Pacific Journal of Mathematics

ON THE THEORY OF SPACES Λ

G. G. LORENTZ

Vol. 1, No. 3 BadMonth 1951

ON THE THEORY OF SPACES Λ

G. G. LORENTZ

1. Introduction. In this paper we discuss properties of the spaces $\Lambda(\phi, p)$, which were defined for the special case $\phi(x) = \alpha x^{\alpha-1}$, $0 < \alpha \leq 1$, in our previous paper [8]. A function $f(x)$, measurable on the interval $(0, l)$, $l < +\infty$ belongs to the class $\Lambda(\phi, p)$ provided the norm $\|f\|$, defined by

$$\|f\| \equiv \left\{\int_0^l \phi(x)\, f^*(x)^p dx\right\}^{1/p}, \tag{1.1}$$

is finite. Here $\phi(x)$ is a given nonnegative integrable function on $(0, l)$, not identically 0, and $f^*(x)$ is the decreasing rearrangement of $|f(x)|$, that is, the decreasing function on $(0, l)$, equimeasurable with $|f(x)|$. (For the properties of decreasing rearrangements see [5, 12, 7, and 8].) We write also $\Lambda(\alpha, p)$ instead of $\Lambda(\phi, p)$ with $\phi(x) = \alpha x^{\alpha-1}$, and $\Lambda(\phi)$ instead of $\Lambda(\phi, 1)$. We shall also consider spaces $\Lambda(\phi, p)$ for the infinite interval $(0, +\infty)$. In §2 we give some simple properties of the spaces Λ, and show in particular that $\Lambda(\phi, p)$ has the triangle property if and only if $\phi(x)$ is decreasing. In §3 we discuss the conjugate spaces $\Lambda^*(\phi, p)$, and show that the spaces $\Lambda(\phi, p)$ are reflexive. In §4 we give a generalization of the spaces $\Lambda(\phi, p)$, and characterize the conjugate spaces in case $p = 1$. In §5 we give applications; we prove that the Hardy-Littlewood majorants $\theta(x, f)$ of a function $f \in \Lambda(\phi, p)$ or $f \in \Lambda^*(\phi, p)$ also belong to the same class. We give sufficient conditions for an integral transformation to be a linear operation from one of these spaces into itself, and apply them to solve the moment problem for the spaces $\Lambda(\phi, p)$ and $\Lambda^*(\phi, p)$.

2. Properties of spaces $\Lambda(\phi, p)$. We shall establish the following result.

THEOREM 1. *The norm $\|f\|$ defined by* (1.1) *has the triangle property if and only if $\phi(x)$ is equivalent to a decreasing function; in this case $f, g \in \Lambda(\phi, p)$*

Received April 11, 1951.

A large part of this investigation was carried out while the author held a fellowship at the Summer Institute of the Canadian Mathematical Congress in 1950.

Pacific J. Math. 1 (1951), 411-429.

implies $f+g \in \Lambda(\phi, p)$.

Proof. (a) Suppose that $\|f\|$ has the triangle property. Let $\delta > 0$, $h > 0$, $a > 0$, and $a + 2h \leq l$. Set

$$f(x) = \begin{cases} 1+\delta & \text{on } (0, a+h) \\ 1 & \text{on } (a+h, a+2h) \\ 0 & \text{on } (a+2h, l), \end{cases} \qquad g(x) = \begin{cases} 1 & \text{on } (0, h) \\ 1+\delta & \text{on } (h, a+2h) \\ 0 & \text{on } (a+2h, l); \end{cases}$$

then

$$(f+g)^*(x) = \begin{cases} 2+2\delta & \text{on } (0, a) \\ 2+\delta & \text{on } (a, a+2h) \\ 0 & \text{on } (a+2h, l). \end{cases}$$

We have $\|f\| = \|g\|$; hence the inequality $\|f+g\| \leq \|f\| + \|g\|$ is equivalent to

$$\left\{(2+2\delta)^p \int_0^a \phi(x)\,dx + (2+\delta)^p \int_a^{a+2h} \phi(x)\,dx\right\}^{1/p}$$
$$\leq 2\left\{(1+\delta)^p \int_0^{a+h} \phi(x)\,dx + \int_{a+h}^{a+2h} \phi(x)\,dx\right\}^{1/p},$$

or to

$$(2+\delta)^p \int_a^{a+2h} \phi(x)\,dx \leq (2+2\delta)^p \int_a^{a+h} \phi(x)\,dx + 2^p \int_{a+h}^{a+2h} \phi(x)\,dx,$$

and thus to

$$\frac{(1+\delta)^p - (1+\frac{1}{2}\delta)^p}{(1+\frac{1}{2}\delta)^p - 1} \int_a^{a+h} \phi(x)\,dx \geq \int_{a+h}^{a+2h} \phi(x)\,dx. \tag{2.1}$$

If $\Phi(x)$ is the integral of ϕ over $(0, x)$, we obtain from (2.1), making $\delta \longrightarrow 0$,

$$\Phi(a+h) \geq \tfrac{1}{2}\left[\Phi(a) + \Phi(a+2h)\right];$$

that is, $\Phi(x)$ is concave, and thus $\phi(x)$ is equivalent to a decreasing function.

