \sum_{n=1}^{\infty} \left| \frac{1}{1 + c_0} \frac{1}{1 + c_1} \cdots \frac{1}{1 + c_{n-1}} \frac{c_n}{1 + c_n} \right| < \infty.$$

Proof. Theorem 3 may be proved by revising the proof of Theorem 2 to use the relations

$$(8) \quad \frac{a_{n+1}}{a_n} = \frac{b_{n+1}}{b_n} \frac{1}{1 + c_n} \quad (n = 0, 1, 2, \dots)$$

instead of (1) or, which amounts to the same thing, replacing $1 + c_k$ by $1/(1 + c'_k)$ in (2) and then removing the primes.

3. Theorem. Our main result is the following.

THEOREM 4. *Let $c_n \neq -1$, $n = 0, 1, 2, \dots$. In order that this sequence be such that the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent or both divergent whenever (1) holds, it is necessary and sufficient that $\sum_{n=0}^{\infty} |c_n| < \infty$.*

Proof. To prove necessity, suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent or both divergent whenever (1) holds. Then, by Theorems 2 and 3, both (2) and (7)

hold. Denoting the n th terms of the series in (2) and (7) by u_n and v_n , we see that, as $n \rightarrow \infty$, we have $u_n \rightarrow 0$ and $v_n \rightarrow 0$ and hence

$$(9) \quad u_n v_n = \frac{c_n^2}{1 + c_n} \rightarrow 0 .$$

This implies that $c_n \rightarrow 0$ and hence that $|1/(1 + c_n)| > 1/2$ for n sufficiently great. This and (7) imply that

$$(10) \quad \sum_{n=1}^{\infty} \left| \frac{1}{1 + c_0} \frac{1}{1 + c_1} \cdots \frac{1}{1 + c_{n-1}} c_n \right| < \infty .$$

If we let $x_n = |(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})|$, then (2) and (10) imply that

$$(11) \quad \sum_{n=1}^{\infty} (x_n + x_n^{-1}) |c_n| < \infty .$$

But the mere fact that $x_n > 0$ implies that $(x_n + x_n^{-1}) \geq 2$, and it follows that $\sum_{n=0}^{\infty} |c_n| < \infty$. This proves necessity. To prove sufficiency, suppose that $\sum_{n=0}^{\infty} |c_n| < \infty$. Then the infinite product $\prod(1 + c_k)$ converges to a number not zero, and this means that each of $(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})$ and $[(1 + c_0)(1 + c_1) \cdots (1 + c_n)]^{-1}$ converges to a number not zero. This and $\sum_{n=0}^{\infty} |c_n| < \infty$ imply (2) and (7). Therefore Theorems 2 and 3 imply that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent or both divergent. This completes the proof of Theorem 4.

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Vari-Type Composition by
Cecile Leonard
Ruth Stafford

With the cooperation of
E. F. Beckenbach
E. G. Straus

Printed in the United States of America by
Edwards Brothers, Inc., Ann Arbor, Michigan

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES
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Pacific Journal of Mathematics

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November, 1951

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