

# Pacific Journal of Mathematics

**CONVOLUTION TRANSFORMS WITH COMPLEX KERNELS**

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# CONVOLUTION TRANSFORMS WITH COMPLEX KERNELS

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1. **Introduction.** In the present paper we shall consider the inversion of a class of convolution transforms with kernel  $G(t)$  of the form

$$(1.1) \quad G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{st} ds \quad (-\infty < t < \infty),$$

$$(1.2) \quad E(s) = \prod_1^{\infty} \left( 1 - \frac{s}{a_k} \right) e^{s/b_k},$$

$a_k = b_k + ic_k$  ( $k = 1, 2, \dots$ ) being a sequence of complex numbers such that

$$(1.3) \quad \sum_{k=1}^{\infty} (1/b_k)^2 < \infty, \quad \sum_{k=1}^{\infty} (c_k/b_k)^2 < \infty.$$

This class of kernels is more extensive than that treated previously by the authors, see [4], [5], [6], and [7]; however the results obtained here are slightly less precise than those which it was possible to obtain there. We shall show essentially that if

$$(1.4) \quad f(x) = \int_{-\infty}^{\infty} G(x-t) d\alpha(t),$$

and if  $x_1$  and  $x_2$  are points of continuity of  $\alpha(t)$ , then

$$(1.5) \quad \lim_{m \rightarrow \infty} \int_{x_1}^{x_2} \left[ \prod_{k=1}^m \left( 1 - \frac{D}{a_k} \right) e^{D/b_k} \right] f(x) dx = \alpha(x_2) - \alpha(x_1).$$

Here  $D$  is the operation of differentiation, and  $e^{D/a}$  that of translation through the

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Received November 20, 1950.  
*Pacific J. Math.* 1 (1951), 211-225.

distance  $1/a$ , so that, for example,

$$\begin{aligned} \left(1 - \frac{D}{a_1}\right) e^{D/b_1} \left(1 - \frac{D}{a_2}\right) e^{D/b_2} f(x) &= f\left(x + \frac{1}{b_1} + \frac{1}{b_2}\right) \\ &\quad - \left(\frac{1}{a_1} + \frac{1}{a_2}\right) f'\left(x + \frac{1}{b_1} + \frac{1}{b_2}\right) \\ &\quad + \frac{1}{a_1 a_2} f''\left(x + \frac{1}{b_1} + \frac{1}{b_2}\right). \end{aligned}$$

If we replace equation (1.2) and inequalities (1.3) by the more special relations

$$(1.6) \quad E(s) = \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{a_k^2}\right),$$

$$(1.7) \quad \lim_{k \rightarrow \infty} b_k/k = \Omega > 0, \quad \sum_{k=1}^{\infty} (c_k/b_k)^2 < \infty,$$

we have in addition the complex inversion formula,

$$(1.8) \quad \lim_{\lambda \rightarrow 1^-} \int_{x_1}^{x_2} dx \frac{1}{2\pi i} \int_{C_\lambda} f(\lambda w + x) K(w) dw = \alpha(x_2) - \alpha(x_1),$$

where

$$(1.9) \quad K(w) = \int_0^\infty E(s) e^{-sw} ds$$

and  $C_\lambda$  is a closed rectifiable curve encircling the segment  $[-i\Omega, i\Omega]$  and lying in the strip  $|\Re w| < \Omega/\lambda$ . The inner integral in formula (1.8) is to be taken in the counterclockwise direction.

As one example we may take

$$\begin{aligned} E(s) &= \frac{\Gamma(1/2 + \nu/2)^2}{\Gamma(1/2 + \nu/2 - s/2) \Gamma(1/2 + \nu/2 + s/2)}, \\ G(t) &= \frac{(e^t + e^{-t})^{-\nu-1}}{\Gamma(1/2 + \nu/2)^2}, \end{aligned}$$

where  $\Re \nu > -1$ . If

$$f(x) = \int_{-\infty}^{\infty} [e^{(x-t)} + e^{-(x-t)}]^{-\nu-1} \Gamma(1/2 + \nu/2)^{-2} d\alpha(t),$$

then

$$\lim_{m \rightarrow \infty} \int_{x_1}^{x_2} \left\{ \prod_{k=1}^m \left[ 1 - \left( \frac{D}{-1/2 + \nu/2 + k} \right)^2 \right] f(x) \right\} dx = \alpha(x_2) - \alpha(x_1);$$

and if  $\Re \nu > 0$ , then

$$\begin{aligned} \lim_{\lambda \rightarrow 1^-} \frac{2^\nu \Gamma(1/2 + \nu/2)^2}{\pi \Gamma(\nu)} \int_{x_1}^{x_2} dx \int_{-\pi/2}^{\pi/2} f(x + i \lambda w) [\cos w]^{\nu-1} dw \\ = \alpha(x_2) - \alpha(x_1). \end{aligned}$$

