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ON SOME TRIGONOMETRIC TRANSFORMS

Otto Szász

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1. Introduction. To a given series $\Sigma_{n=1}^{\infty} u_n$ we consider the transform

$$(1.1) \qquad A_n = \sum_{\nu=1}^{n} u_\nu \frac{\sin \nu t_n}{\nu t_n}, \qquad \text{where } t_n \downarrow 0 \text{ as } n \longrightarrow \infty .$$

It was shown in a previous paper [5, Section 4, Theorem 3] that the transform (1.1) is regular if and only if

$$(1.2) \qquad n t_n = O(1), \qquad \text{as } n \longrightarrow \infty .$$

We shall now consider the transform (1.1) in relation to Cesàro means. In a forthcoming paper Cornelius Lanczos has found independently that the transform (1.1) is very useful in summing Fourier series and derived series, and gave some very interesting examples; he takes $t_n = \pi/n$. Of our results we quote here the following theorem:

THEOREM 1. *In order that the transform* (1.1) *includes* $(C, 1)$ *summability, it is necessary and sufficient that*

$$(1.3) \qquad n t_n = p\pi + \alpha_n, \quad n\alpha_n = O(1), \qquad p \textit{ a positive integer.}$$

We also discuss other triangular transforms which may be generated by "truncation" of well-known summation processes, such as Riemann summability. The transform A_n and the transform D_n (Section 5) are special cases of the general transform

$$\gamma_n = \sum_{\nu=0}^{n} u_\nu \phi(\nu P_n) ,$$

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where $\phi(P)$ is a function of the n-dimensional point $P(x_1, x_2, \cdots, x_n)$, and $P_n \longrightarrow 0$. This transform and many special cases of it were discussed by W. Rogosinski [4]; in particular, the special case $\alpha_n = 0$ of our Theorem 4 is included in his result on page 96. The general approach is essentially the same as in the present paper.

2. Proof of Theorem 1. If we write

$$\sum_{\nu=1}^{n} u_\nu = s_n, \qquad \sum_{\nu=1}^{n} s_\nu = s_n', \qquad \frac{\sin \nu t_n}{\nu t_n} - \frac{\sin (\nu+1) t_n}{(\nu+1) t_n} = \Delta_\nu,$$

$$\frac{\sin \nu t_n}{\nu t_n} - \frac{2 \sin (\nu+1) t_n}{(\nu+1) t_n} + \frac{\sin (\nu+2) t_n}{(\nu+2) t_n} = \Delta_\nu^2,$$

then

$$A_n = \sum_{\nu=1}^{n-1} s_\nu \Delta_\nu + s_n \frac{\sin n t_n}{n t_n}$$

$$= \sum_{\nu=1}^{n-2} s_\nu' \Delta_\nu^2 + s_{n-1}' \Delta_{n-1} + (s_n' - s_{n-1}') \frac{\sin n t_n}{n t_n},$$

or

$$A_n = \sum_{\nu=1}^{n-2} s_\nu' \Delta_\nu^2 + s_{n-1}' \left[\frac{\sin (n-1) t_n}{(n-1) t_n} - \frac{2 \sin n t_n}{n t_n} \right] + s_n' \frac{\sin n t_n}{n t_n}. \tag{2.1}$$

Now $(C, 1)$ summability of $\Sigma_{n=1}^{\infty} u_n$ to s means that

$$n^{-1} s_n' \longrightarrow s, \qquad \text{as } n \longrightarrow \infty. \tag{2.2}$$

If $s_n \equiv 1$, then $A_n = \sin t_n / t_n \longrightarrow 1$.

In order that (2.2) imply $A_n \longrightarrow s$, it is necessary and sufficient [in view of (2.1)] that

$$\text{(2.3)} \qquad \frac{\sin nt_n}{t_n} = O(1)\,, \qquad \frac{\sin\,(n-1)\,t_n}{t_n} = O(1)\,,$$

$$\text{(2.4)} \qquad \sum_{\nu=1}^{n-2} \nu |\Delta_\nu^2| = O(1)\,, \qquad \text{as } n \longrightarrow \infty\,.$$

