

Pacific Journal of Mathematics

ON TOTALLY DIFFERENTIABLE AND SMOOTH FUNCTIONS

HASSLER WHITNEY

ON TOTALLY DIFFERENTIABLE AND SMOOTH FUNCTIONS

HASSLER WHITNEY

1. Introduction. H. Rademacher has proved that a function of n variables satisfying a Lipschitz condition is totally differentiable a. e. (almost everywhere) (see, for instance, Saks, [6, pp. 310-311]). It was discovered by H. Federer (though not stated as a theorem; see [2, p. 442]) that if f is totally differentiable a. e. in the bounded set P , then there is a closed set $Q \subset P$ with the measure $|P - Q|$ as small as desired, such that f is smooth (continuously differentiable) in Q ; that is, the values of f in Q may be extended through space so that the resulting function g is smooth there.

Theorem 1 of the present paper strengthens the latter theorem by showing that f is approximately totally differentiable a. e. in P if and only if Q exists with the above property. The rest of the paper gives further theorems in the direction of Federer's Theorem, as follows.

Suppose the domain of definition of f were a bounded open set P . Then in applying the part (a) \rightarrow (c) of Theorem 1, we might alter f in a set $P - Q$ which included a neighborhood of the boundary of P . In applications, it might be important to keep the values of f in most of a subset close to the boundary of P , or in most of some other subset. That such can be done follows from Theorem 2.

If f satisfies a Lipschitz condition, Theorem 3 shows that g may be made to satisfy a Lipschitz condition also, with a constant which equals a number ρ_n (depending on the number n of variables only) times the constant for f ; in the case of one variable, we may take $\rho_1 = 1$.

If we weaken the assumption on f , assuming only that it is measurable, then Lusin's Theorem shows that we can alter f on a set of arbitrarily small measure, giving a continuous function g . In the other direction, suppose we assume that f (defined in an open set) has continuous m th partial derivatives, and that these derivatives are totally differentiable a. e. Then Theorem 4 shows that we may alter f on a set of arbitrarily small measure, giving a function g which has continuous partial derivatives of order $m + 1$. For the case of one variable, this is essentially a theorem of Marcinkiewicz, [5, Theorem 3].

Examples show that the hypotheses in the theorems cannot be materially

weakened without altering the conclusions. For instance, define a function ϕ of one variable as follows. Let $\phi_0(t)$ be the distance from t to the nearest integer. Using any sufficiently large integer a , set

$$\phi_i(t) = 2^i \phi_0(a^i t)/a^i, \quad \phi(t) = \sum_{i=0}^{\infty} \phi_i(t).$$

Then ϕ satisfies a Lipschitz condition of order $1 - \alpha$, for any $\alpha > 0$; but Property (c) of Theorem 1 is not true for it. If $\Phi(t) = \int_0^t \phi(s) ds$, then Φ is smooth, and its first derivative satisfies a Lipschitz condition of order $1 - \alpha$; but the conclusion of Theorem 4 (with $m = 1$) fails.

2. The theorem for bounded sets. Let $x = (x_1, \dots, x_n)$ denote points of n -space E^n . With the unit vectors e_1, \dots, e_n of a coordinate system, any vector v can be written in the form $\sum v_i e_i$. The length of v is $|v| = (\sum v_i^2)^{1/2}$; $|y - x|$ is the distance from x to y . Given n numbers $f_1(x), \dots, f_n(x)$, set

$$(2.1) \quad F(x) \cdot v = \sum f_k(x) v_k;$$

this is linear in v . If $f(x)$, $f(y)$, and the $f_k(x)$ are defined, set

$$(2.2) \quad e(x, y) = \frac{f(y) - f(x) - F(x) \cdot (y - x)}{|y - x|}$$

for $y \neq x$, and $e(x, x) = 0$. Let $S_z [\alpha(z)]$ denote the set of elements z with the property α . Given f , and so on, as above, set

$$(2.3) \quad H(x, \epsilon) = S_y [e(x, y) < \epsilon].$$

The measurable function f defined in the set P is a. t. d. (approximately totally differentiable) at $x \in P$ in terms of the $f_k(x)$ (see [6, p. 300]) if for each $\epsilon > 0$ the set $H(x, \epsilon)$ has x as a point of density. (Any standard definition of density points may be used for the purposes of this paper.) If this holds, then x is a point of density of P , and the $f_k(x)$ are uniquely determined; if x is a point of density in the direction of each axis, then the $f_k(x)$ are the approximate partial derivatives of f at x . The f_k are measurable (see [6, p. 299]).

THEOREM 1. *Let f be measurable in the bounded set P . Then the four following conditions are equivalent:*

- (a) *The function f is a. t. d. a. e. in P .*
- (b) *The function f is approximately derivable with respect to each variable a. e. in P .*
- (c) *For each $\epsilon > 0$ there is a closed set $Q \subset P$ such that $|P - Q| < \epsilon$ and f is smooth in Q .*
- (d) *There is a sequence of disjoint closed sets Q_1, Q_2, \dots in P such that $|P - Q_1 \cup Q_2 \cup \dots| = 0$ and f is smooth in each Q_i .*

REMARK. If f is assumed totally differentiable a. e. in P , the proof that (c) holds is simplified; see [2, p. 442].

Proof of Theorem 1. For the equivalence of (a) and (b), see [6, pp. 300-303]. Note that (b) is an obvious consequence of (d). We shall prove the equivalence of (a), (c) and (d).

Suppose (c) holds. We choose the disjoint closed sets Q_1, Q_2, \dots in succession so that f is smooth in each and $|P_i| \leq |P|/2^i$, where

$$P_i = P - Q_1 \cup \dots \cup Q_i ,$$

as follows. Having found Q_1, \dots, Q_{i-1} , choose a closed set Q'_i so that f is smooth in Q'_i and $|P - Q'_i| \leq |P|/2^{i+1}$. Let $U_\delta(A)$ denote the δ -neighborhood of the set A . For small enough δ , we may use

$$Q_i = Q'_i - U_\delta(Q_1 \cup \dots \cup Q_{i-1}) .$$

Thus (d) holds.