See [7] and [8], and [9]. A second example is

$$E(s) = \pi 2^s \left[ \cos \frac{\pi \nu}{2} \Gamma \left( \frac{1}{2} - \frac{\nu}{2} - \frac{s}{2} \right) \Gamma \left( \frac{1}{2} + \frac{\nu}{2} - \frac{s}{2} \right) \right]^{-1},$$

$$G(t) = \frac{2}{\pi} \cos \frac{\pi \nu}{2} e^t K_\nu(e^t)$$

for  $-1 < \Re \nu < 1$ . If

$$f(x) = \int_{-\infty}^{\infty} e^{x-t} K_\nu(e^{x-t}) \left( \frac{2}{\pi} \cos \frac{\nu \pi}{2} \right) d\alpha(t),$$

then

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_{x_1}^{x_2} \left\{ \prod_{k=1}^m \left( 1 - \frac{D}{-1/2 - \nu/2 + k} \right) \left( 1 - \frac{D}{-1/2 + \nu/2 + k} \right) f(x) \right\} dx \\ = \alpha(x_2) - \alpha(x_1). \end{aligned}$$

See [2].

**2. Inversion of a class of convolution transforms.** We assume as given throughout this section a sequence,  $\{a_k\}_1^\infty$ , of complex numbers  $a_k = b_k + ic_k$  subject

to the restrictions

$$(2.1) \quad \sum_{k=1}^{\infty} (1/b_k)^2 < \infty, \quad \sum_{k=1}^{\infty} (c_k/b_k)^2 < \infty.$$

We define the entire functions

$$(2.2) \quad E_{m,n}(s) = \prod_{k=m+1}^n (1 - s/a_k) e^{s/b_k},$$

$$E_m(s) = \prod_{k=m+1}^{\infty} (1 - s/a_k) e^{s/b_k},$$

$$F_m(s) = \prod_{k=m+1}^{\infty} |b_k/a_k| (1 - s/b_k) e^{s/b_k}.$$

The definition of  $E_m(s)$  is significant because

$$E_m(s) = \left\{ \prod_{m+1}^{\infty} \left( 1 - \frac{s}{a_k} \right) e^{s/a_k} \right\} \left\{ \exp \sum_{m+1}^{\infty} \frac{ic_k s}{b_k(b_k + ic_k)} \right\},$$

and because the series  $\sum_{m+1}^{\infty} |a_k|^{-2}$ ,  $\sum_{m+1}^{\infty} c_k/b_k(b_k + ic_k)$  converge as a consequence of (2.1) and Schwarz's inequality. Similarly,  $F_m(s)$  is well defined. We define

$$(2.3) \quad P_m(D) = \prod_{k=1}^m (1 - D/a_k) e^{D/b_k} \quad (m = 0, 1, \dots).$$

We also set

$$(2.4) \quad \beta_1(m) = \max_{\substack{b_k < 0 \\ k > m}} (b_k, -\infty), \quad \beta_2(m) = \min_{\substack{b_k > 0 \\ k > m}} (b_k, \infty),$$

**THEOREM 2a.** *Let*

$$G_m(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E_m(s)]^{-1} e^{st} ds \quad (-\infty < t < \infty; m = 0, 1, 2, \dots);$$

then we have

A.  $\int_{-\infty}^{\infty} G_m(t) e^{-st} dt = 1/E_m(s), \quad \beta_1(m) < \Re s < \beta_2(m);$

B.  $\int_{-\infty}^{\infty} |G_m(t)| e^{-\sigma t} dt \leq 1/F_m(\sigma), \quad \beta_1(m) < \sigma < \beta_2(m);$

C.  $P_m(D)G_0(t) = G_m(t);$

D.  $(d/dt)^k G_m(t) = O(e^{\gamma_1 t}), \quad t \rightarrow +\infty,$   
 $= O(e^{\gamma_2 t}), \quad t \rightarrow -\infty \quad (k = 0, 1, \dots),$

for  $\gamma_1 > \beta_1(m)$  and  $\gamma_2 < \beta_2(m)$ .