(b) Suppose that ϕ is decreasing. Instead of (2.1) we can now write

$$\|f\| = \sup_{\phi_r} \left\{\int_0^l \phi_r |f|^p\,dx\right\}^{1/p}, \tag{2.2}$$

the supremum being taken over all possible rearrangements ϕ_r of ϕ. It follows from (2.2) that $f, g \in \Lambda(\phi, p)$ implies $f+g \in \Lambda(\phi, p)$ and $\|f+g\| \leq \|f\| + \|g\|$.

It is now easy to see that, for $\phi(x)$ decreasing, $\Lambda(\phi, p)$ is a Banach space; the completeness may be proved by usual methods (compare [8]). In general, $\Lambda(\phi, p)$ is not uniformly convex. Suppose, for instance, that there is a sequence $\delta_n \longrightarrow 0$ such that

$$\Phi(2\delta_n)/\Phi(\delta_n) \longrightarrow 1. \tag{2.3}$$

This condition is satisfied, for example, if $\phi(x) = x^{-1}\,|\log x|^{-p}$, $p > 1$. We take $f_n(x) = h_n$ on $(0, 2\delta_n)$, $f_n(x) = 0$ on $(2\delta_n, l)$; we take $g_n(x) = h_n$ on $(0, \delta_n)$, $g_n(x) = -h_n$ on $(\delta_n, 2\delta_n)$, and $g_n(x) = 0$ on $(2\delta_n, l)$; and we choose h_n so that

$$\|f_n\|^p = \|g_n\|^p = h_n^p\,\Phi(2\delta_n) = 1.$$

Then we have

$$\tfrac{1}{2}\{f_n(x) + g_n(x)\} = \begin{cases} h_n & \text{on } (0, \delta_n), \\ 0 & \text{elsewhere}, \end{cases}$$

and $(1/2)(f_n - g_n)^*(x)$ is the same function. Therefore

$$\left\|\frac{f_n + g_n}{2}\right\|^p = \left\|\frac{f_n - g_n}{2}\right\|^p = h_n^p\,\Phi(\delta_n) \longrightarrow 1,$$

and so $\Lambda(\phi, p)$ is not uniformly convex. In case of the spaces $\Lambda(\alpha, p)$, the problem remains open.

The remarks made above apply also to the spaces $\Lambda(\phi, p)$ in case of the infinite interval $(0, +\infty)$. We assume in this case that $\int_0^l \phi(x)\,dx < +\infty$ for any $l < +\infty$; the additional hypothesis on $f \in \Lambda(\phi, p)$ is that the rearrangement $f^*(x)$ exists, which is the case if and only if any set $[\,|f(x)| \geq \epsilon]$, $\epsilon > 0$, has finite measure. The completeness of $\Lambda(\phi, p)$ in this case follows from the fact that the set of such f is a closed linear subset of the Banach space of all f for which (2.2) is finite. If

$$\int_0^{+\infty} \phi(x)\,dx = +\infty, \tag{2.4}$$

this subspace coincides with the whole space. Condition (2.4) is in particular satisfied if $\phi(x) = \alpha x^{\alpha - 1}$.

3. Reflexivity of the spaces $\Lambda(\phi, p)$. We shall first give some definitions and lemmas which will be useful in the sequel. If $g(x)$, $g_1(x)$ are two positive functions defined on $(0, l)$, $0 < l \leq +\infty$, we write $g \prec g_1$, if for all finite $0 \leq x \leq l$ we have

$$\int_0^x g(t)\,dt \leq \int_0^x g_1(t)\,dt.$$

Integration by parts readily yields:

LEMMA 1. *If $g \prec g_1$, and f is positive and decreasing on $(0, l)$, then*

$$\int_0^l g f\,dx \leq \int_0^l g_1 f\,dx. \tag{3.1}$$

LEMMA 2. *If $g \prec g_1$, and g, g_1 are positive and decreasing, then also $\psi(g) \prec \psi(g_1)$ for any convex increasing positive function, in particular for $\psi(u) = u^p$, $p \geq 1$.*

For the proof, let $f(x) = \{\psi(g_1(x)) - \psi(g(x))\}/\{g_1(x) - g(x)\}$ if $g(x) \neq g_1(x)$, and let $f(x)$ be equal to one of the derivates of $\psi(u)$ at $u = g(x)$ if $g(x) = g_1(x)$. Then $f(x)$ is the slope of the chord of the curve $v = \psi(u)$ on the interval (u, u_1), $u = g(x)$, $u_1 = g_1(x)$. The slope decreases as both u, u_1 decrease. Therefore $f(x)$ is decreasing and positive. Applying Lemma 1, we obtain

$$\int_0^l f(x)[g(x) - g_1(x)]\,dx \leq 0,$$

which proves our assertion.