Suppose (d) holds. Let Q_i^* be the set of points of density of Q_i , and set $Q^* = Q_1^* \cup Q_2^* \cup \dots$. Then $|P - Q^*| = 0$. Take any $x \in Q^*$; say $x \in Q_i^*$. Since f is smooth in Q_i and x is a point of density of Q_i , f (considered now in P) is a. t. d. at x . Thus f is a. t. d. at all points of Q^* , and (a) holds.

Now given (a), we must prove (c). There is a number $a > 0$ with the following property. For any points x, y , and number r with $|y - x| \leq r$, we have

$$|U_r(x) \cap U_r(y)| \geq 2a |U_r(x)| .$$

For $x \in P$, set $V_i = |U_{1/i}(x)|$, and

$$(2.4) \quad \psi_i(x, \eta) = |U_{1/i}(x) - H(x, \eta)| ,$$

$$(2.5) \quad \phi_i(x) = \text{g.l.b. } S_\eta[\psi_i(x, \eta) < aV_i] .$$

Since $e(x, y)$ is measurable in the pair of variables x, y , it follows that $\psi_i(x, \eta)$ is measurable for fixed η . Also, as a function of η , $\psi_i(x, \eta)$ is monotone and continuous on the left; hence

$$(2.6) \quad \phi_i(x) < \zeta \quad \text{if and only if} \quad \psi_i(x, \zeta) < aV_i.$$

Therefore ϕ_i is measurable.

Let Q_1 be the set of points where f is a.t.d.; then f_1, \dots, f_n are defined in Q_1 . Given $x \in Q_1$ and $\epsilon' > 0$, we may choose $\delta > 0$ so that

$$\psi_i(x, \epsilon') < aV_i \quad \text{if} \quad 1/i < \delta;$$

using (2.6) shows that

$$(2.7) \quad \lim_{i \rightarrow \infty} \phi_i(x) = 0, \quad x \in Q_1.$$

By Lusin's and Egeroff's theorems, there is a closed set $Q \subset Q_1$ such that $|Q_1 - Q| < \epsilon$, the f_k are continuous in Q , and $\phi_i(x) \rightarrow 0$ uniformly in Q . We now prove that for each $\epsilon' > 0$ there is a $\delta > 0$ such that

$$(2.8) \quad e(x, y) < \epsilon' \quad \text{if} \quad x, y \in Q, \quad |y - x| < \delta.$$

Setting $\epsilon_1 = \epsilon'/6$, we may choose δ so that

$$(2.9) \quad |F(y) \cdot v - F(x) \cdot v| \leq \epsilon_1 |v| \quad \text{if} \quad x, y \in Q, \quad |y - x| < 2\delta,$$

$$(2.10) \quad \phi_i(x) < \epsilon_1 \quad \text{if} \quad x \in Q, \quad 1/(i+1) < \delta.$$

Now take any $x, y \in Q$ with $|y - x| < \delta$. Let j be the largest integer such that $1/j \geq |y - x|$, and set

$$R = U_{1/j}(x) \cap U_{1/j}(y); \quad \text{then} \quad |R| \geq 2aV_j.$$

Since $1/(j+1) < |y - x| < \delta$, (2.10) and (2.6) give

$$\psi_j(x, \epsilon_1), \quad \psi_j(y, \epsilon_1) < aV_j.$$

Hence there is a point z in R in neither corresponding set; that is,

$$|z - x|, \quad |z - y| < 1/j; \quad e(x, z), \quad e(y, z) < \epsilon_1.$$

Since $F(x) \cdot v$ is linear in v and $|z - x|, |z - y| < 2|y - x|$, we have

$$\begin{aligned} e(x, y) |y - x| &= |f(y) - f(x) - F(x) \cdot (y - x)| \\ &\leq |f(z) - f(x) - F(x) \cdot (z - x)| \\ &\quad + |f(z) - f(y) - F(y) \cdot (z - y)| + |[F(y) - F(x)] \cdot (z - y)| \\ &\leq \epsilon_1 [|z - x| + |z - y| + |z - y|] < \epsilon' |y - x| \end{aligned}$$

if $y \neq x$, proving (2.8).

This fact, together with the continuity of the $f_k(x)$ in Q , shows that f is “of class C^1 in terms of the f_k in Q ”, as defined in [7] (the definition is given after (6.3), below). Hence, by [7, Lemma 2], we may extend f to be smooth in E^n , completing the proof. (The extension is described in Section 4, below; by use of the results of that section, it is not hard to show that f has the required properties.)

3. The theorem for unbounded sets. We remove the restriction of boundedness in Theorem 1, and give more information about the set in which f may be left unaltered.

THEOREM 2. *Let A_1, A_2, \dots be open sets in E^n such that each has points in common with, at most, a finite number of the others, and let $\epsilon_1, \epsilon_2, \dots$ be positive numbers. Let K be the set of points x such that there is a sequence of distinct sets $A_{\mu_1}, A_{\mu_2}, \dots$ and a sequence of points x_1, x_2, \dots with $x_i \in A_{\mu_i}$ and $x_i \rightarrow x$. Let P be a measurable subset of $E^n - K$, and let f be a. t. d. a. e. in P in terms of the f_k . Then there is a set $Q \subset P$ such that Q is closed in $E^n - K$ and $|(P - Q) \cap A_i| < \epsilon_i$, and there is a smooth function g in $E^n - K$ such that $g(x) = f(x)$ and $\partial g(x)/\partial x_k = f_k(x)$ in Q .*

