Pacific Journal of Mathematics

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Vol. 1, No. 1 November 1951

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1. Introduction. A transformation of the form

$$\tau' = \frac{a\tau + b}{c\tau + d} , \tag{1.1}$$

where a, b, c, d are rational integers satisfying

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb = 1 , \tag{1.2}$$

is called a *modular transformation.* Without loss of generality we may assume $c \geq 0$. A function $f(\tau)$, analytic in the upper halfplane $\mathfrak{J}(\tau) > 0$, and satisfying the functional equation

$$f(\tau) = (c\tau + d)^k \; f\left(\frac{a\tau + b}{c\tau + d}\right), \tag{1.3}$$

is called a *modular form of dimension k.* An example of a modular form is the discriminant

$$\Delta(\tau) = \exp\{2\pi i\tau\} \prod_{m=1}^{\omega} (1 - \exp\{2\pi i m\tau\})^{24} , \tag{1.4}$$

which is of dimension -12; that is, it satisfies the equation*

$$\Delta(\tau') = (c\tau + d)^{12}\Delta(\tau) \ . \tag{1.5}$$

An important role in the theory of modular functions is played by the function

$$\eta(\tau) = \exp\left\{\frac{\pi i \tau}{12}\right\} \prod_{m=1}^{\infty} (1 - \exp\{2\pi i m\tau\}) , \tag{1.6}$$

Received June 12, 1950.

*Cf. Hurwitz [6] ; however, he gives this formula only in homogeneous coordinates.

Pacific J. Math. 1 (1951), 83-95.

which is the 24th root of $\Delta(\tau)$. The transformation formula for this function may be obtained from (1.5) and is conveniently written as:

$$\eta(\tau') = \eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon\sqrt{-i(c\tau + d)}\;\eta(\tau)\,. \tag{1.7}$$

Since we have assumed $c \geq 0$ and $\mathfrak{J}(\tau) > 0$, the radicand has a nonnegative real part. By the square root we always mean the principal branch; that is, $\mathfrak{R}(\sqrt{\;}) > 0$. The ϵ appearing in (1.7) is a 24th root of unity. The purpose of the present paper is to determine this ϵ completely.

Investigations concerning this root of unity were carried out first by Dedekind [2] and later by Tannery and Molk [10] and Rademacher [8; 9]. However, they use the theory of log $\eta(\tau)$, which requires much more than is needed for this purpose. Hurwitz discusses only $[\Delta(\tau)]^{1/12}$ and remarks that the transformation formula of $\eta(\tau)$ can be obtained by means of θ-functions. The investigations of Hermite [5] are likewise not sufficient for our purpose, because he discusses only $\eta^3(\tau)$, and therefore a third root of unity remains still undetermined.

In the following, we shall approach the determination of ϵ directly by investigations of the function $\eta(\tau)$, which, by a well-known formula due to Euler, can be written as the following sum:

$$\begin{aligned}\eta(\tau) &= \exp\left\{\frac{\pi i \tau}{12}\right\} \sum_{\lambda=-\infty}^{+\infty} (-1)^{\lambda} \exp\{\pi i \tau \lambda(3\lambda - 1)\} \\ &= \sum_{\lambda=-\infty}^{+\infty} (-1)^{\lambda} \exp\left\{3\pi i \tau\left(\lambda - \frac{1}{6}\right)^2\right\}.\end{aligned} \tag{1.8}$$

Our starting point is formula (1.8); our principal tools are a Poisson transformation formula and Gaussian sums.