THEOREM 2. *Suppose that $f(x)$, $g(x)$ are positive and decreasing on $(0, l)$, and $f \in \Lambda(\phi, p)$, $p > 1$. Then*

$$\int_0^l f g\,dx \leq \|f\|_\Lambda \inf_{\phi D \succ g} \left\{ \int_0^l \phi D^q\,dx \right\}^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{3.2}$$

where infimum is taken for all decreasing positive $D(x)$ for which $\phi D \succ g$. Moreover, this infimum is equal to the supremum of $\int_0^l fg\,dx$ for all positive decreasing f with $\|f\| \leq 1$, if there is a function D with $\phi D \succ g$ and $\int \phi D^q\,dx < +\infty$, and is to $+\infty$ if there is no such D.

This theorem is due to I. Halperin. For the proofs, see a paper of Halperin appearing in the Canadian Journal of Mathematics and, for a simpler proof, [10].

Inequality (3.2) is a combination of (3.1) and the usual Hölder inequality. For if $g_1 = \phi D \succ g$, then

$$\int_0^l fg\,dx \leq \int_0^l fg_1\,dx = \int_0^l \phi^{1/p} f \phi^{1/q} D\,dx \tag{3.3}$$

$$\leq \|f\| \left\{\int_0^l \phi D^q\,dx\right\}^{1/q}.$$

Here and in the next section, the following theorem will be useful:

THEOREM 3. *Suppose that X is a normed linear space of measurable functions $f(x)$ on $(0,l)$, $0 < l < +\infty$, with the properties:* (i) *X contains all constants*; (ii) *if f_1 is measurable and $|f_1(x)| \leq |f(x)|$, $f \in X$, then $f_1 \in X$ and $\|f_1\| \leq \|f\|$*; (iii) *if $f \in X$ and f_e denotes the characteristic function of the set e, then $\|f f_e\| \to 0$ as* meas $e \to 0$.

Let Y consist of all measurable functions g for which $\int_0^l fg\,dx$ exists for all $f \in X$. Then

$$F(f) = \int_0^l fg\,dx, \qquad g \in Y, \tag{3.4}$$

is the general form of a linear functional on X, and its norm is equal to

$$\|g\| \equiv \sup_{\|f\| \leq 1} \int_0^l fg\,dx < +\infty.$$

Proof. (a) Let $g \in Y$; then $\int_0^l f|g|\,dx$ exists for all $f \in X$, and $\|g\| = \sup \int_0^l f|g|\,dx$, where f runs through all *positive* $f \in X$ with $\|f\| \leq 1$. If $\|g\| = +\infty$, there is a sequence $f_n \geq 0$, $\|f_n\| \leq 1$ such that $\int f_n|g|\,dx > n^3$. Then $f = \sum n^{-2} f_n \in X$, and therefore $\int_0^l f|g|\,dx$ must exist. However $\int f|g|\,dx \geq n^{-2}\int f_n|g|\,dx \geq n$, which is a contradiction. Hence $\|g\| < +\infty$ for $g \in Y$. We see now that for $g \in Y$, $\int fg\,dx$ is a linear functional with norm $\|g\|$.

(b) Suppose that $F(f)$ is a given linear functional on X. By (i) and (ii), any characteristic function $f_e(x)$ belongs to X. Define $G(e) = F(f_e)$; since $|G(e) \leq \|F\|\,\|f_e\| \to 0$ as meas $e \to 0$, there is an integrable $g(x)$ with $G(e) = \int_e g\,dx$. This means that (3.4) holds for $f = f_e$, and therefore also for all step-functions $\bar{f}$ (which are linear combinations of the f_e). For a bounded f, there is a sequence $\bar{f}_n(x) \to f(x)$ uniformly. As $\|\bar{f}_n - f\| \to 0$, this establishes (3.4) for all bounded f. Now suppose $f \in X$ is such that $fg = |f|\,|g|$. Let $f_n(x) = f(x)$ if $|f(x)| \leq n$,

$f_n(x) = 0$ otherwise; then $\|f - f_n\| \longrightarrow 0$ by (iii), and hence $\int_0^l f_n g\, dx = F(f_n)$ has a finite limit. This shows that $\int |f|\, |g|\, dx < +\infty$; therefore $g \in Y$. Repeating the last part of this argument for an arbitrary $f \in A$, we obtain (3.4).

REMARKS. (A) Let X have the additional property: (iv) $f_n(x) \longrightarrow f(x)$ *almost everywhere,* $f_n \in X$, *and* $\|f_n\| \le M$ *imply* $f \in X$. Then the existence of $\int fg\, dx$ for all $g \in Y$ implies $f \in X$.

For taking the subsequence $f_n(x) \longrightarrow f(x)$ of (b), we see that $F_n(g) = \int f_n g\, dx$ is a sequence of linear functionals convergent toward $\int fg\, dx$ for any $g \in Y$. Then the norms $\|F_n\| = \|f_n\|$ are uniformly bounded, and using (iv) we obtain $f \in X$.

(B) Since Y is the conjugate space to X, Y is a Banach space, and Y clearly satisfies (ii). Suppose now that X satisfies (i)–(iv) and that Y satisfies (i) and (iii). Then Remark A and Theorem 3 together imply that X is the conjugate space of Y, in other words that any linear functional $F(g)$ in Y is of the form $F(g) = \int fg\, dx$, $f \in X$ and $\|F\| = \|f\|$.