REMARKS. Clearly K is closed and $K \cap A_i = 0$ for all i . If $Q^* \subset P$, Q^* is closed in $E^n - K$, and for some positive continuous functions $\delta_1(x), \delta_2(x), \dots$ in Q^* ,

$$e(x, y) \leq 1/2^i \quad \text{if } x \in Q^*, \quad |y - x| < \delta_i(x),$$

the proof shows that we may make $Q \supset Q^*$. For instance, we may make Q contain any given set of points of P in which f is totally differentiable and which has no accumulation points in $E^n - K$. On the other hand, we must expect to drop out a neighborhood of the set of points where f is not totally differentiable. Further, we cannot in general keep in Q any given closed set where f is approximately totally differentiable, as is shown by the following example (in one variable):

$$f(t) = t^2 \sin(i/t) (t \neq 0), \quad f(0) = 0.$$

Proof of Theorem 2. For each pair (k, l) of positive integers, let $U_{k,l}$ be the set of points x satisfying the conditions (with a fixed x_0 in E^n)

$$k - 1 < |x - x_0| < k + 1, \quad 1/(l - 1) > \text{dist}(x, K) > 1/(l + 1);$$

for $k = 1$ or $l = 1$, we drop out the first inequalities. If K is void, the index l is

not needed, and the situation is simpler. The $U_{k,l}$ are bounded open sets covering $E^n - K$, and each one touches at most eight others. Arrange them in a sequence U'_1, U'_2, \dots .

For each i , let $\lambda_{i,1}, \lambda_{i,2}, \dots$ be the (finite or infinite) set of numbers such that $U'_{\lambda_{i,k}} \cap A_i \neq \emptyset$. Since the \bar{U}'_j are compact and in $E^n - K$, each touches at most a finite number of the A_i ; hence for given j , there is at most a finite number of values of i such that $\lambda_{i,k} = j$ for some k . Let ϵ'_j be the smallest of the numbers $\epsilon_i/2^k$, using these values of i and corresponding k .

Considering f and the f_k in $P \cap U'_j$ alone, apply the proof of Theorem 1 to find a closed set $Q_j \subset P \cap U'_j$ such that $|P \cap U'_j - Q_j| < \epsilon'_j$, and such that f is of class C^1 in terms of the f_k in Q_j . Set

$$V_j = U'_j - Q_j, \quad V = \cup_j V_j, \quad Q = E^n - K - V.$$

Then V is open, Q is closed in $E^n - K$, and $Q \cap U'_j \subset Q_j$. Now

$$(P - Q) \cap A_i = P \cap V \cap A_i = \cup_j (P \cap V_j \cap A_i),$$

$$|P \cap V_j| = |P \cap U'_j - Q_j| < \epsilon'_j.$$

Since $V_j \subset U'_j$, $P \cap V_j \cap A_i$ is void unless $j = \lambda_{i,k}$ for some k . Hence

$$|(P - Q) \cap A_i| \leq \sum_j |P \cap V_j \cap A_i| < \sum_k \epsilon'_{\lambda_{i,k}} \leq \sum_k \epsilon_i/2^k = \epsilon_i.$$

Since each U'_j is open and $Q \cap U'_j \subset Q_j$, f is clearly of class C^1 in terms of the f_k in Q . Hence, as before, we may extend the values of f in Q through $E^n - K$, as required. (We are applying [7, Lemma 2] in an open set; the change required in the proof is very simple. Or we could use [7, Theorem III].)

4. The theorem for Lipschitz functions. The following theorem has two parts, corresponding to the two theorems above.

THEOREM 3. *For each positive integer n there is a number ρ_n (we may take $\rho_1 = 1$) with the following properties.*

(a) *Let f be defined and satisfy a Lipschitz condition in the bounded closed set $P \subset E^n$:*

$$(4.1) \quad |f(y) - f(x)| \leq N|y - x|, \quad x, y \in P.$$

Then for each $\epsilon > 0$ there is a closed set $Q \subset P$ such that $|P - Q| < \epsilon$, and there is a smooth function g in E^n satisfying a Lipschitz condition (see (4.15)) with the constant $\rho_n N$, such that $g = f$ in Q .

(b) Let the A_i , ϵ_i , and K be as in Theorem 2. Let P be closed in $E^n - K$ (it may have accumulation points in K). Let f be defined in P and satisfy (4.1). Then there is a set $Q \subset P$ which is closed in P (and hence in $E^n - K$) such that $|(P - Q) \cap A_i| < \epsilon_i$; and there is a function g satisfying (4.15) in E^n which is smooth in $E^n - Q^*$, where $Q^* = \bar{Q} - Q$, such that $g = f$ in Q .

(c) We may take Q [in either (a) or (b)] so that f is totally differentiable in Q in terms of functions f_1, \dots, f_n ; we may then take g so that $\partial g / \partial x_k = f_k$ in Q ($k = 1, \dots, n$).

(d) Given a positive continuous function $\eta(x)$ in $E^n - K$ [in E^n , for case (a)], we may make

$$(4.2) \quad |g(x) - f(x)| < \eta(x), \quad x \in P.$$

REMARKS. It is no restriction to take P closed (or closed in $E^n - K$). For if P is not closed, it is easily seen that we may extend f (uniquely) over \bar{P} so that it is continuous there; then (4.1) now holds in \bar{P} . (We can in fact extend f to satisfy (4.1) in E^n ; see [3] or [4].) Note that, in (b), $Q^* \subset \bar{P} - P$; if K is void, then Q^* is void, and g is smooth in E^n . As an immediate consequence of (4.15), we have

$$(4.3) \quad |\sum v_k \partial g(x) / \partial x_k| \leq \rho_n N \quad \text{if} \quad x \in E^n - Q^*, \quad |v| = 1.$$

The hypothesis of total differentiability a. e. in P , together with

$$|\sum v_k f_k(x)| \leq N |v|$$

where the f_k are defined, is not enough to give the theorem (unless, for instance, $P = E^n$), as simple examples show. (Compare the examples in H. Whitney [8].) If we wish to prove (4.3) rather than (4.15), the proof may be slightly simplified; of course (4.15) follows from (4.3) if $Q^* = 0$ (hence if $K = 0$). See also the remarks following Theorem 2.