The first condition of (2.3) [in view of (1.2)] is equivalent to

$$\sin nt_n = O(t_n) = O(1/n)\,;$$

hence

$$nt_n = p\pi + \alpha_n\,, \qquad n\alpha_n = O(1)\,.$$

The second condition of (2.3) now reduces to

$$\cos nt_n \sin t_n = O(t_n)\,,$$

or

$$\cos \alpha_n \sin t_n = O(n^{-1})\,,$$

which is satisfied. Finally

$$\frac{\sin \nu t}{\nu} = \int_0^t \cos \nu x\, dx = \Re \int_0^t e^{i\nu x}\, dx\,;$$

hence

$$\text{(2.5)} \qquad t_n \Delta_\nu^2 = \Re \int_0^{t_n} \Delta^2 e^{i\nu x}\, dx = \Re \int_0^{t_n} e^{i\nu x}(1 - e^{ix})^2\, dx\,,$$

and

$$\text{(2.6)} \qquad t_n |\Delta_\nu^2| < \int_0^{t_n} |1 - e^{ix}|^2\, dx = 4 \int_0^{t_n} (\sin x/2)^2\, dx$$

$$< \int_0^{t_n} x^2\, dx < t_n^3\,.$$

It follows that

$$\sum_{\nu=1}^{n-2} \nu\,|\Delta_\nu^2| < t_n^2 \sum_{\nu=1}^{n} \nu < n^2 t_n^2 = O(1)\,, \qquad \text{as } n \longrightarrow \infty\,.$$

This proves Theorem 1.

We can show by an example that the transform A_n may be more powerful than $(C, 1)$. In (1.3) let $p = 1$, $n\alpha_n = -\pi/2$; the series $\Sigma_{\nu=1}^{\infty} (-1)^{n-1} n$ (that is, $u_n = (-1)^n n$) is not summable $(C, 1)$, but summable $(C, 2)$ to $1/4$. Now

$$t_n A_n = \sum_{\nu=1}^{n} (-1)^{\nu-1} \sin \nu t_n$$

$$= \frac{\sin t_n - (-1)^n [\sin nt_n + \sin (n+1) t_n]}{|1 + e^{it}|^2},$$

where $nt_n = \pi - \pi/2n$. Hence, as $n \longrightarrow \infty$,

$$A_n \sim 1/4 + o(1) .$$

An even more striking example is $u_n = (-1)^{n-1} n^2$.

3. Summation by harmonic polynomials. We get a more powerful method if we introduce the harmonic polynomial

$$h_n(\rho, t) = \sum_{\nu=1}^{n} u_\nu \rho^\nu \frac{\sin \nu t}{\nu}, \tag{3.1}$$

and the corresponding transform

$$B_n = \sum_{\nu=1}^{n} u_\nu \rho_n^\nu \frac{\sin \nu t_n}{\nu t_n}, \qquad \rho_n \longrightarrow 1, \quad t_n \downarrow 0, \tag{3.2}$$

or

$$B_n = t_n^{-1} h_n(\rho_n, t_n) .$$

Let

$$s_n^k = \sum_{\nu=0}^{n} s_\nu \gamma_{n-\nu}^{k-1},$$

where

$$\gamma_n^k = \frac{(k+1)\cdots(k+n)}{n!} \sim \frac{n^k}{\Gamma(k+1)} \ ;$$

we also write

$$\Delta^k v_\nu = \sum_{r=1}^{k} (-1)^r \binom{k}{r} v_{\nu+r} \ ,$$

and

$$\sigma_n^k = \frac{s_n^k}{\gamma_n^k} \ .$$

Now (C, k) summability of the sequence $\{s_n\}$ to s is defined by

$$\lim_{n\to\infty} \sigma_n^k = s \ .$$

We quote the following elementary theorem [cf. 6, Theorem 1], which is included in a more general result of Mazur [1, Theorem X]:

LEMMA 1. *Let k be a given positive integer, and let*

$$T_n = \sum_{\nu=0}^{n} a_{n,\nu} s_\nu \ , \qquad n = 0, 1, 2, \cdots .$$

In order that $\lim T_n$ *exist, whenever the sequence $\{s_n\}$ is (C, k) summable to s, it is necessary and sufficient that*:

$$\sum_{\nu=0}^{n} \gamma_\nu^k \, |\Delta^k a_{n,\nu}| = O(1) \ , \qquad a_{n,\nu} = 0 \ \text{for} \ \nu > n \ ; \tag{3.3}$$

$$\lim_{n\to\infty} \gamma_\nu^k \, \Delta a_{n,\nu} = \alpha_\nu \ \textit{exists} \ , \qquad \nu = 0, 1, 2, \cdots ; \tag{3.4}$$

$$\lim_{n\to\infty} \sum_{\nu=0}^{n} a_{n,\nu} = \beta \ \textit{exists} \ . \tag{3.5}$$

