

# Pacific Journal of Mathematics

**ON THE LERCH ZETA FUNCTION**

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# ON THE LERCH ZETA FUNCTION

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1. **Introduction.** The function  $\phi(x, a, s)$ , defined for  $\Re s > 1$ ,  $x$  real,  $a \neq$  negative integer or zero, by the series

$$(1.1) \quad \phi(x, a, s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi i x}}{(a+n)^s},$$

was investigated by Lipschitz [4; 5], and Lerch [3]. By use of the classic method of Riemann,  $\phi(x, a, s)$  can be extended to the whole  $s$ -plane by means of the contour integral

$$(1.2) \quad I(x, a, s) = \frac{1}{2\pi i} \int_C \frac{z^{s-1} e^{az}}{1 - e^{z+2\pi i x}} dz,$$

where the path  $C$  is a loop which begins at  $-\infty$ , encircles the origin once in the positive direction, and returns to  $-\infty$ . Since  $I(x, a, s)$  is an entire function of  $s$ , and we have

$$(1.3) \quad \phi(x, a, s) = \Gamma(1-s) I(x, a, s),$$

this equation provides the analytic continuation of  $\phi$ . For integer values of  $x$ ,  $\phi(x, a, s)$  is a meromorphic function (the Hurwitz zeta function) with only a simple pole at  $s = 1$ . For nonintegral  $x$  it becomes an entire function of  $s$ . For  $0 < x < 1$ ,  $0 < a < 1$ , we have the functional equation

$$(1.4) \quad \begin{aligned} &\phi(x, a, 1-s) \\ &= \frac{\Gamma(s)}{(2\pi)^s} \{ e^{\pi i(s/2-2ax)} \phi(-a, x, s) + e^{\pi i(-s/2+2a(1-x))} \phi(a, 1-x, s) \}, \end{aligned}$$

first given by Lerch, whose proof follows the lines of the first Riemann proof of the functional equation for  $\zeta(s)$  and uses Cauchy's theorem in connection with the contour integral (1.2).

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Received March 4, 1951.

*Pacific J. Math.* 1 (1951), 161-167.

In the present paper, §2 contains a proof of (1.4) based on the transformation theory of theta-functions. This proof is of particular interest because the usual approach (Riemann's second method) does not lead to the functional equation (1.4) as might be expected but to a different functional relationship (equation (2.4) below). Further properties of  $\phi(x, a, s)$ , having no analogue in the case of  $\zeta(s)$ , are needed to carry this method through to obtain (1.4).

In §3 we evaluate the function  $\phi(x, a, s)$  for negative integer values of  $s$ . These results are expressible in closed form by means of a sequence of functions  $\beta_n(a, e^{2\pi ix})$  which are polynomials in  $a$  and rational functions in  $e^{2\pi ix}$ . These functions are closely related to Bernoulli polynomials; their basic properties also are developed here.

2. Functional Equation for  $\phi(x, a, s)$ . The theta-function

$$\vartheta_3(y|\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2i n y)$$

has the transformation formula [6, p. 475]

$$\vartheta_3(y|\tau) = (-i\tau)^{-1/2} \exp\left(\frac{y^2}{\pi i \tau}\right) \vartheta_3\left(\frac{y}{\tau} \middle| \frac{-1}{\tau}\right).$$

If we let

$$\theta(x, a, z) = \exp(-\pi a^2 z) \vartheta_3(\pi x + \pi i a z | iz) = \sum_{n=-\infty}^{\infty} \exp(2n\pi i x - \pi z(a + n)^2),$$

then we have the functional equation

$$(2.1) \quad \theta(a, -x, 1/z) = [\exp(2\pi i a x)]_z^{1/2} \theta(x, a, z).$$

The key to Riemann's second method is the formal identity

$$(2.2) \quad \pi^{-s/2} \Gamma(s/2) \sum_{n=1}^{\infty} a_n f_n^{-s/2} = \int_0^{\infty} z^{s/2-1} \sum_{n=1}^{\infty} a_n \exp(-\pi z f_n) dz.$$

Taking first  $a_n = \exp[(2\pi i(n-1)x)]$ ,  $f_n = (n-1+a)^2$  in (2.2), and then  $a_n = \exp(-2\pi i n x)$ ,  $f_n = (n-a)^2$ , we obtain

$$(2.3) \quad \pi^{-s/2} \Gamma(s/2) \{ \phi(x, a, s) + \exp(-2\pi i x) \phi(-x, 1 - a, s) \} \\ = \left( \int_1^\infty + \int_0^1 \right) z^{s/2-1} \theta(x, a, z) dz .$$

In the second integral in (2.3) we apply (2.1) and replace  $z$  by  $1/z$ . Denoting the expression in braces by  $\Lambda(x, a, s)$ , replacing  $s$  by  $1 - s$ ,  $x$  by  $-a$ ,  $a$  by  $x$ , using  $\theta(-a, x, z) = \theta(a, -x, z)$ , and the relation

$$\pi^{1/2-s} \Gamma(s/2) / \Gamma\left(\frac{1-s}{2}\right) = 2(2\pi)^{-s} \cos(\pi s/2) \Gamma(s) ,$$

we are led to

$$(2.4) \quad \Lambda(x, a, 1 - s) = 2(2\pi)^{-s} \cos(\pi s/2) \Gamma(s) \exp(-2\pi i a x) \Lambda(-a, x, s) .$$