Proof of Theorem 3. To prove the theorem, we first note that (a) is contained in (b); use $A_1 = E^n$, $\epsilon_1 = \epsilon$. Next, (d) will follow at once from (4.1) and (4.15) if we make sure that each point of P is sufficiently close to some point of Q ; this will clearly be the case if, in applying the proof of Theorem 2, we take the ϵ' small enough. Also, just as in Theorem 2, (c) will hold. It remains to show that we can obtain the properties in (b), using the proof of Theorem 2. We do this here, except for showing that we can make $\rho_1 = 1$.

We must examine the proof of [7, Lemma 2]. First, since f is totally differentiable a. e. in P [6, p. 311], we may choose Q as in the proof of Theorem 2;

recall that the f_k are continuous in Q . We shall use a cubical subdivision of $E^n - \bar{Q}$, essentially as in [7]. For each integer s (in [7], only $s \geq 0$ was used), let K'_s be the set of all cubes of edge length $1/2^s$, the coordinates of whose corners are integral multiples of $1/2^s$. Let K''_s consist of the cubes of K'_s whose distances from \bar{Q} are at least $6n^{1/2}/2^s$. Let K_s consist of the cubes of K''_s which are not in cubes of K''_{s-1} . Take any cube $C \in K_s$; suppose $C \subset C'$, $C' \in K'_{s-1}$. Then $\text{dist}(C', \bar{Q}) < 6n^{1/2}/2^{s-1}$. Therefore, clearly

$$(4.4) \quad 6n^{1/2}/2^s \leq \text{dist}(C, \bar{Q}) < 13n^{1/2}/2^s, \quad C \in K_s.$$

Take $C \in K_s, C' \in K_{s+2}$. Then each point of C' is within

$$n^{1/2}/2^{s+2} + 13n^{1/2}/2^{s+2}$$

from \bar{Q} ; hence

$$(4.5) \quad \text{dist}(C, C') \geq (5/2)n^{1/2}/2^s, \quad C \in K_s, \quad C' \in K_{s+2}.$$

Let y^1, y^2, \dots be the set of all corners of all cubes of all K_s . Choose $x^\nu \in \bar{Q}$ with $|x^\nu - y^\nu| = \text{dist}(y^\nu, \bar{Q})$. Let b_ν be the largest length of edge of any cube of any K_s with y^ν as a corner, and let I_ν be the cube defined by

$$|x_i - y_i^\nu| \leq b_\nu (i = 1, \dots, n).$$

Let ϕ'_0 be a smooth function which is positive within a fixed unit cube and is zero outside; by a translation and similarity transformation, define ϕ'_ν , positive within I_ν and zero outside. Set $\phi_\nu = \phi'_\nu / \sum \phi'_\lambda$; then ϕ_ν is positive within I_ν and zero outside, and $\sum \phi_\nu = 1$ in $E^n - \bar{Q}$. Since there is at most some fixed number of shapes of cubes (of some K_s , and K_{s+1} perhaps) forming any I_ν , there is clearly a number $M_n \geq 1$ with the following property (compare [7, Section 10]): taking $|v| = 1$,

$$(4.6) \quad |\sum v_i \partial \phi_\nu / \partial x_i| < 2^s M_n \text{ if } \phi_\nu(y) \neq 0 \text{ for some } y \in C \in K_s.$$

Extend f to be continuous in \bar{P} (if $\bar{P} \neq P$); (4.1) still holds. For any $x^* \in \bar{Q}$ and any $x \in E^n$, set

$$(4.7) \quad \psi(x; x^*) = f(x^*) + \sum f_i(x^*)(x_i - x_i^*);$$

this is the value at x of the linear function approximating to f at x^* . Then set

$$(4.8) \quad g(x) = \sum \phi_\nu(x) \psi(x; x^\nu), \quad x \in E^n - \bar{Q}.$$

It is not hard to show that if $g = f$ in \bar{Q} , then g is smooth in $E^n - Q^*$ and $\partial g / \partial x_i = f_i$ in Q ; see the proof of [7, Lemma 2]. We must still prove (4.15).

Take first any x and x' in $E^n - \bar{Q}$; say for definiteness that

$$(4.9) \quad x \in C \in K_s, \quad x' \in C' \in K_{s'}, \quad s \geq s'.$$

Let x^* be a point of \bar{Q} nearest to x . Since $\sum \phi_\nu(x) = 1$, we may write

$$g(x) = f(x^*) + \sum_{\nu} \phi_{\nu}(x)[f(x^{\nu}) - f(x^*)] + \sum_{\nu, i} \phi_{\nu}(x)f_i(x^{\nu})(x_i - x_i^{\nu}).$$

Hence

$$(4.10) \quad g(x') - g(x) = \sum_{\nu} [\phi_{\nu}(x') - \phi_{\nu}(x)][f(x^{\nu}) - f(x^*)] \\ + \sum_{\nu, i} [\phi_{\nu}(x') - \phi_{\nu}(x)]f_i(x^{\nu})(x_i' - x_i^{\nu}) + \sum_{\nu, i} \phi_{\nu}(x)f_i(x^{\nu})(x_i' - x_i).$$

We shall find a bound for each non-zero term. First we show that

$$(4.11) \quad |\phi_{\nu}(x') - \phi_{\nu}(x)| |f(x^{\nu}) - f(x^*)| \leq 64NM_n n^{1/2} |x' - x|.$$

Consider first any ν such that $\phi_{\nu}(x) \neq 0$. Then by (4.6),

$$(4.12) \quad |\phi_{\nu}(x') - \phi_{\nu}(x)| < 2^s M_n |x' - x|.$$

Also, since

$$|x^* - x| \leq \text{diam}(C) + \text{dist}(C, \bar{Q}) \leq 14n^{1/2}/2^s, \\ |y^{\nu} - x| \leq 2 \text{diam}(C) \leq 2n^{1/2}/2^s, \\ |y^{\nu} - x^{\nu}| \leq |x^* - y^{\nu}| \leq 16n^{1/2}/2^s,$$

we have

$$|x^{\nu} - x^*| \leq 32n^{1/2}/2^s,$$

and hence

$$|f(x^{\nu}) - f(x^*)| \leq 32Nn^{1/2}/2^s.$$

These relations give (4.11). Next consider any ν such that $\phi_\nu(x') \neq 0$. Then (using inequalities like those above) we obtain

$$\begin{aligned} |x^\nu - x^*| &\leq |x^\nu - x'| + |x' - x| + |x - x^*| \\ &\leq 18n^{1/2}/2^{s'} + |x' - x| + 14n^{1/2}/2^s \leq 32n^{1/2}/2^{s'} + |x' - x|. \end{aligned}$$