(C) The above results hold for the interval $(0, +\infty)$ if the conditions (i)–(iii) [and eventually (iv)] are true for functions vanishing outside of a finite interval, and also (v) *for any* $f \in X$, $\|f - f^l\| \longrightarrow 0$ *as* $l \longrightarrow \infty$, *where* f^l *is defined by* $f^l(x) = f(x)$ *on* $(0, l)$ *and* $f^l(x) = 0$ *on* $(l, +\infty)$.

Applying these general results to the space $\Lambda(\phi, p)$ in case of a finite interval, we see that (i) and (ii) are satisfied. Condition (iii) follows from

$$\|h_e\|^p \le \int_0^{\text{meas } e} \phi f^{*p}\, dx \longrightarrow 0, \qquad \text{meas } e \longrightarrow 0,$$

[$h_e(x)$ is the function $f(x) f_e(x)$], and (iv) from (2.2) and Fatou's theorem. We obtain the result that *the space* $\Lambda^*(\phi, p)$ *conjugate to* $\Lambda(\phi, p)$ *consists of all measurable functions* g *such that there is a decreasing positive* D *with* $\phi D > g^*$ *and* $\int_0^l \phi D^q\, dx < +\infty$; *further,*

$$\|g\|_{\Lambda^*} = \inf_{\phi D > g^*} \left\{ \int_0^l \phi D^q\, dx \right\}^{1/q}. \tag{3.5}$$

For it follows from Theorem 2 that

$$\left| \int_0^l f g\, dx \right| \le \int_0^l f^* g^*\, dx \le \|f\|_\Lambda \, \|g\|_{\Lambda^*},$$

and that $\|g\|_{\Lambda^*}$ is the supremum of the integral $\int fg\,dx$ for all $\|f\|\leq 1$.

Now if $g(x) = C > 0$ is a constant, we take an $l_1 > 0$ with $\phi(l_1) > 0$ and $C_1 = Cl[l_1\phi(l_1)]^{-1}$. Then $\int_0^{l_1} C_1\phi(x)\,dx \geq Cl$; and if $D(x) = C_1$ on $(0,l_1)$, $D(x) = 0$ on (l_1,l), then $\phi D \succ g$. Therefore Λ^* satisfies (i). Also (iii) holds, for if $h_e(x) = g(x)f_e(x)$, $g \in \Lambda^*$, $g^* \prec \phi D$, then $h_e^* \prec \phi D_1$, where $D_1(x) = D(x)$ on $(0, \text{meas } e)$, $D_1(x) = 0$ on $(\text{meas } e, l)$, and

$$\|h_e\|_{\Lambda^*}^q \leq \int_0^l \phi D_1^q\,dx = \int_0^{\text{meas } e} \phi D^q\,dx \longrightarrow 0, \qquad \text{meas } e \longrightarrow 0.$$

We have proved the theorem:

THEOREM 4. *The space $\Lambda(\phi,p)$, $p > 1$, is reflexive. Its conjugate is defined by* (3.5).

We now consider the case of an infinite interval and assume $\int_0^\infty \phi\,dx = +\infty$. Then $f \in \Lambda(\phi,p)$ implies $f^*(x) \longrightarrow 0$ for $x \longrightarrow \infty$. If $a > 0$ is fixed and l sufficiently large, then the function $|f^l(x)|$ of (v) will take values $\geq f^*(a)$ only on a set of arbitrarily small measure. In view of (iii), condition (v) will follow for $\Lambda(\phi,p)$, if we can show that the norm of the function $f^*(a+x)$, $0 \leq x < +\infty$, tends to 0 as $a \longrightarrow \infty$, or even if this is true for some sequence $a \longrightarrow \infty$. This norm does not exceed

$$\left\{\int_0^\infty \phi(x)f^*(a+x)^p\,dx\right\}^{1/p} = \left\{\int_0^\infty \phi(x)f^*(x)^p\left[\frac{f^*(x+a)}{f^*(x)}\right]^p dx\right\}^{1/p} \longrightarrow 0,$$

as the integrand has the majorant ϕf^{*p}, and $f^*(x+a)/f^*(x) \longrightarrow 0$ for $a \longrightarrow \infty$.

To prove (v) for $\Lambda^*(\phi,p)$, we need a result going beyond Lemma 1, namely that if g and D are decreasing and positive, and $\phi D \succ g$, then there is another such function D_0 for which $\phi D \succ \phi D_0 \succ g$, and that except for certain open intervals I where D_0 is constant, $\int_0^x \phi D_0\,dt = \int_0^x g\,dt$. (This fact is proved in the paper of Halperin, mentioned at the beginning of this section and in [10]). As before, we have to prove that if $g \in \Lambda^*(\phi,p)$ is positive and decreasing, then the norm of the function $h(x) = g(x+a)$, $x \geq 0$, tends to 0 as $a \longrightarrow \infty$ for certain values of a. There is a D with $\phi D \succ g$ and $\int_0^\infty \phi D^q\,dx < +\infty$; and, by Lemma 2, $\int_0^\infty \phi D_0^q\,dx < +\infty$. As $\int_0^\infty \phi dx = +\infty$, we deduce that $D_0(x) \longrightarrow 0$ for $x \longrightarrow \infty$. Therefore