We then have $\lim T_n = \beta s + \sum_{\nu=0}^{\infty} \alpha_\nu(\sigma_\nu^k - s)$. Since then the transform T_n

is convergence preserving we must have (3.5) and:

$$\lim_{n\to\infty} a_{n,\nu} \text{ exists}, \qquad \nu = 0, 1, 2, \cdots;$$

hence (3.4) and (3.5) hold, so that the conditions of Lemma 1 reduce to (3.3). In the case of the transform B_n, we have

$$a_{n,n} = \rho_n^n \frac{\sin nt_n}{nt_n},$$

$$a_{n,\nu} = \rho_n^\nu \frac{\sin \nu t_n}{\nu t_n} - \rho_n^{\nu+1} \frac{\sin (\nu+1) t_n}{(\nu+1) t_n}, \qquad \nu = 1, 2, \cdots, n-1;$$

hence

$$a_{n,\nu} \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty.$$

To satisfy (3.3) we must have

$$n^k \rho_n^n \frac{\sin nt_n}{nt_n} = O(1), \tag{3.6}$$

$$n^k \rho_n^{n-1} \frac{\sin (n-1) t_n}{(n-1) t_n} = O(1), \tag{3.7}$$

$$\cdots$$

$$n^k \rho_n^{n-k} \frac{\sin (n-k) t_n}{(n-k) t_n} = O(1),$$

and

$$\sum_{\nu=1}^{n-k-1} \nu^k \left| \Delta^{k+1} \rho_n^\nu \frac{\sin \nu t_n}{\nu t_n} \right| = O(1). \tag{3.8}$$

Assume first that $k = 0$; then our conditions become:

$$\rho_n^n \frac{\sin nt_n}{nt_n} = O(1), \tag{3.9}$$

and

$$\sum_{\nu=1}^{n-1} \rho_n^\nu \left| \frac{\sin \nu t_n}{\nu t_n} - \rho_n \frac{\sin (\nu+1) t_n}{(\nu+1) t_n} \right| = O(1). \tag{3.10}$$

We now prove the lemma:

LEMMA 2. *If*

$$\rho_n^n = O(1)\,, \quad \frac{1-\rho_n^n}{1-\rho_n}\, t_n = O(1)\,, \qquad as\ t_n \downarrow 0\,, \quad \rho_n \longrightarrow 1\,, \tag{3.11}$$

then B_n is a regular transform.

Clearly (3.9) holds, and we need only to show that (3.10) also holds.

If $\rho_n > 1$, then $\rho_n^\nu < \rho_n^n$, $\nu = 0, 1, \cdots, n-1$; if on the other hand $\rho_n \leq 1$, then $\rho_n^\nu \leq 1$. Hence, in either case,

$$\max_{0 \leq \nu \leq n} \rho_n^\nu = O(1)\,, \qquad \text{as } n \longrightarrow \infty.$$

We have

$$\sum_{\nu=1}^{n} \rho^\nu \left| \frac{\sin \nu t}{\nu} - \rho\, \frac{\sin\,(\nu+1)\,t}{\nu+1} \right| \leq \sum_{\nu=1}^{n} \rho^\nu \left| \frac{\sin \nu t}{\nu} - \frac{\sin\,(\nu+1)\,t}{\nu+1} \right|$$
$$+ (1-\rho) \sum_{\nu=1}^{n} \left| \frac{\sin\,(\nu+1)\,t}{\nu+1} \right| \rho^\nu\,;$$

the second term is $O(t)$, and

$$\frac{\sin \nu t}{\nu} - \frac{\sin\,(\nu+1)\,t}{\nu+1} = \int_0^t [\cos \nu x - \cos\,(\nu+1)x]\, dx = O(t^2)\,,$$

so that

$$\sum_{\nu=1}^{n} \rho^\nu \left| \frac{\sin \nu t}{\nu} - \frac{\sin\,(\nu+1)\,t}{\nu+1} \right| = O\left(t^2\, \frac{1-\rho^n}{1-\rho} \right).$$

Thus (3.10) is satisfied and Lemma 2 holds.