Conclusion A is an immediate consequence of Hamburger's theorem; see [4, pp.141-144]. We define  $g(u) = e^{u-1}$  for  $-\infty < u \leq 1$ , and  $g(u) = 0$  for  $1 < u < \infty$ , and we set

$$g_k(t) = a_k \operatorname{sgn} b_k \{ \exp[ic_k(t - b_k^{-1})] \} g(b_k t).$$

It is immediately verifiable that

$$\int_{-\infty}^{\infty} e^{-st} g_k(t) dt = \left[ \left( 1 - \frac{s}{a_k} \right) e^{s/b_k} \right]^{-1},$$

for  $-\infty < \Re s < b_k$  if  $b_k > 0$ , and for  $b_k < \Re s < \infty$  if  $b_k < 0$ . Let

$$g_1 * g_2(t) = \int_{-\infty}^{\infty} g_1(t-u)g_2(u) du,$$

and so on; then by the convolution theorem for the bilateral Laplace transform we have

$$\int_{-\infty}^{\infty} g_{m+1} * g_{m+2} * \dots * g_n(t) e^{-st} dt = [E_{m,n}(s)]^{-1}$$

for  $\beta_1(m) < \Re s < \beta_2(m)$ . From the complex inversion formula for the bilateral Laplace transform we obtain

$$g_{m+1} * g_{m+2} * \dots * g_n(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E_{m,n}(s)]^{-1} e^{st} ds.$$

Since

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \{ [E_m(s)]^{-1} - [E_{m,n}(s)]^{-1} \} e^{st} ds = 0$$

for  $-\infty < t < \infty$ , it follows that

$$\lim_{n \rightarrow \infty} g_{m+1} * \dots * g_n(t) = G_m(t) \quad (-\infty < t < \infty).$$

See [4; pp.139-145]. It is easily seen that

$$\int_{-\infty}^{\infty} |g_k(t)| e^{-st} dt = [(1 - s/b_k) e^{s/b_k} |b_k/a_k|]^{-1},$$

for  $-\infty < \Re s < b_k$  if  $b_k > 0$ , or for  $b_k < \Re s < \infty$  if  $b_k < 0$ . By Fatou's lemma we have

$$\begin{aligned} \int_{-\infty}^{\infty} |G_m(t)| e^{-\sigma t} dt &\leq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} |g_{m+1} * \dots * g_n(t)| e^{-\sigma t} dt, \\ &\leq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} |g_{m+1}| * \dots * |g_n(t)| e^{-\sigma t} dt \\ &\leq [F_m(\sigma)]^{-1}, \end{aligned}$$

so that conclusion B is established.

Conclusion C follows from the identity

$$P_m(D) e^{st} = e^{st} \prod_{k=1}^m (1 - sa_k^{-1}) e^{s/a_k}.$$

Conclusion D may be established by shifting the line of integration in the integral defining  $G_m(t)$  to  $\Re s = \gamma_1$  and  $\Re s = \gamma_2$ . See [4; pp.152-154].

In what follows we shall write  $G(t)$  for  $G_0(t)$ .

**THEOREM 2b.** *If*

- (a)  $G(t)$  is defined as in Theorem 2a,
- (b)  $\beta_1(0) < c < \beta_2(0)$ ,  $c + \gamma_1 > \beta_1(0)$ ,  $c + \gamma_2 < \beta_2(0)$ ,

(c)  $\alpha(t)$  is of bounded variation on every finite interval,  $\alpha(t) = O(e^{\gamma_1 t})$  as  $t \rightarrow -\infty$ ,  $\alpha(t) = O(e^{\gamma_2 t})$  as  $t \rightarrow +\infty$ ,

(d)  $P_m(D)$  is defined as in equation (2.3),

(e)  $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\alpha(t)$ ,

(f)  $x_1$  and  $x_2$  are points of continuity of  $\alpha(t)$ ,

then

$$\lim_{m \rightarrow \infty} \int_{x_1}^{x_2} e^{-cx} [P_m(D)f(x)] dx = \alpha(x_2) - \alpha(x_1).$$