2. Application of a Poisson formula. We introduce a new variable z with $\mathfrak{R}(z) > 0$ by the substitution*

$$\tau' = \frac{iz}{c} + \frac{a}{c}\,, \qquad c > 0;\;\; (a, c) = 1\,, \tag{2.1}$$

* This requires $c \neq 0$, but the case $c = 0$ is trivial.

and obtain, from (1.8)

$$\begin{aligned}(2.2)\qquad \eta\left(\frac{a}{c}+\frac{iz}{c}\right) &= \sum_{\lambda=-\infty}^{+\infty} (-1)^{\lambda} \exp\left\{\frac{3\pi i}{c}(a+iz)\left(\lambda-\frac{1}{6}\right)^2\right\} \\ &= \sum_{j \bmod 2c} \exp \pi i\left\{j+\frac{3a}{c}\left(j-\frac{1}{6}\right)^2\right\} \\ &\qquad \times \sum_{q=-\infty}^{+\infty} \exp\left\{-\frac{3\pi z}{c}\left(2cq+j-\frac{1}{6}\right)^2\right\}.\end{aligned}$$

To the inner sum,

$$F_c(z) = \sum_{q=-\infty}^{+\infty} \exp\left\{-12\pi cz\left(q+\frac{6j-1}{12c}\right)^2\right\},$$

we apply Poisson's formula (cf. [11]),

$$\sum_{m=-\infty}^{+\infty} \exp\{-\pi(m+\alpha)^2 t\} = \frac{1}{\sqrt{t}} \sum_{m=-\infty}^{+\infty} \exp\left\{2\pi i m\alpha - \frac{\pi m^2}{t}\right\}, \qquad \Re(t) > 0,$$

and obtain

$$F_c(z) = \frac{1}{2\sqrt{3cz}} \sum_{q=-\infty}^{+\infty} \exp\left\{2\pi i q\,\frac{6j-1}{12c} - \frac{\pi q^2}{12cz}\right\}.$$

Putting this in (2.2), we get:

$$(2.3)\qquad \eta\left(\frac{a}{c}+\frac{iz}{c}\right) = \frac{1}{\sqrt{3cz}} \sum_{q=-\infty}^{+\infty} \exp\left\{\frac{-\pi q^2}{12cz}\right\} T_q(c),$$

where

$$\begin{aligned}T_q(c) &= \frac{1}{2} \sum_{j \bmod 2c} \exp \pi i\left\{j+\frac{3a}{c}\left(j-\frac{1}{6}\right)^2 + q\,\frac{6j-1}{6c}\right\} \\ &= \frac{1}{2}\exp \pi i\left\{\frac{a-2q}{12c}\right\}[1+\exp \pi i\{3ac+c-a+q\}] \\ &\qquad \times \sum_{j=1}^{c} \exp\left\{\frac{\pi i}{c}[3aj^2+j(c-a+q)]\right\}.\end{aligned}$$

But, a and c being coprime, and thus

$$3ac + c - a \equiv 1 \pmod 2,$$

only the T_q with odd subscripts actually appear so that we have

$$(2.4)\quad T_{2r+1}(c) = \exp \pi i \left\{\frac{a - 4r - 2}{12c}\right\} \sum_{j=1}^{c} \exp\left\{\frac{\pi i}{c} \left[3aj^2 + j(c-a+1+2r)\right]\right\}.$$

In order to have a complete square in the exponent we multiply each term of the sum by

$$\exp \pi i \left\{j \frac{ad-1}{c}(c-1+2r)\right\} = \exp \pi i \{jb(c+1)\}.$$

As we do not wish to change T_{2r+1} by this multiplication, we have to assume that, for c even, b also is even. Using the abbreviation

$$(2.5)\quad \beta = cd + d - 1,$$

we obtain from (2.4):

$$(2.6)\quad T_{2r+1}(c) = \exp \pi i \left\{\frac{a-4r-2}{12c}\right\} \sum_{j=1}^{c} \exp\left\{\frac{\pi i a}{12c}\left[36j^2 + 12j(cd+d-1+2rd)\right]\right\}$$

$$= \exp \pi i \left\{\frac{a - a\beta^2 - 2}{12c} - \frac{r}{3c}(ad^2 r + ad\beta + 1)\right\}$$

$$\times \sum_{j=1}^{c} \exp\left\{\frac{\pi i a}{12c}(6j + \beta + 2rd)^2\right\}.$$