In the present case, (4.12) holds with s' . Suppose first that

$$|x' - x| \leq 32n^{1/2}/2^{s'}.$$

Then

$$|f(x^\nu) - f(x^*)| \leq N \cdot 64n^{1/2}/2^{s'},$$

and (4.11) follows. If $|x' - x| > 32n^{1/2}/2^{s'}$, then $|x^\nu - x^*| < 2|x' - x|$, and since $|\phi_\nu(x') - \phi_\nu(x)| \leq 2$ and $4N < 64NM_n n^{1/2}$, (4.11) follows again.

Next we show that

$$(4.13) \quad |\phi_\nu(x') - \phi_\nu(x)| |f_i(x^\nu)| |x'_i - x_i^\nu| \leq 40NM_n n^{1/2} |x' - x|.$$

We may suppose that $\phi_\nu(x') \neq 0$, in which case $|x' - x^\nu| \leq 18n^{1/2}/2^{s'}$, or $\phi_\nu(x) \neq 0$, in which case $|x' - x^\nu| \leq 18n^{1/2}/2^s + |x' - x|$; in either case,

$$|x'_i - x_i^\nu| \leq |x' - x^\nu| \leq 18n^{1/2}/2^{s'} + |x' - x|.$$

First suppose that $|x' - x| \leq 2n^{1/2}/2^{s'}$. Then, by (4.5), $s \leq s' + 1$. Hence, using (4.6) with s or s' we get

$$|\phi_\nu(x') - \phi_\nu(x)| \leq 2^{s'+1} M_n |x' - x|;$$

since $|f_i(x^\nu)| \leq N$, (4.13) follows. Next suppose that $|x' - x| > 2n^{1/2}/2^{s'}$. Then $|\phi_\nu(x') - \phi_\nu(x)| \leq 2$, and $|x'_i - x_i^\nu| < 10|x' - x|$, giving (4.13) again.

Finally, we have

$$(4.14) \quad |\phi_\nu(x) f_i(x^\nu)(x'_i - x_i)| \leq N|x' - x|.$$

There is clearly a number c_n such that for any x , there are at most c_n values

of ν such that $\phi_\nu(x) \neq 0$. In the three groups of terms in (4.10), there are at most $2c_n$, $2c_n n$, and $c_n n$ non-zero terms respectively. Hence, by (4.11), (4.13) and (4.14), we have

$$|g(x') - g(x)| \leq c_n N M_n n^{1/2} (128 + 80n + n) |x' - x|,$$

which gives

$$(4.15) \quad |g(x') - g(x)| \leq \rho_n N |x' - x|, \quad \rho_n = 209 c_n M_n n^{3/2}.$$

If x and x' are in \bar{Q} , (4.15) follows from (4.1), since $g = f$ (or the extended f) in \bar{Q} . Suppose finally that $x \in \bar{Q}$, $x' \in E^n - \bar{Q}$ (or *vice versa*). Let x'' be the last point of the segment xx' in \bar{Q} . Then (4.1) holds for x and x'' , and (4.15) holds for x'' and x' , with x''' in $x''x'$ and arbitrarily close to x'' ; hence (4.15) holds in all cases, and the proof is complete.

5. Lipschitz functions of one variable. We must prove Theorem 3, with $n = 1$, $\rho_1 = 1$. The proof is elementary in nature; we do not need [7]. Find a closed subset Q_1 of $E^1 - K$ (or of E^1 , in case (a)) as in the proof of Theorem 2 (or Theorem 1, if we are only using (a)). Now (4.1) holds in Q_1 , f_1 is continuous in Q_1 , and f is smooth in terms of f_1 in Q_1 (see Section 3, above), that is, for each $x \in Q_1$ and each $\epsilon' > 0$ there is a $\delta > 0$ such that

$$(5.1) \quad |f(x'') - f(x') - (x'' - x') f_1(x')| \leq \epsilon' |x'' - x'|$$

if $|x'' - x|, |x' - x| < \delta, \quad x', x'' \in Q_1.$

Let primes on functions denote differentiation. We shall find a set Q which is closed in $E^1 - K$, with $|Q_1 - Q| < \epsilon^*$ in case (a) or $|(Q_1 - Q) \cap A_i| < \epsilon_i^*$ in case (b), and a function g which satisfies (4.1) in E^1 and is smooth in $E^1 - K$, and such that $g = f$ and $g' = f_1$ in Q ; for ϵ^* or the ϵ_i^* small enough, Q and g have the required properties.

Let I_1, I_2, \dots be the closed intervals whose interiors fill out $E^1 - Q_1 \cup K$. Extend f through E^1 so that (4.1) holds there; see [3] or [4]. Set $g_0 = f$ in $Q_1 \cup K$, and let g_0 be linear in the I_k , so that g_0 is continuous in the closed intervals. Then g_0 is continuous in E^1 , and satisfies (4.1) there.

We shall need the following lemma.

LEMMA 1. *Let ϕ be defined and satisfy (4.1) in the closed interval $[a^*, b]$, and let ϕ be linear in the subinterval $[a, b]$. Then there is an arbitrarily small*

interval $[a', a'']$ about a such that $\phi'(a')$ exists, and there is a function ψ in $[a^*, b]$ which equals ϕ in $[a^*, a'] \cup [a'', b]$ and is smooth in $[a', b]$, and is such that $\psi'(a') = \phi'(a')$, and for x in $[a', a'']$, $\psi'(x)$ lies between $\phi'(a')$ and $\phi'(a'')$.