$$\int_0^x \phi D_0\,dx = o[\Phi(x)].$$

On intervals I, $\int_0^x \phi D_0\, dt$ is of the form $C\Phi(x) + C_1$, where $\Phi(x) = \int_0^x \phi\, dt$. If an I extends to $+\infty$, we have $C = 0$, that is $\int_0^x \phi D_0\, dt = C_1$ for all large x. and $D_0(x)$ is necessarily 0 for all such x. In this case also $g(x) = 0$ for all large x, and our assertion is trivial. If, on the other hand, there are arbitrarily large values a which do not belong to any I, then we have for these a,

$$\int_0^a \phi D_0\, dt = \int_0^a g\, dt .$$

It follows that $\int_0^x \phi D_0\, dt \geq \int_0^x g\, dt$, $x \geq a$, or $\phi(x + a) D_0(x + a) \succ g(x + a)$, and this implies $\phi(x) D_0(x + a) \succ g(x + a)$. Therefore,

$$\|h\|^q \leq \int_0^\infty \phi(x) D_0(x + a)^q\, dx = \int_0^\infty \phi(x) D_0(x)^q \left[\frac{D_0(x + a)}{D_0(x)}\right]^q dx \longrightarrow 0$$

for $a \longrightarrow \infty$. We obtain in this way:

THEOREM 5. *The space* $\Lambda(\phi, p)$, $p > 1$, $l = \infty$ *is reflexive; its conjugate is given by* (3.5).

4. A generalization. There is an obvious generalization of the spaces $\Lambda(\phi, p)$. Consider a class C of functions $\phi(x) \geq 0$ integrable over $(0, l)$, and let $X(C, p)$ consist of all those functions $f(x)$ for which

$$\|f\| = \sup_{\phi \in C} \left\{\int_0^l \phi\, |f|^p\, dx\right\}^{1/p} < +\infty. \tag{4.1}$$

A special type of these spaces is obtained if C is chosen to consist of all integrable positive functions $\phi(x)$ whose integrals $\phi_1(e)$ satisfy the condition

$$\phi_1(e) \leq \Phi(e) , \tag{4.2}$$

where $\Phi(e)$ is a given positive finite set function of measurable sets $e \subset (0, l)$. We may then assume that

$$\Phi(e) = \sup_{\phi_1} \phi_1(e) . \tag{4.3}$$

(A full characterization of set functions $\Phi(e)$ which may be represented in form (4.3) by means of a class of positive additive ϕ_1 will be given by the author elsewhere [9].) In particular, let $\phi_0(x)$ be a fixed decreasing positive function, and let $\Phi(e) = \int_0^{\text{meas}\, e} \phi_0\, dx$; then condition (4.2) is equivalent to the condition

$$\phi^*(x) \leq \phi_0(x) \,.$$

Therefore, in this case the norm (4.1) is equal to (1.1), and so $X(\Phi, p) = \Lambda(\phi_0, p)$.

For the space $X(\Phi, p)$, the condition $\|f\| = 0$ is equivalent to $f(x) = 0$ almost everywhere if and only if $\Phi(e) > 0$ for any set e of positive measure. Suppose now that $\Phi(e)$, defined by (4.3), vanishes on certain sets e with meas $e > 0$. There is then [2, p. 80, Theorem 15] a least measurable set e_0 which contains any such set e up to a null set; and e_0 is a union of a properly chosen denumerable set of these sets e. Hence $\phi_1(e_0) = 0$, and $\Phi(e_0) = 0$. It is easy to see that in this case $\|f\| = 0$ is equivalent to $f(x) = 0$ almost everywhere on $(0, l) - e_0$, and that the values of $f(x)$ on e_0 have no significance whatsoever for $\|f\|$. Omitting e_0 from $(0, l)$, we do not change the space $X(\phi, p)$, and we obtain a $\Phi(e)$ satisfying the above condition. In the sequel, ϕ is assumed to have this property.

The spaces $X(\Phi, p)$ are normed linear spaces. Their completeness may be proved by usual methods, if for instance $F(e)$ has the property that meas $e \longrightarrow 0$ implies $\Phi(e) \longrightarrow 0$ and if $l < +\infty$.