In the sum appearing here, j can be taken as running over any full residue system mod c, because $\beta \equiv c \pmod 2$ and therefore the sum remains unchanged if j is replaced by $j + c$. Consequently, β can be chosen arbitrarily, mod 6, and $T_{2r+1}(c)$ can be simplified by the substitution $r = 3\mu + \nu$. We note that

$$\exp \pi i \left\{\frac{-r}{3c}(ad^2 r + ad\beta + 1)\right\}$$

$$= \exp \pi i \left\{\frac{-\mu}{c}(3\mu d + 3\mu bcd + 2d\nu + bc\beta + cd + d)\right.$$

$$\left. - \frac{\nu}{3c}(d\nu + bcd\nu + bc\beta + cd + d)\right\};$$

and considering

$$\exp\{-\pi i\mu(b\beta+d+3\mu bd)\} = \exp\{-\pi i\mu(bcd-b+d)\} = \exp\{\pi i\mu\},$$

we obtain

$$T_{6\mu+2\nu+1}(c) = \exp\pi i\left\{\frac{a-a\beta^2-2}{12c}-\frac{\nu}{3}\left[bd\nu+d+b\beta+\frac{d}{c}(\nu+1)\right]\right.$$
$$\left.-\frac{\mu}{c}\left[3\mu d+d(1+2\nu)+c\right]\right\} H_{a,c}(\beta+2\nu d)$$

with the abbreviation

$$(2.7)\qquad H_{a,c}(\beta) = \sum_{j \bmod c} \exp\left\{\frac{\pi i a}{12c}(6j+\beta)^2\right\}, \qquad \beta \equiv c \pmod 2.$$

Looking back to (2.3), we see that the result we have obtained so far may be written as:

$$(2.8)\quad \eta\left(\frac{a}{c}+\frac{iz}{c}\right) = \frac{1}{\sqrt{3cz}}\exp\pi i\left\{\frac{a-a\beta^2-2}{12c}\right\}$$
$$\times\sum_{\nu=0}^{2}\exp\left\{\frac{-\pi i\nu}{3}\left[bd\nu+d+b\beta+\frac{d}{c}(\nu+1)\right]\right\}U_\nu(z)\,H_{a,c}(\beta+2d\nu),$$

with

$$U_\nu(z) = \sum_{\mu=-\infty}^{+\infty}\exp\left\{\pi i\left[\mu-\frac{3d}{c}\mu^2-\frac{d}{c}\mu(2\nu+1)\right]-\frac{3\pi}{cz}\left(\mu+\frac{2\nu+1}{6}\right)^2\right\}.$$

These expressions are easy to sum, since, according to (1.8), we have

$$U_0(z) = \sum_{\mu=-\infty}^{+\infty}\exp\left\{\pi i\left[\mu-\frac{3d}{c}\left(\mu^2+\frac{\mu}{3}\right)\right]-\frac{3\pi}{cz}\left(\mu+\frac{1}{6}\right)^2\right\}$$
$$= \exp\left\{\frac{\pi id}{12c}\right\}\eta\left(-\frac{d}{c}+\frac{i}{cz}\right);$$

and, replacing μ by $-\mu-1$, we see that

$$U_1(z) = -U_1(z), \quad \text{or} \quad U_1(z) = 0\ ,$$

$$U_2(z) = -\exp\left\{\pi i\ \frac{2d}{c}\right\} U_0(z)\ .$$

Now, by the meaning of z in (2.1), we get

$$-\frac{d}{c} + \frac{i}{cz} = \frac{-d\tau' + b}{c\tau' - a} = \tau\ ,$$

and have therefore:

$$\eta(\tau') = \frac{1}{\sqrt{3c}} \exp \pi i \left\{\frac{a(1-\beta^2)-2+d}{12c}\right\} \times \left[H_{a,c}(\beta) - \exp\left\{\frac{-2\pi i}{3}\ (d+2bd+b\beta)\right\} \times H_{a,c}(\beta+4d)\right] \sqrt{-i(c\tau+d)}\ \ \eta(\tau)\ . \tag{2.9}$$