From assumption (c) and from conclusion D of Theorem 2a we may show, using integration by parts, that each of the integrals

$$\int_{-\infty}^{\infty} G_m(x-t)e^{ct}d\alpha(t)$$

converges uniformly for  $x$  in any finite interval. Since  $P_m(D)G(t) = G_m(t)$  by conclusion C of Theorem 2a, it follows (see [4; pp.167-170]) that

$$(2.5) \quad P_m(D)f(x) = \int_{-\infty}^{\infty} G_m(x-t)e^{ct}d\alpha(t) \quad (-\infty < x < \infty).$$

Multiplying by  $e^{-cx}$  and integrating by parts, we have

$$\begin{aligned} e^{-cx}P_m(D)f(x) &= - \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial t} [G_m(x-t)e^{-c(x-t)}] \right\} \alpha(t) dt \\ &= \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial x} [G_m(x-t)e^{-c(x-t)}] \right\} \alpha(t) dt. \end{aligned}$$

Since this integral converges uniformly for  $x$  in any finite interval, we obtain

$$\begin{aligned} &\int_{x_1}^{x_2} e^{-cx}P_m(D)f(x) dx \\ &= \int_{x_1}^{x_2} dx \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial x} [G_m(x-t)e^{-c(x-t)}] \right\} \alpha(t) dt \\ &= \int_{-\infty}^{\infty} \alpha(t) dt \int_{x_1}^{x_2} \left\{ \frac{\partial}{\partial x} [G_m(x-t)e^{-c(x-t)}] \right\} dx \end{aligned}$$



$$\begin{aligned}
&= \int_{-\infty}^{\infty} \{G_m(x_2 - t)e^{-c(x_2 - t)} - G_m(x_1 - t)e^{-c(x_1 - t)}\} \alpha(t) dt \\
&= \int_{-\infty}^{\infty} G_m(x_2 - t)e^{-c(x_2 - t)} \alpha(t) dt - \int_{-\infty}^{\infty} G_m(x_1 - t)e^{-c(x_1 - t)} \alpha(t) dt.
\end{aligned}$$

We thus need only show that if  $x$  is a point of continuity of  $\alpha(t)$  we have

$$(2.6) \quad \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} G_m(x - t)e^{-c(x - t)} \alpha(t) dt = \alpha(x).$$

We shall first show that for any  $\epsilon > 0$  we have

$$(2.7) \quad \lim_{m \rightarrow \infty} \int_{|t| \geq \epsilon} G_m(t)e^{-ct} \alpha(x - t) dt = 0.$$

Using assumptions (a) and (b) we see that it is enough to prove that for any  $\delta$  with  $\beta_1(0) < \delta < \beta_2(0)$ , we have

$$(2.8) \quad \lim_{m \rightarrow \infty} \int_{|t| \geq \epsilon} |G_m(t)| e^{-\delta t} dt = 0.$$

Choose  $\eta > 0$  so small that  $\beta_1(0) < \delta - 2\eta < \delta + 2\eta < \beta_2(0)$ . For  $|t| \geq \epsilon$  we have

$$e^{-\delta t} \leq \frac{e^{-\delta t} (\sinh \eta t)^2}{(\sinh \epsilon \eta)^2},$$

so that it is enough to prove that

$$\lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} |G_m(t)| e^{-\delta t} [\sinh \eta t]^2 dt = 0.$$

Using conclusions A and B of Theorem 2a we see that

$$\begin{aligned}
&\int_{-\infty}^{\infty} |G_m(t)| e^{-\delta t} [\sinh \eta t]^2 dt \\
&\leq \frac{1}{4} \left[ \frac{1}{F_m(\delta + 2\eta)} + \frac{1}{F_m(\delta - 2\eta)} - \frac{2}{E_m(\delta)} \right] = o(1) \quad (m \rightarrow +\infty),
\end{aligned}$$

and equation (2.8) follows from this. We assert that

$$(2.9) \quad \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} G_m(t)e^{-ct} dt = 1$$

$$(2.10) \quad \limsup_{m \rightarrow \infty} \int_{-\infty}^{\infty} |G_m(t)| e^{-ct} dt = 1.$$