Thus Riemann's method gives us a functional equation for  $\Lambda$  instead of (1.4). At this point we introduce the differential-difference equations satisfied by  $\phi$ , namely:

$$(2.5) \quad \frac{\partial \phi(x, a, s)}{\partial a} = -s \phi(x, a, s + 1)$$

and

$$(2.6) \quad \frac{\partial \phi(x, a, s)}{\partial x} + 2\pi i a \phi(x, a, s) = 2\pi i \phi(x, a, s - 1) .$$

The first of these follows at once from (1.1). To obtain (2.6) we first write

$$\phi(x, a, s) = \exp(-2\pi i a x) \sum_{n=0}^\infty \frac{\exp[2\pi i(n+a)x]}{(n+a)^s}$$

before differentiating with respect to  $x$ . The equations hold for all  $s$  by analytic continuation.

The proof of (1.4) as a consequence of (2.4) now proceeds as follows. We differentiate both sides of (2.4) with respect to the variable  $a$ , using (2.5) on the left and (2.6) on the right, and replace  $s$  by  $s + 1$  in the resulting equation. This

leads to the relation

$$\begin{aligned} & \phi(x, a, 1-s) - \exp(-2\pi i x) \phi(-x, 1-a, 1-s) \\ &= 2i(2\pi)^{-s} \sin(\pi s/2) \Gamma(s) \\ & \quad \times [\exp(-2\pi i a x) \phi(-a, x, s) - \exp(-2\pi i a(1-x)) \phi(a, 1-x, s)]. \end{aligned}$$

Adding this equation to (2.4) gives the desired relation (1.4).

This method has already been used by N. J. Fine [1] to derive the functional equation of the Hurwitz zeta function. Fine's proof uses (2.5) with  $x = 0$ . In our proof of (1.4) it is essential that  $x \neq 0$  since we have occasion to interchange the variables  $x$  and  $a$ , and  $\phi(x, a, s)$  is not regular for  $a = 0$ ; hence Fine's proof is not a special case of ours. Furthermore, putting  $x = 0$  in (1.4) does not yield the Hurwitz functional equation, although this can be obtained from (1.4) as shown elsewhere by the author.

3. Evaluation of  $\phi(x, a, -n)$ . If  $x$  is an integer, then  $\phi(x, a, s)$  reduces to the Hurwitz zeta function  $\zeta(s, a)$  whose properties are well known [6, pp. 265-279]. For nonintegral  $x$  the analytic character of  $\phi$  is quite different from that of  $\zeta(s, a)$ , and in what follows we assume that  $x$  is not an integer.

The relation (2.6) can be used to compute recursively the values of  $\phi(x, a, s)$  for  $s = -1, -2, -3, \dots$ . As a starting point we compute the value at  $s = 0$  by substituting in (1.2). The value of the integral reduces to the residue of the integrand at  $z = 0$  and gives us

$$\phi(x, a, 0) = \frac{1}{1 - \exp(2\pi i x)} = (i/2) \cot \pi x + 1/2.$$

Using (2.6) we obtain

$$\phi(x, a, -1) = (a/2)(i \cot \pi x + 1) - (1/4) \csc^2 \pi x,$$

$$\phi(x, a, -2) = (a^2/2)(i \cot \pi x + 1/4) - (a/2) \csc^2 \pi x - (i/4) \cot \pi x \csc^2 \pi x.$$

If we put  $s = -n$  in (1.2) and use Cauchy's residue theorem we obtain, for  $n \geq 0$ , the relation

$$\phi(x, a, -n) = -\frac{\beta_{n+1}(a, e^{2\pi i x})}{n+1},$$

where  $\beta_n(a, \alpha)$  is defined by the generating function

$$(3.1) \quad z \frac{e^{az}}{\alpha e^z - 1} = \sum_{n=0}^{\infty} \frac{\beta_n(a, \alpha)}{n!} z^n .$$

When  $\alpha = 1$ ,  $\beta_n(a, \alpha)$  is the Bernoulli polynomial  $B_n(a)$ . For our purposes we assume  $\alpha \neq 1$ , and in the remainder of this section we give the main properties of the functions  $\beta_n(a, \alpha)$ .