Comparing this with (1.7), we see that we have obtained so far:

$$\begin{aligned} \epsilon &= \frac{1}{\sqrt{3c}} \exp \pi i \left\{\frac{a(1-\beta^2)-2+d}{12c}\right\} \\ &\quad \times \left[H_{a,c}(\beta) - \exp\left\{\frac{-2\pi i}{3}\ (d+2bd+b\beta)\right\} H_{a,c}(\beta+4d)\right] \\ &= \frac{1}{\sqrt{3c}} \exp \pi i \left\{\frac{bd(1-c^2)-cd}{12} + \frac{(1-d)(b+ad)}{6}\right\} \\ &\quad \times \left[H_{a,c}(\beta) - \exp\left\{\frac{-2\pi i}{3}\ (d+2bd+b\beta)\right\} H_{a,c}(\beta+4d)\right] \end{aligned} \tag{2.91}$$

and it remains to be shown that this is a root of unity.

3. Reduction to Gaussian sums. The sums $H_{a,c}(\beta)$ which appear in (2.91) are defined in (2.7) only for $\beta \equiv c \pmod 2$. In this section, however, it will be more convenient to consider the more general sums*

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*We have used the letters h and k instead of a and c in order to indicate that the investigations of this section are independent from our previous results.

$$H_{h,k}(\gamma) = \frac{1}{2} \sum_{j \bmod 2k} \exp\left\{\frac{\pi i h}{12k} (6j + \gamma)^2\right\}, \tag{3.1}$$

with no restriction on γ. These sums can be expressed in terms of Gaussian sums

$$G(h, k) = \sum_{j \bmod k} \exp\left\{\frac{2\pi i h}{k} j^2\right\}. \tag{3.2}$$

Comparing the definitions (3.1) and (3.2) one finds immediately that:

$$H_{h,k}(0) + H_{h,k}(1) + H_{h,k}(2) + H_{h,k}(3) + H_{h,k}(4) + H_{h,k}(5) = \frac{1}{4} G(h, 24k),$$

$$H_{h,k}(0) + H_{h,k}(2) + H_{h,k}(4) = \frac{1}{2} G(h, 6k),$$

$$H_{h,k}(0) + H_{h,k}(3) = \frac{1}{4} G(3h, 8k).$$

If we consider that

$$H_{h,k}(-\gamma) = H_{h,k}(\gamma) = H_{h,k}(\gamma + 6n),$$

we get the following relations:

$$H_{h,k}(0) = \frac{1}{2} G(3h, 2k), \tag{3.31}$$

$$H_{h,k}(3) = \frac{1}{4} G(3h, 8k) - \frac{1}{2} G(3h, 2k), \tag{3.32}$$

$$H_{h,k}(2) = \frac{1}{4} G(h, 6k) - \frac{1}{4} G(3h, 2k), \tag{3.33}$$

$$H_{h,k}(1) = \frac{1}{8} G(h, 24k) - \frac{1}{8} G(3h, 8k) - \frac{1}{4} G(h, 6k) + \frac{1}{4} G(3h, 2k). \tag{3.34}$$

In order to obtain the sums $H_{h,k}(\gamma)$ explicitly, the following rules concerning Gaussian sums may be useful.*

* For the formulas (3.41)–(3.47) see [1] or [3]; (3.46) may also be found in [7].

As elementary consequences of the definition (3.2) we have:

$$G(mh, mk) = mG(h,k) \qquad m > 0 \tag{3.41}$$

$$G(h, k_1k_2) = G(hk_1, k_2)\ G(hk_2, k_1) \qquad (k_1, k_2) = 1 \tag{3.42}$$

$$G(m^2h, k) = G(h,k) \qquad (m,k) = 1 \tag{3.43}$$

$$G(h, m^2k) = mG(h,k) \qquad (m,h) = 1\,; \quad m > 0 \text{ and odd.} \tag{3.44}$$

The following results, due to Gauss [4], are a little deeper:

$$G(h_1h_2, k) = \left(\frac{h_1}{k}\right) G(h_2, k) \qquad (h_1h_2, k) = 1\,, \quad k \text{ odd} \tag{3.45}$$

$$G(1,k) = \sqrt{k}\ i^{[(k-1)/2]^2} \qquad k \text{ odd} \tag{3.46}$$

$$G(h, 2^\alpha) = \begin{cases} 0 & h \text{ odd}\,, \quad \alpha = 1 \\ 2^{(\alpha+1)/2} \left(\dfrac{2}{h}\right)^{\alpha+1} e^{\pi i h/4} & h \text{ odd}\,, \quad \alpha \geq 2\,. \end{cases} \tag{3.47}$$

The symbol $\left(\frac{h}{k}\right)$ is the Jacobi symbol.