Note that the condition $\rho_n^n = O(1)$ is equivalent to $n(\rho_n - 1) < c$, a positive constant (see [5, p. 73]); furthermore, if $nt_n = O(1)$, then clearly the second condition of (3.11) holds.

Next let $k = 1$; we shall prove the theorem:

THEOREM 2. *If* (3.11) *holds, and if*

$$\rho_n^n \sin nt_n = O(t_n), \qquad n \longrightarrow \infty, \tag{3.12}$$

then B_n *includes* $(C, 1)$.

The conditions (3.6)–(3.8) now become:

$$\rho_n^n \sin nt_n = O(t_n),$$

$$\rho_n^n \sin (n-1)\, t_n = O(t_n),$$

and

$$\sum_{\nu=1}^{n-2} \nu \left| \Delta^2 \rho_n^\nu \frac{\sin \nu t_n}{\nu} \right| = O(t_n), \qquad \text{as } n \longrightarrow \infty. \tag{3.13}$$

Clearly, we need only to show that (3.13) is satisfied. Now

$$\Delta^2 \rho^\nu \frac{\sin \nu t}{\nu} = \Delta^2 \rho^\nu \int_0^t \cos \nu x \, dx = \Re \Delta^2 \int_0^t \rho^\nu e^{i\nu x} \, dx$$

$$= \Re \int_0^t \rho^\nu e^{i\nu x} (1 - 2\rho e^{ix} + \rho^2 e^{2ix}) \, dx$$

$$= \Re \int_0^t \rho^\nu e^{i\nu x} (1 - \rho e^{ix})^2 \, dx .$$

Hence

$$\left| \Delta^2 \rho^\nu \frac{\sin \nu t}{\nu} \right| < \rho^\nu \int_0^t |1 - \rho e^{ix}|^2 \, dx < \rho^\nu t \{(1-\rho)^2 + \rho t^2\} ;$$

it follows from (3.11) that

$$\sum_{\nu=1}^{n} \nu \left| \Delta^2 \rho_n^\nu \frac{\sin \nu t_n}{\nu t_n} \right| < \{(1-\rho_n)^2 + \rho_n t_n^2\} \sum_{\nu=1}^{n} \nu \rho_n^\nu = O(1) .$$

This proves (3.13) and Theorem 2.

4. **Comparison of** B_n **and** (C, k), $k \geq 2$. We wish to prove the following theorem:

THEOREM 3. *Suppose that* (3.11) *holds and that*

$$n^{k-1}\rho_n^n \sin nt_n = O(t_n)\,, \tag{4.1}$$

$$n^{k-1}\rho_n^n \cos nt_n = O(1)\,, \qquad \rho_n \longrightarrow 1\,, \qquad t_n \downarrow 0\,, \tag{4.2}$$

then B_n *includes* (C, k) *summability.*

Now (3.6) holds because of (4.1), and then (3.7) follows from (4.2). It remains to prove (3.8). We have

$$\Delta^{k+1}\rho^\nu \frac{\sin \nu t}{\nu} = \Delta^{k+1}\rho^\nu \int_0^t \cos \nu x \, dx = \Delta^{k+1}\Re \int_0^t \rho^\nu e^{i\nu x}\, dx$$

$$= \Re \int_0^t \rho^\nu e^{i\nu x}(1-\rho e^{ix})^{k+1}\, dx\,;$$

hence

$$\left|\Delta^{k+1}\rho^\nu \frac{\sin \nu t}{\nu}\right| < \rho^\nu \int_0^t |1-\rho e^{ix}|^{k+1}\, dx \tag{4.3}$$

$$< \rho^\nu \int_0^t \{(1-\rho)^2 + \rho t^2\}^{(k+1)/2}\, dx$$

$$= O(\rho^\nu t\{(1-\rho)^{k+1} + t^{k+1}\})\,.$$

It follows that

$$\sum_{\nu=1}^{n} \nu^k \left|\Delta^{k+1}\rho_n^\nu \frac{\sin \nu t_n}{\nu t_n}\right| = O\left(\sum_{\nu=1}^{n} \nu^k \rho_n^\nu\{(1-\rho_n)^{k+1} + t_n^{k+1}\}\right) \tag{4.4}$$

$$= O\left((1-\rho_n)^{k+1} \sum_{\nu=1}^{n} \nu^k \rho_n^\nu\right) + O\left(t_n^{k+1} \sum_{\nu=1}^{n} \nu^k \rho_n^\nu\right)\,.$$