The spaces $X(C, p)$ satisfy the conditions (i), (ii), and (iv) of 3 [(iv) follows easily by Fatou's theorem]. Condition (iii) is not fulfilled in general. We can however enforce (iii) by defining the spaces $\Lambda(C, p)$ and $\Lambda(\Phi, p)$ to consist of all those functions $f \in X(C, p)$ or $f \in X(\Phi, p)$, respectively, for which $\|f f_e\| \longrightarrow 0$ with meas $e \longrightarrow 0$ in X. Then the conjugate space $\Lambda^*(C, p)$ and all linear functionals in $\Lambda(C, p)$ are given by Theorem 3. We conclude this section by describing the spaces $\Lambda^*(\Phi, 1)$ more precisely:

THEOREM 6. *If $f \in \Lambda(\Phi, 1)$, then*

$$\left| \int_0^l f g \, dx \right| \leq \|f\| \sup_{\Phi(e) > 0} \frac{1}{\Phi(e)} \int_e |g| \, dx \,, \tag{4.4}$$

and the left integral exists provided the right side is finite; moreover, the supremum $M(g)$ in the right side is equal to the supremum of $\int_0^l fg \, dx$ for all $f \in \Lambda(\Phi, 1)$ with $\|f\| \leq 1$.

Proof. Consider the function $\phi_0(x) = M(g)^{-1} \, |g(x)|$; then

$$\int_0^l |f| \, |g| \, dx = M(g) \int_0^l \phi_0 \, |f| \, dx \leq M(g) \, \|f\|_\Lambda \,,$$

since

$$\int_e \phi_0(x)\,dx = M(g)^{-1}\int_e g(x)\,dx \le \Phi(e)\,, \qquad e \subset (0,\,l)\,.$$

This proves (4.4). On the other hand, if e is an arbitrary subset of $(0,l)$ with $\Phi(e) > 0$, then the function $f(x) = \Phi(e)^{-1} f_e(x)\ \mathrm{sign}\, g(x)$ has norm 1 in $\Lambda(\phi, 1)$, and

$$\int_0^l fg\,dx = \Phi(e)^{-1}\int_e |g|\,dx\,.$$

Therefore the integral $\int_0^l fg\,dx$ takes values arbitrarily close to $M(g)$.

From Theorems 3 and 6 we deduce that the space $\mathrm{M}(\Phi, 1) = \Lambda^*(\Phi, 1)$ consists of all $g(x)$ for which

$$\|g\| = \sup_e \left\{\Phi(e)^{-1}\int_e |g(x)|\,dx\right\} < +\infty. \tag{4.5}$$

In particular, the space $\mathrm{M}(\phi)$, conjugate to $\Lambda(\phi)$, is given by

$$\|g\|_{\mathrm{M}(\phi)} = \sup_e \left\{\phi_1(e)^{-1}\int_e |g|\,dx\right\}. \tag{4.6}$$

It is easy to see that the expression (4.6) is the limit, for $p \longrightarrow 1$, of the norm of g in the space $\Lambda^*(\phi, p)$, $p > 1$.

5. Applications. We shall make three applications.

5.1. *Hardy-Littlewood majorants.* We take in this section $l = 1$. We write

$$\theta(x, f) = \sup_{0 \le y \le 1} \frac{1}{y - x}\int_x^y |f(t)|\,dt\,, \tag{5.1}$$

and denote by $\theta_1(x, f)$ and $\theta_2(x, f)$ the supremum of the same expression for $0 \le y < x$ or $x < y \le 1$, respectively. Then

$$\theta(x, f) \le \max\,\{\theta_1(x, f),\ \theta_2(x, f)\}\,. \tag{5.2}$$

On the other hand, it is well known [5, p. 291] that

$$\theta_1^*(x, f) \le \theta(x, f^*) = \frac{1}{x}\int_0^x f^*(t)\,dt\,, \tag{5.3}$$

and this is also true with θ_2 in place of θ_1. From (5.2) we derive, for any $p \ge 1$,

$$\theta^p(x, f) \leq \theta_1^p(x, f) + \theta_2^p(x, f) \, .$$

It follows that

$$\theta^*(x, f)^p \leq (\theta_1^p + \theta_2^p)^* \prec (\theta_1^p)^* + (\theta_2^p)^* = \theta_1^{*p} + \theta_2^{*p} \leq 2\theta(x, f^*)^p \, ;$$

that is,

$$\theta^*(x, f)^p \prec 2\theta(x, f^*)^p \, . \tag{5.4}$$

We shall make repeated use of the inequality of Hardy [12, p.72] :

$$\int_0^l x^{s-p} F(x)^p \, dx \leq \left(\frac{p}{p-s-1}\right)^p \int_0^l x^s f(x)^p \, dx \, , \tag{5.5}$$

where $p > 1$, $s < p - 1$, $0 < l \leq +\infty$, and $F(x)$ is the integral of the positive function $f(x)$.

In our present situation it follows from (5.3) and (5.5), if $p > 1$, that

$$\int_0^x \theta(t, f^*)^p \, dt \leq \left(\frac{p}{p-1}\right)^p \int_0^x f^*(t)^p \, dt \, ;$$

and, by Lemma 1,

$$\int_0^1 \phi(x) \theta^*(x, f)^p \, dx \leq 2\left(\frac{p}{p-1}\right)^p \int_0^1 \phi(x) f^*(x)^p \, dx \, . \tag{5.6}$$

This is case (i) of the following theorem:

THEOREM 7. (i) *If* $f \in \Lambda(\phi, p)$ *and* $p > 1$, *then also* $\theta(x, f) \in \Lambda(\phi, p)$; (ii) *if* $f^*(x) \log (1/x) \in \Lambda(\phi)$, *then* $\theta(x, f) \in \Lambda(\phi)$; (iii) *if* $f \in \Lambda(\phi)$, *and* $\phi(x)$ *is decreasing with respect to* $x^{-\delta}$ *for some* $\delta > 0$, *then* $\theta(x, f) \in \Lambda(\phi)$.