The following discussion may be restricted to the case $\gamma \equiv k \pmod 2$, which will be sufficient for our purpose. Furthermore, we put* throughout $k = 2^\lambda k_1$ (k_1 being odd), and have then to distinguish whether 3 does or does not divide k_1.

Assume first $3 \mid k_1$. Then we find, using (3.41) and (3.44), that

$$H_{h,k}(1) = 0\,, \qquad H_{h,k}(2) = 0\,; \tag{3.51}$$

and, applying (3.41), (3.42), (3.44), (3.45), and (3.47), we obtain:

$$H_{h,k}(0) = 2^{\lambda/2}\left(\frac{2}{h}\right)^\lambda \exp\left\{\frac{3}{4}\pi i h k_1\right\} G(2h, 3k_1)\,, \tag{3.52}$$

$$H_{h,k}(3) = \exp\left\{\frac{3}{4}\pi i h k\right\}\ G(2h, 3k)\,. \tag{3.53}$$

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*We do this in order to avoid the reciprocity law for Gaussian sums which would require additional distinctions concerning the sign of h.

As a consequence of (3.46) we have:

$$G(1,\,3k) = \sqrt{3k}\ \ \exp\left\{\frac{\pi i}{8}\ (3k-1)^2\right\} = -\sqrt{3}\ \ \exp\left\{\frac{-\pi i k}{2}\right\}\ G(1,k)\ ,$$

and therefore, according to (3.45),

$$G(2h, 3k) = \left(\frac{2h}{3k}\right)\ G(1, 3k) = -\left(\frac{2h}{3}\right)\ \sqrt{3}\ \ \exp\left\{\frac{-\pi i k}{2}\right\}\ G(2h, k)\ .$$

This formula enables us to express (3.52) and (3.53) in the single formula:

$$(3.6)\quad H_{h,k}\,(k) = \sqrt{3}\ 2^{\lambda/2}\ \left(\frac{h}{3}\right)\ \exp \pi i \left\{\frac{k_1\,(h-1)}{2} + \frac{hk_1}{4} + \lambda\frac{h^2-1}{8}\right\}\ G(2h, k_1)\ .$$

In case $3\nmid k_1$, by use of (3.42) and (3.43) we can express the more complicated sums $H_{h,k}(1)$ and $H_{h,k}(2)$ by $H_{h,k}(3)$ and $H_{h,k}(0)$, respectively:

$$(3.71)\qquad H_{h,k}\,(1) = \exp\left\{\frac{4}{3}\ \pi\, ihk\right\}\ H_{h,k}\,(3)\ ,$$

$$(3.72)\qquad H_{h,k}\,(2) = \exp\left\{\frac{4}{3}\ \pi\, ihk\right\} H_{h,k}\,(0)\ .$$

More generally, the following recursion formula holds:

$$(3.73)\qquad H_{h,k}\,(\gamma+2n) = \exp\left\{\frac{\pi i}{3}\ (\gamma+n)\ nhk\right\}\ H_{h,k}\,(\gamma)\ .$$

In order to compute $H_{h,k}(0)$ and $H_{h,k}(3)$, we apply (3.42), (3.43), (3.45), and (3.47) to obtain:

$$H_{h,k}\,(3) = \left(\frac{k}{3}\right) \exp \pi i \left\{\frac{k-1}{2} + \frac{3hk}{4}\right\}\ G(2h, k)\ ,$$

$$H_{h,k}\,(0) = \left(\frac{k}{3}\right)\ 2^{\lambda/2}\left(\frac{2}{h}\right)^{\lambda} \exp \pi i \left\{\frac{k_1-1}{2} + \frac{3hk_1}{4}\right\}\ G(2h, k_1)\ .$$