Here the first term is $O(1)$ by Lemma 2 of [6]; finally

$$t_n^{k+1} \sum_{\nu=1}^{n} \nu^k \rho_n^\nu = O\left(t_n \sum_{\nu=1}^{n} \rho_n^\nu\right)^{k+1} = O(1)\,.$$

This proves Theorem 3.

An interesting special case is $t_n = \pi/n$; the conditions now reduce to the single condition

$$n^{k-1}\rho_n^n = O(1)\,.$$

If, in particular, $n^k\rho_n^n = O(1)$ for *all* k, then B_n includes all (C, k).

Observe that by Lemma 1 of [6] the condition $n^k\rho_n^n = O(1)$ is equivalent to

$$\limsup\,\{n(\rho_n - 1) + k\,\log n\} < +\infty\,.$$

Note also that (4.1) and (4.2) imply:

$$n^{k-1}\rho_n^n = O(1)\,.$$

5. Truncated Riemann summability. The series $\sum_{\nu=0}^{\infty} u_\nu$ is called (R, k) summable to s if the series

$$u_0 + \sum_{n=1}^{\infty}\left(\frac{\sin nt}{nt}\right)^k u_n = R_k(t) \tag{5.1}$$

converges in some interval $0 < t < t_0$, and if

$$R_k(t) \longrightarrow s\,, \qquad \text{as } t \longrightarrow 0\,.$$

For $k = 1$ it is sometimes called Lebesgue summability. The method (R, k) is regular for $k \geq 2$ and, in fact, it is more powerful than $(C, k-2)$; for $k = 2$, it was employed by Riemann in the theory of trigonometric series. We generate from it by truncation the triangular series to sequence transform $(u_0 = 0)$:

$$D_n = \sum_{\nu=1}^{n} u_\nu \left(\frac{\sin \nu t_n}{\nu t_n}\right)^k = \sum_{\nu=1}^{n-1} s_\nu \Delta\left(\frac{\sin \nu t_n}{\nu t_n}\right)^k + s_n\left(\frac{\sin n t_n}{n t_n}\right)^k\,;$$

k is a positive integer. We assume $k \geq 2$; it is then easy to show that D_n is a regular transformation.

From Lemma 1 we find for (C, k) to be included in D_n the conditions:

$$t_n^{-k}(\sin \overline{n-\nu}\,t_n)^k = O(1)\,, \qquad \text{for } \nu = 0, 1, \cdots, k; \tag{5.2}$$

$$\sum_{\nu=1}^{n-k-1} \nu^k \left|\Delta^{k+1}\left(\frac{\sin \nu t_n}{\nu t_n}\right)^k\right| = O(1)\,, \qquad n \longrightarrow \infty\,. \tag{5.3}$$

It follows from (5.2) (see Section 2) that we must have

(5.4) $$nt_n = p\pi + \alpha_n \,, \qquad n\,\alpha_n = O(1)\,, \qquad p \text{ a positive integer };$$

now (5.2) reduces to

$$t_n \sin\,(\alpha_n - \nu t_n) = O(1)\,, \qquad \nu = 0, 1, \cdots, k\,,$$

and this is satisfied in view of (5.4).

To show that now (5.3) also holds, we employ a lemma, due to Obreschkoff [2, p. 443]:

LEMMA 3. *We have*

$$\left| \Delta^m \left(\frac{\sin \nu t}{\nu t} \right)^k \right| \le M \, \frac{t^{m-k}}{\nu^k}\,,$$

where M is independent of t and ν.

It now follows that

$$\sum_{\nu=1}^{n} \nu^k \left| \Delta^{k+1} \left(\frac{\sin \nu t_n}{\nu t_n} \right)^k \right| = O(nt_n) = O(1)\,, \qquad n \longrightarrow \infty\,.$$

This yields the following theorem:

THEOREM 4. *If $nt_n = p\pi + \alpha_n$, p a positive integer, $n\alpha_n = O(1)$, then the transform*

$$\sum_{\nu=1}^{n} u_\nu \left(\frac{\sin \nu t_n}{\nu t_n} \right)^k = D_n$$

includes (C, k) summability (k a positive integer).