To prove (ii) we observe that (5.4) with $p = 1$ and Lemma 1 imply

$$\|\theta\|_{\Lambda(\phi)} = \int_0^1 \phi(x) \theta^*(x, f) \, dx \leq 2\int_0^1 \phi(x) \frac{1}{x} dx \int_0^x f^*(t) \, dt$$

$$= 2\int_0^1 f^*(t) \, dt \int_t^1 \frac{\phi(x)}{x} dx \leq 2\int_0^1 \phi(t) f^*(t) \log \frac{1}{t} dt < +\infty .$$

Finally, if the hypothesis of (iii) holds, that is if $\phi(x) = x^{-\delta} D(x)$ with a decreasing positive D, then the preceeding inequality gives

$$\|\theta\| \leq 2 \int_0^1 f^*(t) D(t) \int_t^1 x^{-\delta-1} dx \leq 2\,\delta^{-1} \int_0^1 \phi(t) f^*(t)\, dt\,.$$

THEOREM 8. (i) *If* $f^*(x) \log(1/x) \in \Lambda^*(\phi, p)$, $p \geq 1$, *then* $\theta(x,f) \in \Lambda^*(\phi, p)$; (ii) *if* $f \in \Lambda^*(\alpha, p)$, $p > 1$, *then* $\theta(f) \in \Lambda^*(\alpha, p)$.

Proof. (i) Let $p > 1$ [the case $p = 1$, $\Lambda^*(\phi, p) = M(\phi)$ is simpler]. By (5.4), and since $\theta(x, f^*)$ decreases, we have

$$\|\theta(f)\|^q \leq 2^q \|\theta(f^*)\|^q = 2^q \inf_{\phi D \succ \theta(f^*)} \int_0^1 \phi(x) D(x)^q\, dx\,.$$

But by (5.3), we have

$$\int_0^x \theta(u, f^*)\, du = \int_0^x f^*(t)\, dt \int_t^x \frac{du}{u} \leq \int_0^x f^*(t) \log \frac{1}{t}\, dt\,,$$

which means that $\theta(x, f^*) \prec f^*(x) \log(1/x) = h(x)$; hence

$$\|\theta(f)\|^q \leq 2^q \inf_{\phi D \succ h} \int_0^1 \phi D^q\, dx = 2^q \|h\|^q < +\infty\,.$$

(ii) Let $f \in \Lambda^*(\alpha, p)$; because of (5.4) we may assume that $f = f^*$, that is, that f is positive and decreasing. Suppose $f \prec \phi D$ and $\int_0^1 \phi D^q\, dx < +\infty$ with $\phi(x) = \alpha x^{\alpha-1}$. Then by (5.3) we have

$$\theta(x, f) = \frac{1}{x} \int_0^x f(t)\, dt \leq \frac{\alpha}{x} \int_0^x t^{\alpha-1} D(t)\, dt$$

$$= \alpha x^{\alpha-1} \frac{1}{x^\alpha} \int_0^x t^{\alpha-1} D(t)\, dt = \phi(x) D_1(x)\,,$$

say. The function $D_1(x)$ is positive and decreasing, as

$$D_1'(x) = -\alpha x^{-\alpha-1} \int_0^x t^{\alpha-1} D\, dt + x^{-1} D(x)$$

$$\leq -\alpha x^{-\alpha-1} D(x) \int_0^x t^{\alpha-1}\, dt + x^{-1} D(x) = 0\,.$$

Therefore, by Hardy's inequality, we have

$$\|\theta(f)\|^q \leq \alpha \int_0^1 x^{\alpha-1} D_1^q\, dx = \alpha \int_0^1 x^{(1-\alpha)(q-1)} \left\{\frac{1}{x} \int_0^x t^{\alpha-1} D\, dt\right\}^q dx$$

$$\leq C \int_0^1 x^{(1-\alpha)(q-1)+(\alpha-1)q} D(x)^q\, dx = C \int_0^1 x^{\alpha-1} D^q\, dx$$

with some constant C. Thus $\theta(f) \in \Lambda^*$, which proves (ii).