These results are immediate consequences of conclusions A and B of Theorem 2a. Now  $x$  being fixed and  $\eta > 0$  being given, let us choose  $\epsilon > 0$  so small that  $|\alpha(t) - \alpha(x)| \leq \eta$  for  $|t - x| \leq \epsilon$ . We have

$$\int_{-\infty}^{\infty} G_m(x - t) e^{-c(x-t)} \alpha(t) dt - \alpha(x) = I_1 + I_2 + I_3,$$

where

$$I_1 = \alpha(x) \left[ \int_{-\infty}^{\infty} G_m(x - t) e^{-c(x-t)} dt - 1 \right]$$

$$I_2 = \int_{|t| \geq \epsilon} G_m(t) e^{-ct} [\alpha(x - t) - \alpha(x)] dt$$

$$I_3 = \int_{|t| \leq \epsilon} G_m(t) e^{-ct} [\alpha(x - t) - \alpha(x)] dt.$$

We have  $\lim_{m \rightarrow \infty} I_1 = 0$  by equation (2.9),  $\lim_{m \rightarrow \infty} I_2 = 0$  by equation (2.7), and  $\limsup_{m \rightarrow \infty} |I_3| \leq \eta$  by equation (2.10). Since  $\eta$  is arbitrary our demonstration is complete.

**3. Complex inversion formulas.** In this section we restrict our attention to a much more special class of kernels. We suppose that

$$(3.1) \quad b_k > 0, \quad b_k \sim \frac{\Omega k}{\pi}, \quad \sum_{k=1}^{\infty} \left( \frac{c_k}{b_k} \right)^2 < \infty.$$

We define

$$(3.2) \quad E(s) = \prod_{k=1}^{\infty} \left( 1 - \frac{s}{a_k^2} \right)^2,$$

$$(3.3) \quad H(\lambda, s) = \prod_{k=1}^{\infty} \left[ \lambda^2 + (1 - \lambda^2) \frac{|a_k| b_k}{b_k^2 - s^2} \right] \quad (0 \leq \lambda < 1).$$

The product (3.3) is defined for  $s \neq b_k$  ( $k = 1, 2, \dots$ ) since it can be rewritten as

$$H(\lambda, s) = \frac{\prod_{k=1}^{\infty} \left[ 1 - \frac{\lambda^2 s^2}{b_k^2} + (1 - \lambda^2) \frac{|a_k| - b_k}{b_k} \right]}{\prod_{k=1}^{\infty} \left( 1 - \frac{s^2}{b_k^2} \right)},$$

and assumption (a) implies that  $\sum_1^{\infty} [|a_k| - b_k] b_k^{-1}$  is convergent. We define

$$(3.4) \quad \beta = \min b_k.$$

THEOREM 3a. *If*

$$G(\lambda, w) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{sw} E(\lambda s)}{E(s)} ds \quad (0 \leq \lambda < 1),$$

then

- A.  $G(\lambda, w)$  is analytic for  $|w| < \Omega(1 - \lambda)$ ;
- B.  $\int_{-\infty}^{\infty} G(\lambda, t) e^{-st} dt = E(\lambda s)/E(s), \quad -\beta < \Re s < \beta$ ;
- C.  $(d/dw)^k G(\lambda, w) = O(e^{\gamma_1 u}) \quad (u \rightarrow +\infty)$   
 $= O(e^{\gamma_2 u}) \quad (u \rightarrow -\infty) \quad (k = 0, 1, \dots),$

where  $\gamma_1 > -\beta, \gamma_2 < \beta$ , uniformly for  $|v| < \Omega(1 - \lambda) - \epsilon, \epsilon > 0$ . (Here  $w = u + iv$ .)

$$D. \quad \int_{-\infty}^{\infty} |G(\lambda, t)| e^{-\sigma t} dt \leq H(\lambda, \sigma), \quad -\beta < \sigma < \beta.$$

We shall write  $G(t)$  for  $G(0, t)$ .