Writing  $\beta_n(\alpha)$  instead of  $\beta_n(0, \alpha)$  we obtain from (3.1):

$$(3.2) \quad \beta_n(a, \alpha) = \sum_{k=0}^n \binom{n}{k} \beta_k(\alpha) a^{n-k} \quad (n \geq 0) ,$$

from which we see that the functions  $\beta_n(a, \alpha)$  are polynomials in the variable  $a$ . The defining equation (3.1) also leads to the difference equation

$$(3.3) \quad \alpha \beta_n(a + 1, \alpha) - \beta_n(a, \alpha) = n \alpha^{n-1} \quad (n \geq 1) .$$

Taking  $a = 0$  we obtain, for  $n = 1$ , the relation

$$(3.4) \quad \alpha \beta_1(1, \alpha) = 1 + \beta_1(\alpha)$$

while for  $n \geq 2$  we have

$$(3.5) \quad \alpha \beta_n(1, \alpha) = \beta_n(\alpha) .$$

Putting  $a = 1$  in (3.2) now allows us to compute the functions  $\beta_n(\alpha)$  recursively by means of

$$(3.6) \quad \beta_n(1, \alpha) = \sum_{k=0}^n \binom{n}{k} \beta_k(\alpha)$$

and (3.4), (3.5). From (3.1) we obtain  $\beta_0(\alpha) = 0$ ; the next few functions are found to be :

$$\beta_1(\alpha) = \frac{1}{\alpha - 1} , \quad \beta_2(\alpha) = -\frac{2\alpha}{(\alpha - 1)^2} , \quad \beta_3(\alpha) = \frac{3\alpha(\alpha + 1)}{(\alpha - 1)^3} ,$$

$$\beta_4(\alpha) = -\frac{4\alpha(\alpha^2 + 4\alpha + 1)}{(\alpha - 1)^4} , \quad \beta_5(\alpha) = \frac{5\alpha(\alpha^3 + 11\alpha^2 + 11\alpha + 1)}{(\alpha - 1)^5} ,$$

$$\beta_6(\alpha) = -\frac{6\alpha(\alpha^4 + 26\alpha^3 + 66\alpha^2 + 26\alpha + 1)}{(\alpha - 1)^6}.$$

The general formula is

$$(3.7) \quad \beta_n(\alpha) = \frac{n\alpha}{(\alpha - 1)^n} \sum_{s=1}^{n-1} (-1)^s s! \alpha^{s-1} (\alpha - 1)^{n-1-s} \mathfrak{B}_{n-1}^{(s)},$$

where  $\mathfrak{B}_k^{(j)}$  are Stirling numbers of the second kind defined by

$$\mathfrak{B}_k^{(j)} = \frac{\Delta^j 0^k}{j!},$$

with

$$\Delta^j 0^n = (\Delta^j x^n)_{x=0}, \quad \Delta^j 0^n = 0 \quad \text{if } j > n, \quad \Delta^0 0^0 = 1,$$

in the usual notation of finite differences. (A short table of Stirling numbers is given in [2].)

To prove (3.7) we put

$$g(z, \alpha) = \frac{1}{\alpha e^z - 1} = \frac{1}{\alpha - 1} \left( 1 + \sum_{n=1}^{\infty} \left( \frac{\alpha}{1 - \alpha} \right)^n (e^z - 1)^n \right).$$

Using Herschel's theorem [2, p.73] which expresses  $(e^z - 1)^n$  as a power series in  $z$  we obtain

$$(\alpha - 1) g(z, \alpha) = 1 + \sum_{m=1}^{\infty} \sum_{s=1}^m \left( \frac{\alpha}{1 - \alpha} \right)^s \frac{s!}{m!} \mathfrak{B}_m^{(s)} z^m.$$

Comparing with

$$g(z, \alpha) = \sum_0^{\infty} \beta_n(\alpha) \frac{z^n}{n!}$$

we get (3.7).

The following further properties of the numbers  $\beta_n(a, \alpha)$ , which closely resemble well-known formulas for Bernoulli polynomials, are easy consequences of

the above :

$$\frac{\partial^p}{\partial a^p} \beta_n(a, \alpha) = \frac{n!}{(n-p)!} \beta_{n-p}(a, \alpha) \quad (0 \leq p \leq n),$$

$$\beta_n(a+b, \alpha) = \sum_{k=0}^n \binom{n}{k} \beta_k(a, \alpha) b^{n-k},$$

$$\int_a^b \beta_n(t, \alpha) dt = \frac{\beta_{n+1}(b, \alpha) - \beta_{n+1}(a, \alpha)}{n+1} \quad (n \geq 0).$$

Taking  $a = b - 1$  and using (3.3), we can also use this last equation to obtain the functions  $\beta_n(a, \alpha)$  recursively by successive integration of polynomials.

As a final result, taking  $a = 0, 1, 2, \dots, m-1$  in (3.3) and summing we obtain

$$(3.8) \quad \sum_{a=0}^{m-1} a^n = \frac{\alpha-1}{n+1} \sum_{a=1}^m \beta_{n+1}(a, \alpha) + \frac{\beta_{n+1}(m, \alpha) - \beta_{n+1}(\alpha)}{n+1},$$

a generalization of the famous formula giving  $\sum a^n$  in terms of Bernoulli polynomials. This result is somewhat surprising because of the appearance of the parameter  $\alpha$  on the right. (When  $\alpha = 1$ , (3.8) reduces to the Bernoulli formula.)

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$8.00; single issues, \$2.50. Spécial price to individual faculty members of supporting institutions and to members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

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