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RATIO TESTS FOR CONVERGENCE OF SERIES

RALPH PALMER AGNEW

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1. Introduction. The following theorem was proved and used by Jehlke [2] to obtain elegant improvements of the classic tests of Gauss and Weierstrass for convergence of series of real and of complex terms.

THEOREM 1. *If the terms of two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are such that*

$$(1) \quad \frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n} (1 + c_n) \quad (n = 0, 1, \dots),$$

where $\sum_{n=0}^{\infty} c_n$ is absolutely convergent, then the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent or both divergent.

It is the main object of this note to prove that Theorem 1 is a best possible theorem in that no hypothesis weaker than the hypothesis that $\sum_{n=0}^{\infty} |c_n| < \infty$ is sufficient to imply the conclusion of the theorem. The final result, Theorem 4, is obtained from two preliminary theorems, Theorems 2 and 3, which seem to have independent interest.

2. Preliminary theorems. We first establish the following result.

THEOREM 2. *Let $c_n \neq -1$, $n = 0, 1, 2, \dots$. In order that the sequence $\{c_n\}$ be such that $\sum_{n=0}^{\infty} b_n$ converges whenever (1) holds and $\sum_{n=0}^{\infty} a_n$ converges, it is necessary and sufficient that*

$$(2) \quad \sum_{n=1}^{\infty} |(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})c_n| < \infty.$$

Proof. To prove Theorem 2, let (1) hold. Then

$$(3) \quad \frac{b_{n+1}}{a_{n+1}} = \frac{b_n}{a_n} (1 + c_n) \quad (n = 0, 1, 2, \dots),$$

and hence

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$$(4) \quad \frac{b_n}{a_n} = \frac{b_0}{a_0} (1 + c_0)(1 + c_1) \cdots (1 + c_{n-1}) \quad (n = 1, 2, \dots).$$

Let

$$(5) \quad p_n = \frac{b_0}{a_0} (1 + c_0)(1 + c_1) \cdots (1 + c_{n-1}) \quad (n = 1, 2, \dots).$$

Then $b_n = p_n a_n$. But by a well-known theorem of Hadamard [1], $\sum_{n=0}^{\infty} p_n a_n$ converges whenever $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} |p_{n+1} - p_n| < \infty$. But (5) implies that

$$(6) \quad p_{n+1} - p_n = \frac{b_0}{a_0} (1 + c_0)(1 + c_1) \cdots (1 + c_{n-1}) c_n,$$

and the conclusion of Theorem 2 follows.

THEOREM 3. *Let $c_n \neq -1$, $n = 0, 1, 2, \dots$. In order that the sequence $\{c_n\}$ be such that $\sum_{n=0}^{\infty} a_n$ converges whenever (1) holds and $\sum_{n=0}^{\infty} b_n$ converges, it is necessary and sufficient that*

$$(7) \quad \sum_{n=1}^{\infty} \left| \frac{1}{1 + c_0} \frac{1}{1 + c_1} \cdots \frac{1}{1 + c_{n-1}} \frac{c_n}{1 + c_n} \right| < \infty.$$

Proof. Theorem 3 may be proved by revising the proof of Theorem 2 to use the relations

$$(8) \quad \frac{a_{n+1}}{a_n} = \frac{b_{n+1}}{b_n} \frac{1}{1 + c_n} \quad (n = 0, 1, 2, \dots)$$

instead of (1) or, which amounts to the same thing, replacing $1 + c_k$ by $1/(1 + c'_k)$ in (2) and then removing the primes.

3. Theorem. Our main result is the following.

THEOREM 4. *Let $c_n \neq -1$, $n = 0, 1, 2, \dots$. In order that this sequence be such that the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent or both divergent whenever (1) holds, it is necessary and sufficient that $\sum_{n=0}^{\infty} |c_n| < \infty$.*

Proof. To prove necessity, suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent or both divergent whenever (1) holds. Then, by Theorems 2 and 3, both (2) and (7)

hold. Denoting the n th terms of the series in (2) and (7) by u_n and v_n , we see that, as $n \rightarrow \infty$, we have $u_n \rightarrow 0$ and $v_n \rightarrow 0$ and hence

$$(9) \quad u_n v_n = \frac{c_n^2}{1 + c_n} \rightarrow 0 .$$

This implies that $c_n \rightarrow 0$ and hence that $|1/(1 + c_n)| > 1/2$ for n sufficiently great. This and (7) imply that

$$(10) \quad \sum_{n=1}^{\infty} \left| \frac{1}{1 + c_0} \frac{1}{1 + c_1} \cdots \frac{1}{1 + c_{n-1}} c_n \right| < \infty .$$

If we let $x_n = |(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})|$, then (2) and (10) imply that

$$(11) \quad \sum_{n=1}^{\infty} (x_n + x_n^{-1}) |c_n| < \infty .$$

But the mere fact that $x_n > 0$ implies that $(x_n + x_n^{-1}) \geq 2$, and it follows that $\sum_{n=0}^{\infty} |c_n| < \infty$. This proves necessity. To prove sufficiency, suppose that $\sum_{n=0}^{\infty} |c_n| < \infty$. Then the infinite product $\prod(1 + c_k)$ converges to a number not zero, and this means that each of $(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})$ and $[(1 + c_0)(1 + c_1) \cdots (1 + c_n)]^{-1}$ converges to a number not zero. This and $\sum_{n=0}^{\infty} |c_n| < \infty$ imply (2) and (7). Therefore Theorems 2 and 3 imply that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent or both divergent. This completes the proof of Theorem 4.

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