We assert that

$$(3.5) \quad \log |E(\sigma + i\tau)| \sim \Omega |\tau| \quad (\tau \rightarrow \pm \infty)$$

uniformly for  $\sigma$  in any finite interval. We define

$$E * (s) = \prod_{k=1}^{\infty} (1 - s^2 b_k^{-2}).$$

We have

$$\frac{E(s)}{E * (s)} = \prod_{k=1}^{\infty} \frac{a_k^2}{b_k^2} \prod_{k=1}^{\infty} \left\{ 1 + \frac{a_k^2 - b_k^2}{b_k^2 - s^2} \right\},$$

from which it follows that

$$(3.6) \quad \lim_{s \rightarrow \infty} \frac{E(s)}{E * (s)} = \prod_{k=1}^{\infty} \frac{a_k^2}{b_k^2}$$

uniformly for  $0 < \epsilon \leq |\arg s| \leq \pi - \epsilon$ . From [1, pp.267-279] we have that  $\log |E * (\sigma + i\tau)| \sim \Omega |\tau|$  as  $\tau \rightarrow \pm \infty$ , uniformly for  $\sigma$  in any finite interval. Relation (3.5) now follows.

Conclusion A follows immediately from (3.5) and the definition of  $G(\lambda, w)$ . Conclusion B is a consequence of (3.5) and Hamburger's Theorem. The two conclusions C are obtained by shifting the line of integration in the integral defining  $G(\lambda, t)$  to  $\Re s = \gamma_1$ , and to  $\Re s = \gamma_2$ , respectively. See [6, pp.688-691]. To establish conclusion D we introduce the functions

$$G_n(\lambda, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \frac{\prod_{k=1}^n \left( 1 - \frac{\lambda^2 s^2}{a_k^2} \right)}{\prod_{k=1}^{\infty} \left( 1 - \frac{s^2}{a_k^2} \right)} ds.$$

It is immediate that

$$\lim_{n \rightarrow \infty} G_n(\lambda, t) = G(\lambda, t) \quad (-\infty < t < \infty).$$

We define

$$h_k(\lambda, t) = \lambda^2 j(t) + (1 - \lambda^2) \frac{a_k}{2} \int_{-\infty}^t e^{-a_k |u|} du,$$

where  $j(t) = 0$  for  $-\infty < t < 0$ ;  $j(0) = 1/2$ ;  $j(t) = 1$  for  $0 < t < \infty$ . It is easily verified that for  $-b_k < \sigma < b_k$  we have

$$\int_{-\infty}^{\infty} e^{-st} dh_k(\lambda, t) = \frac{1 - \lambda^2 s^2 / a_k^2}{1 - s^2 / a_k^2}.$$

Just as in §2 we may show that

$$G_n(\lambda, t) = \lim_{m \rightarrow \infty} \frac{d}{dt} [h_1(\lambda, t) * \dots * h_n(\lambda, t) * h_{n+1}(0, t) * \dots * h_m(0, t)].$$

Here  $h_1 * h_2(t) = \int_{-\infty}^{\infty} h_1(t - u) dh_2(u)$ . Note that this differs from the convention employed in §2. By Fatou's lemma,

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-\sigma t} |G_n(\lambda, t)| dt \\ & \leq \liminf_{m \rightarrow \infty} \int_{-\infty}^{\infty} e^{-\sigma t} |dh_1(\lambda, t) * \dots * h_n(\lambda, t) * h_{n+1}(0, t) * \dots * h_m(0, t)| \\ & \leq \liminf_{m \rightarrow \infty} \prod_{k=1}^n \int_{-\infty}^{\infty} e^{-\sigma t} |dh_k(\lambda, t)| \prod_{k=n+1}^m \int_{-\infty}^{\infty} e^{-\sigma t} |dh_k(0, t)| \\ & \leq \prod_{k=1}^n \int_{-\infty}^{\infty} e^{-\sigma t} |dh_k(\lambda, t)| \prod_{k=n+1}^{\infty} \int_{-\infty}^{\infty} e^{-\sigma t} |dh_k(0, t)|. \end{aligned}$$

By Fatou's lemma, again,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\sigma t} |G(\lambda, t)| dt & \leq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} e^{-\sigma t} |G_n(\lambda, t)| dt \\ & \leq \prod_{k=1}^{\infty} \int_{-\infty}^{\infty} e^{-\sigma t} |dh_k(\lambda, t)| \\ & \leq \prod_{k=1}^{\infty} \left[ \lambda^2 + (1 - \lambda^2) \frac{|a_k| b_k}{b_k^2 - \sigma^2} \right]. \end{aligned}$$

This completes the proof of the theorem.