We use the notation $\Delta\phi(x, y) = [\phi(y) - \phi(x)]/(y - x)$. If ϕ is linear in some interval $[x_0, b]$ with $x_0 < a$, we may set $\psi = \phi$. If not, we may choose $c < a$ arbitrarily close to a so that $\Delta\phi(c, a) \neq \Delta\phi(a, b)$. Suppose for definiteness that $\Delta\phi(c, a) < \Delta\phi(a, b)$. Take $a'' > a$ arbitrarily close to a . Because of (4.1), ϕ is absolutely continuous, and $\phi(a'') - \phi(c) = \int_c^{a''} \phi'(x) dx$. Hence there is a point a' in $[c, a]$ such that $\phi'(a')$ exists and

$$\phi'(a') < \Delta\phi(a', a'') < \Delta\phi(a'', b) = \phi'(a'') ;$$

that is, the tangents at a' and a'' intersect at a point x' between a' and a'' . Using these tangents except near x' , and smoothing near x' , gives the required ψ .

We return to the theorem. Let x_1, x_2, \dots be the set of end points of the intervals I_k . Let I_1'' be an interval about x_1 , of length $< \epsilon_1''$ for some ϵ_1'' (see below), with one end point interior to an interval I_k with x_1 as end point. Apply the lemma (or the lemma with x replaced by $-x$) to find an interior interval $I_1' = [a_1', a_1'']$ about x_1 , and using $\phi = g_0$ in I_1'' , define $\psi = g$ in I_1' . We may require that neither a_1' nor a_1'' is any x_i . In general, having found disjoint intervals I_1', \dots, I_{j-1}' , let x_h be the first point of the sequence which is in none of these, and let I_j'' be an interval about x_h , of length $< \epsilon_j''$, disjoint from the preceding I_i' . Apply the lemma as before to find I_j' , and define g in I_j' . Set $g = g_0$ elsewhere in E^1 . Let Q be the set of points of Q_1 interior to no I_j' . For small enough ϵ_j'' , the inequalities with ϵ^* or ϵ_j^* hold. We shall show that g is smooth in $E^1 - K$ and $g' = f_1$ in Q ; the other properties of g are clear.

Clearly g' is continuous in a neighborhood of any point interior to an I_j' or an I_k , that is, in $E^1 - KUQ$. Now take any $x \in Q$; we shall show that $g'(x) = f_1(x)$ and g' is continuous at x , considering only points $x' \geq x$ for which $g'(x')$ is defined. The same fact holds for $x' \leq x$, and this will complete the proof. By definition of g , this is true if x is the left hand end point of some I_j' ; suppose this is not the case.

Given $\epsilon' > 0$, choose δ so that (5.1) holds, and so that $|f_1(x') - f_1(x)| < \epsilon'$ for $x' \in Q_1, |x' - x| < \delta$. Choose $y > x$ in Q_1 within δ of x . Now any difference quotient of f , with points in $[x, y] \cap Q_1$, is within $2\epsilon'$ of $f_1(x)$; hence

clearly any difference quotient of g_0 in $[x, y]$ is within $2\epsilon'$ of $f_1(x)$. Hence, for any x' in $[x, y]$ such that $g'_0(x')$ exists, $|g'_0(x') - f_1(x)| < 2\epsilon'$. Because of the last property in the lemma, $|g'(x') - f_1(x)| < 2\epsilon'$ if $g'(x')$ exists. Since $g(x') - g(x) = \int_x^{x'} g'(t) dt$, this shows that $g'(x)$ exists (as a right hand derivative) and equals $f_1(x)$, and proves the required continuity.

6. Functions with totally differentiable m th partial derivatives. We shall prove a theorem corresponding to (a) \rightarrow (c) in Theorem 1; the extension to the case corresponding to Theorem 2 is clear.

THEOREM 4. *Let f and its partial derivatives of order $\leq m$ be defined in a bounded open set $P \subset E^n$, and let each m th partial derivative be totally differentiable a.e. in P . Then for each $\epsilon > 0$ there is a closed set $Q \subset P$ such that $|P - Q| < \epsilon$, and there is a function g with continuous $(m + 1)$ th partial derivatives in E^n such that all partial derivatives of f of order $\leq m + 1$ exist in Q and equal those of g there. In particular, $g = f$ in Q .*

Because of Theorem 1, we may suppose $m \geq 1$. We use the notation of [7]; thus

$$\frac{f_k(x)}{k!} (x' - x)^k = \frac{f_{k_1 \dots k_n}(x_1, \dots, x_n)}{k_1! \dots k_n!} (x'_1 - x_1)^{k_1} \dots (x'_n - x_n)^{k_n}$$

[do not confuse with the earlier $f_k(x)$], $\sigma_k = k_1 + \dots + k_n$, and so on. Also

$$f_{k_1 \dots k_n} = \partial^{\sigma_k} f / \partial x_1^{k_1} \dots \partial x_n^{k_n}$$

where defined.

Take any k with $\sigma_k = m - 1$, and any integers i and j . Since $\partial f_k / \partial x_i$ and $\partial f_k / \partial x_j$ are defined in P and are totally differentiable a.e. in P , it follows that their partial derivatives $\partial^2 f_k / \partial x_i \partial x_j$ and $\partial^2 f_k / \partial x_j \partial x_i$ exist a.e. in P ; by a theorem of Currier [1], these are equal a.e. in P . Where this is so for all i, j , it is clear that we may define f_k with $\sigma_k = m + 1$ uniquely. Let P' be the subset of P in which the f_k exist for $\sigma_k \leq m + 1$, and each f_k ($\sigma_k = m$) is totally differentiable in terms of the f_l ($l_i \geq k_i, \sigma_l = m + 1$); then $|P - P'| = 0$. As seen in Section 2, the f_k are measurable.