Applying this on (3.71) and (3.72), and considering

$$\exp \pi i \left\{\frac{4}{3}\ hk + \frac{3}{4}\ hk_1\right\} = \exp \pi i \left\{\frac{hk}{12} + \frac{3}{4}\ h(k_1 - k)\right\}\ ,$$

we can combine (3.71) and (3.72) into:

$$(3.8)\quad H_{h,k}(k) = 2^{\lambda/2}\left(\frac{k}{3}\right)$$
$$\times \exp \pi i\left\{\frac{hk}{12} + \frac{3}{4}h(k_1-k) + \frac{k_1-1}{2} + \lambda\frac{h^2-1}{8}\right\} G(2h, k_1) \, .$$

4. Determination of the root of unity. Now we go back to our result (2.9) and consider the following expression:

$$(4.1)\qquad \rho = \frac{1}{\sqrt{3c}} \exp\left\{\frac{\pi i}{6}(1-d)(b+ad)\right\}$$
$$\times \left[H_{a,c}(\beta) - \exp\left\{\frac{-2\pi i}{3}(d+2bd+b\beta)\right\} H_{a,c}(\beta+4d)\right] .$$

According to the results of the preceding section, we have to distinguish whether c is divisible by 3 or not and to keep in mind that $c = 2^{\lambda}c_1$, c_1 odd.

Let us assume first $3 \mid c$; according to (3.51) we know that:

$$H_{a,c}(\beta) = H_{a,c}(dc+d-1) = 0 \qquad \text{if } d \equiv -1 \pmod 3 ,$$
$$H_{a,c}(\beta+4d) = H_{a,c}(dc+5d-1) = 0 \qquad \text{if } d \equiv +1 \pmod 3 .$$

Therefore we have:

$$(4.2)\qquad \rho = \left(\frac{d}{3}\right)\frac{1}{\sqrt{3c}} \exp \pi i\left\{\frac{1}{6}(1-d)(b+ad) + \frac{2}{3}(d-1)(1+b)\right\} H_{a,c}(c)$$
$$= \left(\frac{a}{3}\right)\frac{1}{\sqrt{3c}} \exp\left\{\frac{\pi i}{2}(d-1)(b+ad)\right\} H_{a,c}(c) .$$

Considering that

$$\exp\left\{\frac{\pi i}{2}\left[(d-1)(b+ad+c) + (a-1)(c_1-c)\right]\right\} = 1 ,$$

and therefore that

$$\exp \pi i \left\{\frac{1}{2}\,(d-1)(b+ad) + \frac{1}{2}\,(a-1)\,c_1\right\}$$

$$= \exp\left\{\frac{\pi i}{2}\,[(d-1)(b+ad+c) + (a-1)(c_1-c) - c(d-a)]\right\}$$

$$= \exp\left\{\frac{\pi i}{6}\,c(d-a)\right\},$$

we get from (4.2) and (3.6):

$$\rho = \frac{1}{\sqrt{c_1}}\,\exp \pi i \left\{\frac{a}{4}\,(c_1-c) + \frac{cd}{6} + \frac{ac}{12} + \lambda\frac{a^2-1}{8}\right\} G(2a, c_1)\,. \tag{4.3}$$

In case $3\nmid c$, we can apply (3.73), which gives us

$$H_{a,c}\,(\beta + 4d) = \exp\left\{\frac{2\,\pi i}{3}\,(\beta + 2d)\;acd\right\}\,H_{a,c}\,(\beta)$$

$$= \exp\left\{\frac{2\,\pi i}{3}\,(b\beta + 2bd + d - c)\right\}\,H_{a,c}\,(\beta)\,,$$

and obtain from (4.1):

$$\rho = \frac{1}{\sqrt{3c}}\,\exp\left\{\frac{\pi i}{6}\,(1-d)(b+ad)\right\}\left[1 - \exp\left\{\frac{-2\pi i\,c}{3}\right\}\right]\,H_{a,c}\,(\beta)$$

$$= \frac{1}{\sqrt{c}}\left(\frac{c}{3}\right)\,\exp \pi i \left\{\frac{1}{6}\,(1-d)(b+ad) - \frac{1}{2} + \frac{2c}{3}\right\} H_{a,c}\,(\beta)\,.$$