We define

$$(3.7) \quad K(w) = \int_0^\infty E(s)e^{-sw} ds .$$

It follows from relations (3.1) that given  $\epsilon > 0$ , for all sufficiently large  $r$  we have  $\log |E * (re^{i\theta})| \leq (\epsilon + |\sin \theta|)\Omega r$ . See [1, pp.267-279]. From equation (3.6) it follows that

$$\log |E(re^{i\theta})| \leq (\epsilon + |\sin \theta|)\Omega r$$

for  $r$  sufficiently large. Using this inequality and rotating the line of integration in the integral defining  $K(w)$  we can show that  $K(w)$  is analytic and single valued in the  $w$ -plane except on the segment  $[-i\Omega, i\Omega]$ . It may also be shown, see [1, pp.295-311], that if  $C$  is a closed rectifiable curve encircling  $[-i\Omega, i\Omega]$  then

$$(3.8) \quad E(s) = \frac{1}{2\pi i} \int_C K(w)e^{sw} dw ,$$

the integration proceeding in the counterclockwise direction.

LEMMA 3b. *If  $C_\lambda$  is a closed rectifiable curve encircling  $[-i\Omega, i\Omega]$  and contained in the strip  $|v| < \Omega/\lambda$ , then*

$$\frac{1}{2\pi i} \int_{C_\lambda} G(\lambda w + x - t)K(w) dw = G(\lambda, x - t) ,$$

the integration proceeding in the counterclockwise direction.

We have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{C_\lambda} G(\lambda w + x - t)K(w) dw \\ &= \frac{1}{2\pi i} \int_{C_\lambda} K(w) dw \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{s(\lambda w + x - t)} ds \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{s(x-t)} ds \frac{1}{2\pi i} \int_{C_\lambda} K(w)e^{\lambda sw} dw \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left[ \frac{E(\lambda s)}{E(s)} \right] e^{s(x-t)} ds \\
 &= G(\lambda, x - t).
 \end{aligned}$$

THEOREM 3c. *If*

- (a)  $G(t)$  is defined as in Theorem 3a
- (b)  $-\beta < c < \beta$ ,  $-\beta < c + \gamma_1$ ,  $c + \gamma_2 < \beta$
- (c)  $\alpha(t)$  is of bounded variation on every finite interval and

$$\alpha(t) = (e^{\gamma_1 t}) \quad (t \rightarrow +\infty), \quad \alpha(t) = (e^{\gamma_2 t}) \quad (t \rightarrow -\infty)$$

- (d)  $f(w) = \int_{-\infty}^{\infty} G(w - t)e^{ct} d\alpha(t)$
- (e)  $K(w)$  is defined as in equation (3.7)
- (f)  $C_\lambda$  is defined as in Lemma 3b
- (g)  $x_1$  and  $x_2$  are points of continuity of  $\alpha(t)$ , then

$$\lim_{\lambda \rightarrow 1^-} \int_{x_1}^{x_2} e^{-cx} dx \frac{1}{2\pi i} \int_{C_\lambda} f(\lambda w + x)K(w) dw = \alpha(x_2) - \alpha(x_1).$$

It follows from assumption (c) and from conclusion C of Theorem 3a that the integral defining  $f(w)$  converges uniformly for  $w$  in any compact set contained in the strip  $|\Im w| < \Omega$ . Hence

$$\begin{aligned}
 &\frac{1}{2\pi i} \int_{C_\lambda} f(\lambda w + x)K(w) dw \\
 &= \int_{-\infty}^{\infty} e^{ct} d\alpha(t) \frac{1}{2\pi i} \int_{C_\lambda} G(\lambda w + x - t)K(w) dw \\
 &= \int_{-\infty}^{\infty} G(\lambda, x - t)e^{ct} d\alpha(t)
 \end{aligned}$$

by Lemma 3b. The proof may now be completed exactly in the manner of Theorem 2b.

4. Remark. If it is assumed that the roots of  $E(s)$  occur in conjugate pairs, then equation (1.5) can be established under conditions less restrictive than (1.3). A discussion of this case is given in the Master's thesis of Mr. A. O. Garder [3], written under the direction of one of us.

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