As in [7], let $\psi_k(x'; x)$, for $\sigma_k \leq m$, be the value at x' of the polynomial of degree at most $m - \sigma_k$ which has the same value and partial derivatives of order $\leq m - \sigma_k$ at x as f_k . Then

$$\psi_k(x'; x) = \sum_{\sigma_l \leq m - \sigma_k} \frac{f_{k+l}(x)}{l!} (x' - x)^l .$$

Let $R_k(x'; x)$ be the corresponding remainder in Taylor's expansion:

$$(6.2) \quad R_k(x'; x) = f_k(x') - \psi_k(x'; x), \quad \sigma_k \leq m.$$

Define ψ'_k and R'_k similarly for $x \in P'$, with m replaced by $m + 1$. We shall say a remainder R'_k is of order m' at x^0 if the following is true. For each $\epsilon' > 0$ there is a $\delta > 0$ such that

$$(6.3) \quad |R'_k(x; x^0)| \leq \epsilon' |x - x^0|^{m'} \quad \text{if} \quad |x - x^0| < \delta.$$

Recall from [7] that in a closed set, f is of class C^{m+1} in terms of the f_k ($\sigma_k \leq m + 1$) if and only if each R'_k is of order $m + 1 - \sigma_k$ uniformly in a neighborhood of each point.

With the help of Lemma 2 below, we prove Theorem 4 as follows. By Lusin's Theorem, there is a closed set $Q' \subset P'$ with $|P' - Q'| < \epsilon/2$ such that each $f_k(x)$ with $\sigma_k = m + 1$ is continuous in Q' ; that is, R'_k ($\sigma_k = m + 1$) is of order 0 in Q' . For each integer i and each $x^0 \in Q'$, let $\delta_i(x^0)$ be the upper bound of numbers $\delta \leq 1$ such that (6.3) holds with $m' = m - \sigma_k + 1$, $\epsilon' = 1/2^i$, for all k with $\sigma_k \leq m + 1$. Then by the lemma, $\delta_i(x) > 0$ in Q' . As in Section 2, we see that the $\delta_i(x)$ are measurable. Find sets Q_i as in Section 2, and set $Q = Q_1 \cap Q_2 \cap \dots$. Then clearly f is of class C^{m+1} in Q in terms of the f_k , and hence [7, Lemma 2] f may be extended from Q over E^n so that

$$\partial^{\sigma_k} f / \partial x_1^{k_1} \dots \partial x_n^{k_n} = f_k$$

in Q . This extension is the required g . There remains to prove

LEMMA 2. *Let P be open, let*

$$\partial^{\sigma_k} f / \partial x_1^{k_1} \dots \partial x_n^{k_n} = f_k$$

in P for $\sigma_k \leq m$, let $f_k(x^0)$ be defined for $\sigma_k = m + 1$, and let the f_k ($\sigma_k = m$) be totally differentiable in terms of the f_l ($l_i \geq k_i, \sigma_l = m + 1$) at x^0 . Define $R'_k(x; x^0)$ as above. Then R'_k is of order $m - \sigma_k + 1$ at x^0 if $\sigma_k \leq m - 1$.

Note that the hypothesis shows that R'_k for $\sigma_k = m$ is of order 1 at x^0 .

Suppose we have proved Lemma 2 for the case that $f_k(x^0) = 0$ for all k , $\sigma_k \leq m + 1$. Then it holds for the general case. For set

$$\bar{f}_k(x) = f_k(x) - \psi'_k(x; x^0) = R'_k(x; x^0) \quad (\sigma_k \leq m + 1);$$

then $\bar{f}_k(x^0) = 0$. Also, since $\bar{\psi}'_k(x; x^0) = 0$ (using the \bar{f}_l), $\bar{R}'_k(x; x^0) = R'_k(x; x^0)$.

Since the \bar{R}'_k are of order $m - \sigma_k + 1$ at x^0 for $\sigma_k = m$, the lemma shows that this is true also for $\sigma_k < m$. Thus $R'_k = \bar{R}'_k$ is of order $m - \sigma_k + 1$ at x^0 , as required.

We shall need Taylor's Theorem with exact remainder:

LEMMA 3. Let ϕ be a function of one variable such that $\phi^{(h)} = d^h\phi/dx^h$ exists for $h \leq m'$ in an interval and is bounded. Then

$$\begin{aligned} \phi(t_1) &= \sum_{h=0}^{m'} \frac{\phi^{(h)}(t_0)}{h!} (t_1 - t_0)^h \\ &\quad + \frac{1}{(m' - 1)!} \int_{t_0}^{t_1} (t_1 - s)^{m'-1} [\phi^{(m')}(s) - \phi^{(m')}(t_0)] ds . \end{aligned}$$

Since $\phi^{(m')}$ is bounded, $\phi^{(m'-1)}$ satisfies a Lipschitz condition; hence for any smooth α , $\beta = \alpha\phi^{(m'-1)}$ is absolutely continuous, and

$$\int_a^b (d\beta/dt)dt = \beta(b) - \beta(a) .$$

Therefore the usual proof applies.

We return to Lemma 2, assuming $f_k(x^0) = 0$ ($\sigma_k \leq m + 1$). Set

$$x^i = (x_1, \dots, x_i, x_{i+1}^0, \dots, x_n^0) ;$$

then $x^n = x$. Take any $i > 0$, and any k with $\sigma_k \leq m - 1$. Set $m' = m - \sigma_k$, $k(i) = (k_1, \dots, k_i + m', \dots, k_n)$, and

$$x^i(s) = (x_1, \dots, x_{i-1}, s, x_{i+1}^0, \dots, x_n^0) .$$

Then $x^i(x_i^0) = x^{i-1}$, $x^i(x_i) = x^i$. For some $\delta_1 > 0$, the $f_k(x)$ ($\sigma_k \leq m$) are bounded for $|x - x^0| < \delta_1$. Lemma 3 gives

$$\begin{aligned} f_k(x^i) &= \psi_k(x^i; x^{i-1}) \\ &\quad + \frac{1}{(m' - 1)!} \int_{x_i^0}^{x_i} (x_i - s)^{m'-1} \{f_{k(i)}[x^i(s)] - f_{k(i)}(x^{i-1})\} ds . \end{aligned}$$