Now we apply (3.37) once more, putting

$$H_{a,c}\,(\beta) = H_{a,c}\,(c + \beta - c) = \exp\left\{\frac{\pi i}{3}\left(c + \frac{\beta-c}{2}\right)\frac{\beta-c}{2}\,ac\right\}\,H_{a,c}\,(c)$$

$$= \exp\left\{\frac{\pi i}{12}\,(\beta^2 - c^2)\;ac\right\}\,H_{a,c}\,(c)\,.$$

Using (3.8) and considering

$$\exp\left\{\frac{\pi i}{12}\,(\beta^2-c^2)\;ac\right\}$$

$$=\exp\left\{\frac{\pi i}{12}\,\left[ac(c^2-1)(d^2-1)+2ac(d-1)(cd+d)\right]\right\}$$

$$=\exp\left\{\frac{\pi i}{6}\,(d-1)(bc+c+b+c^2)\right\}$$

$$=\exp\pi i\left\{\frac{1}{6}\,(d-1)(b+ad)-\frac{1}{2}\,(d-1)(c^2-1)\;+\frac{c}{6}\,(d-1)\right\},$$

$$\exp\left\{\frac{\pi i}{2}\,[(a-1)(c_1-c)\;-\;(d-1)(c^2-1)]\right\}\;=1\,,$$

we see that the expression for ρ becomes again (4.3). Therefore, we have in all cases:

$$\epsilon=\exp\pi i\left\{\frac{1}{12}\,\left[bd(1-c^2)+c(a+d)\right]+a\,\frac{c_1-c}{4}+\lambda\frac{a^2-1}{8}\right\}\times\frac{1}{\sqrt{c_1}}\,G(2a,c_1)\,, \tag{4.4}$$

with the only restriction that, for even c, b also has to be even.

In order to show that our formula (4.4) holds even if this condition is not satisfied, we put

$$\tau'=\frac{a\tau+b}{c\tau+d}\,, \qquad\qquad c \text{ even},\quad b \text{ odd},$$

$$\tau^*=\frac{(a+c)\;\tau+(b+d)}{c\tau+d}=\tau'+1\,.$$

Then, for τ^*, formula (4.4) holds; considering

$$\eta(\tau+1)=\exp\left\{\frac{-\pi i}{12}\right\}\;\eta(\tau)\,,$$

which is an immediate consequence of (1.6), we find:

$$\eta(\tau^*) = \epsilon^*\eta(\tau) = \exp\left\{\frac{-\pi i}{12}\right\} \eta(\tau') = \exp\left\{\frac{-\pi i}{12}\right\} \epsilon\eta(\tau)$$

$$\epsilon = \exp\left\{\frac{\pi i}{12}\right\} \epsilon^* . \tag{4.5}$$

Now, if we compute ϵ^* by means of (4.4), and then ϵ, using (4.5), the result will be exactly the same as we get computing ϵ directly by means of (4.4).

Finally, we can omit the Gaussian sums in (4.3) and, using (3.45) and (3.46), obtain:

$$\epsilon = \left(\frac{a}{c_1}\right) \times \exp \pi i \left\{\frac{1}{12}\left[bd(1-c^2) + c(a+d)\right] + \frac{1-c_1}{4} + a\,\frac{c-c_1}{4} + \lambda\frac{a^2-1}{8}\right\} . \tag{4.6}$$

This formula agrees with the one given by Tannery and Molk [10, p. 112] .

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Vari-Type Composition by
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Printed in the United States of America by
Edwards Brothers, Inc., Ann Arbor, Michigan

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES

Pacific Journal of Mathematics

Vol. 1, No. 1 November, 1951