6. A converse theorem. We shall establish the following result.

THEOREM 5. *If*

(6.1) $$\liminf \left| \frac{\sin nt_n}{nt_n} \right|^k = \lambda > 1/2\,,$$

then the transform D_n is equivalent to convergence.

It follows from (6.1) that $\limsup nt_n < 2^{1/k}$; hence (see Sections 1 and 5) the transform D_n is regular. We now wish to show that $D_n \longrightarrow s$ implies $s_n \longrightarrow s$; we follow a device used by R. Rado [3].

Assume first that $s = 0$, and that $s_n = 0(1)$; then

$$0 \leq \limsup_{n\to\infty} |s_n| = \delta < \infty ,$$

and we shall show that $\delta = 0$. To a given $\epsilon > 0$ choose $n = n(\epsilon)$ so that $|s_\nu| < \delta + \epsilon$ for $\nu \geq n$. Next choose $m > n$ and such that $|s_m| > \delta - \epsilon$. We have

$$s_m \left(\frac{\sin mt_m}{mt_m}\right)^k = D_m - \sum_{\nu=1}^{m-1} s_\nu \Delta_\nu ,$$

where

$$\Delta_\nu = \left(\frac{\sin \nu t_m}{\nu t_m}\right)^k - \left(\frac{\sin (\nu+1)\, t_m}{(\nu+1)\, t_m}\right)^k ;$$

hence, as $mt_m < \pi$, we have

$$|s_m| \left|\frac{\sin mt_m}{mt_m}\right|^k < |D_m| + \left|\sum_{\nu=1}^{n-1} s_\nu \Delta_\nu\right| + \left|\sum_{\nu=n}^{m-1} s_\nu \Delta_\nu\right|$$

$$< o(1) + (\delta + \epsilon) \left\{\left(\frac{\sin nt_m}{nt_m}\right)^k - \left(\frac{\sin mt_m}{mt_m}\right)^k\right\} .$$

It follows that

$$\delta - \epsilon < |s_m| < o(1) + (\delta + \epsilon) \{1/\lambda - 1 + o(1)\} .$$

But $1/\lambda < 2$, and ϵ is arbitrarily small; hence $\delta = 0$.

We next assume $s = 0$ and $\limsup |s_n| = \infty$; choose $\epsilon > 0$ and ω large. Denote by $m = m(\omega)$ the least m for which $|s_m| > \omega$; then

$$\omega < |s_m| < o(1) + \omega\{1/\lambda - 1 + o(1)\} .$$

But this is impossible for $\lambda > 1/2$, small ϵ, and large m. This proves our theorem for $s = 0$. Finally, applying this result to the sequence $\{s_n - s\}$ and its transform completes the proof of Theorem 5.

7. Application to Fourier series. Suppose that $f(x)$ is a Lebesgue integrable function of period 2π, and let

$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \Sigma\, u_n(x)\,; \tag{7.1}$$

we may assume here $a_0 = 0$. Now (cf. [7, p. 27])

$$F(x) = \int_0^x f(t)\, dt = C + \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \frac{1}{n}\,,$$

where

$$C = \sum_{n=1}^{\infty} \frac{1}{n} b_n\,.$$

It is known [7, p. 55] that at every point x where $F'(x)$ exists and is finite, the series (6.1) is summable (C, r), $r > 1$, to the value $F'(x)$.

It now follows from Theorem 3 for $k = 2$ and $t_n = \pi/n$ that if $n\rho_n^n = O(1)$, then

$$\sum_{\nu=1}^{n} u_\nu(x)\, \rho_n^\nu \frac{\sin \nu\pi/n}{\nu\pi/n} \longrightarrow F'(x)\,.$$

Furthermore, Theorem 4 yields, for $k = 2$, that if

$$nt_n = p\pi + \alpha_n\,, \qquad n\alpha_n = O(1)\,,$$

then

$$\sum_{\nu=1}^{n} u_\nu(x) \left(\frac{\sin \nu t_n}{\nu t_n}\right)^2 \longrightarrow F'(x)\,.$$

An analogous theorem holds for higher derivatives (cf. [7, p. 257]).

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