Since $f_l(x^i(s)) = R'_l(x^i(s); x^0)$, and so on, the definition of R_k gives

$$\begin{aligned} R_k(x^i; x^{i-1}) &= \frac{1}{(m' - 1)!} \int_{x_i^0}^{x_i} (x_i - s)^{m'-1} \\ &\quad \times \{R'_{k(i)}[x^i(s); x^0] - R'_{k(i)}(x^{i-1}; x^0)\} ds . \end{aligned}$$

For a certain ϵ^* chosen below, choose $\delta \leq \delta_1$ so that

$$|R'_l(x'; x^0)| \leq \epsilon^* |x' - x^0| \quad \text{if } \sigma_l = m, \quad |x' - x^0| < \delta.$$

Then if $|x - x^0| < \delta$, using $|x^i(s) - x^0| \leq |x - x^0|$ for $x_i^0 \leq s \leq x_i$, and so on, gives

$$|R_k(x^i; x^{i-1})| \leq \frac{2\epsilon^* |x - x^0|}{(m' - 1)!} \int_{x_i^0}^{x_i} (x_i - s)^{m'-1} ds \leq \frac{2\epsilon^* |x - x^0|^{m'+1}}{m'!}.$$

Now in [7, (6.3)], subtract $f_k(x'')$ from both sides, and change x, x', x'' to x^{i-1}, x^i, x respectively; this gives

$$R_k(x; x^{i-1}) - R_k(x; x^i) = \sum_{\sigma_l \leq m - \sigma_k} \frac{R_{k+l}(x^i; x^{i-1})}{l!} (x - x^i)^l.$$

Hence,

$$|R_k(x; x^{i-1}) - R_k(x; x^i)| \leq 2\epsilon^* |x - x^0|^{m - \sigma_k + 1} \sum_{\sigma_l \leq m - \sigma_k} \frac{1}{(m - \sigma_k - \sigma_l)! l!}.$$

Let A_k denote the sum, and let A be the largest A_k . Since $R'_k(x; x^0) = R_k(x; x^0)$ for the case at hand, adding the inequalities for $i = 1, \dots, n$ gives

$$|R'_k(x; x^0)| \leq 2nA \epsilon^* |x - x^0|^{m - \sigma_k + 1}.$$

Given $\epsilon' > 0$, set $\epsilon^* = \epsilon' / (2nA)$, and choose δ accordingly; this inequality then completes the proof.

REFERENCES

1. A. E. Currier, *Proof of the fundamental theorems on second order cross partial derivatives*, Trans. Amer. Math. Soc. 35 (1933), 245-253.
2. H. Federer, *Surface Area II*, Trans. Amer. Math. Soc. 55 (1944), 438-456.
3. M. D. Kirszbraun, *Über die zusammenziehenden und Lipschitzschen Transformationen*, Fund. Math. 22 (1934), 77-108.
4. E. J. MacShane, *Extension of range of functions*, Bull. Amer. Math. Soc. 40 (1934), 837-842.
5. J. Marcinkiewicz, *Sur les séries de Fourier*, Fund. Math. 27 (1936), 38-69.

6. S. Saks, *Theory of the integral*, Second Edition, Monografie Matematyczne, Warszawa-Lwow, 1937.

7. H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63-89.

8. ———, *On the extension of differentiable functions*, Bull. Amer. Math. Soc. 50 (1944), 76-81.

HARVARD UNIVERSITY

EDITORS

HERBERT BUSEMANN
University of Southern California
Los Angeles 7, California

R. M. ROBINSON
University of California
Berkeley 4, California

E. F. BECKENBACH, Managing Editor
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

R. P. DILWORTH
HERBERT FEDERER
MARSHALL HALL

P. R. HALMOS
HEINZ HOPF
R. D. JAMES

BØRGE JESSEN
PAUL LÉVY
GEORGE PÓLYA

J. J. STOKER
E. G. STRAUS
KÔSAKU YOSIDA

SPONSORS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA, BERKELEY
UNIVERSITY OF CALIFORNIA, DAVIS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
UNIVERSITY OF CALIFORNIA, SANTA BARBARA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NATIONAL BUREAU OF STANDARDS,
INSTITUTE FOR NUMERICAL ANALYSIS

Vari-Type Composition by
Cecile Leonard
Ruth Stafford

With the cooperation of
E. F. Beckenbach
E. G. Straus

Printed in the United States of America by
Edwards Brothers, Inc., Ann Arbor, Michigan

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES
COPYRIGHT 1951 BY PACIFIC JOURNAL OF MATHEMATICS

Ralph Palmer Agnew, <i>Ratio tests for convergence of series</i>	1
Richard Arens and James Dugundji, <i>Topologies for function spaces</i>	5
B. Arnold, <i>Distributive lattices with a third operation defined</i>	33
R. Bing, <i>Concerning hereditarily indecomposable continua</i>	43
David Dekker, <i>Generalizations of hypergeodesics</i>	53
A. Dvoretzky, A. Wald and J. Wolfowitz, <i>Relations among certain ranges of vector measures</i>	59
Paul Erdős, F. Herzog and G. Pirani, <i>Schlicht Taylor series whose convergence on the unit circle is uniform but not absolute</i>	75
Whilhelm Fischer, <i>On Dedekind's function $\eta(\tau)$</i>	83
Werner Leutert, <i>The heavy sphere supported by a concentrated force</i>	97
Ivan Niven and H. Zuckerman, <i>On the definition of normal numbers</i>	103
L. Paige, <i>Complete mappings of finite groups</i>	111
Otto Szász, <i>On a Tauberian theorem for Abel summability</i>	117
Olga Taussky, <i>Classes of matrices and quadratic fields</i>	127
F. Tricomi and A. Erdélyi, <i>The asymptotic expansion of a ratio of gamma functions</i>	133
Hassler Whitney, <i>On totally differentiable and smooth functions</i>	143