It should be remarked that $f^* \log(1/x)$ behaves very much like $f^* \log^+ f^*$:

(a) *If* $f^* \log(1/x)$ *belongs to* $\Lambda^*(\phi, p)$, $p \geq 1$, *then* $f \log^+ |f|$ *belongs to* $\Lambda^*(\phi, p)$. For if $p > 1$ [the case $p = 1$ is similar but simpler], there is a $D(x)$ with $f^* \log(1/x) \prec \phi D$ and $\int_0^1 \phi D\, dx < +\infty$. Then also $f^*(\delta) \log(1/x) \prec \phi D$ on $(0, \delta)$; in particular,

$$f^*(\delta) \int_0^\delta \log \frac{1}{x}\, dx \leq \int_0^\delta \phi D\, dx \leq 1$$

if δ is small. Therefore $f^*(\delta) \leq \delta^{-1}$ for all small δ, which shows that

$$f \log^+ |f| \in \Lambda^*(\phi, p).$$

(b) Now suppose $\phi(x)$ is such that, for some $\delta > 0$, we have $\int_0^1 \phi(x) x^{-\delta} dx < +\infty$. *If* $f \log^+ |f|$ *belongs to* $\Lambda^*(\phi, p)$, $p \geq 1$, *then* $f^* \log(1/x)$ *also does.* In fact, by Young's inequality [5, p. 111; or 11, p. 64], for the pair of inverse functions $\phi(u) = \log^+ u$, $\psi(v) = e^v$, we obtain $ab \leq a \log^+ a + e^b$ $(a, b \geq 0)$ and therefore

$$f^* \log \frac{1}{x} \leq \delta^{-1} f^* \log^+ (\delta^{-1} f^*) + x^{-\delta} \leq \delta^{-1} f^* \log^+ \frac{1}{\delta} + \delta^{-1} f^* \log^+ f^* + x^{-\delta}$$

$$\leq A f^* \log^+ f^* + B + x^{-\delta}$$

for some constants A, B.

It follows from these remarks, that Theorem 7 (ii) may be regarded as a generalization of the theorem of Hardy-Littlewood [12, p. 245] that $f \log^+ |f| \in L$ implies $\theta(f) \in L$.

Theorems 7 and 8 have many applications which may be derived in the same way as the corresponding results for the spaces L^p (see [12, p. 246]). As an example, we give the following result. Let $k > 0$, and let $\sigma_n^{(k)}(x, f)$ denote the Cesàro sum of order k of the Fourier series of a function $f(x)$. If $\theta(x, f)$ is taken for the interval $(0, 4\pi)$, we have: *if* $f(x)$ *satisfies one of the hypotheses of Theorems 7 or 8, then* $|\sigma_n^{(k)}(x, f)| \leq C_k \theta(x, f)$, $n = 0, 1, \cdots$. We may give another formulation of this result. In the spaces $\Lambda(\phi, p)$ and $\Lambda^*(\phi, p)$ we introduce a partial ordering, writing $f_1 \leq f_2$ if $f_1(x) \leq f_2(x)$ almost everywhere. With this ordering, Λ and Λ^* become Banach lattices for which the order convergence $f_n \longrightarrow f$ is identical with the convergence $f_n(x) \longrightarrow f(x)$ almost everywhere and the existence of a function $h(x)$ of the lattice such that $|f_n(x)| \leq h(x)$ almost everywhere. This is an immediate consequence of the fact that the lattices Λ, Λ^* satisfy the condition (ii) of Theorem 3 (see [6, pp. 154-156]). Then the above result implies that $\sigma_n^{(k)} \longrightarrow f$ in order in the corresponding space. Theorems of this section may also be used to obtain analogues of theorems of Hardy [3] and Bellman [1] for spaces Λ and Λ^*; see Petersen [11].

5.2. *Integral transformations.* Let $K(x, t)$ be measurable on the square $0 \leq x \leq 1$, $0 \leq t \leq 1$, and let

$$F(x) = \int_0^1 K(x, t) f(t) \, dt. \tag{5.7}$$

THEOREM 9. *Suppose that there is a constant M such that*

(i) $$\int_0^1 |K(x, t)| \, dt \leq M \quad \textit{almost everywhere};$$

(ii) *for any rearrangement $\phi_r(x)$ of $\phi(x)$, the function $h_r(t) = \int_0^1 \phi_r(x) K(x, t) \, dx$ belongs to $\mathrm{M}(\phi)$ and has a norm not exceeding M. Then* (5.7) *is a linear operator of norm $\leq M$ mapping $\Lambda(\phi, p)$ into itself. Condition* (ii) *may also be replaced by*

(iii) $$\int_0^1 |K(x, t)| \, dx \leq M \quad \textit{almost everywhere}.$$

Proof. Condition (ii) is equivalent to

$$h_r^*(t) \prec M\phi(t). \tag{5.8}$$

Assuming $f \in \Lambda(\phi, p)$, $p > 1$, we have

$$\begin{aligned}
\int_0^1 \phi_r(x) \, |F(x)|^p \, dx &\leq \int_0^1 \phi_r \, dx \left\{ \int_0^1 |K| \, |f(t)| \, dt \right\}^p \\
&\leq \int_0^1 \phi_r \, dx \int_0^1 |K| \, |f|^p \, dt \left\{ \int_0^1 |K| \, dt \right\}^{p/q} \\
&\leq M^{p/q} \int_0^1 |f(t)|^p \, dt \int_0^1 \phi_r(x) \, |K(x, t)| \, dx \\
&\leq M^{p/q} \int_0^1 h_r^*(t) f^*(t)^p \, dt;
\end{aligned}$$

by (5.8) and Lemma 1, this is

$$\leq M^{1+p/q}\int_0^1 \phi(t)f^*(t)^p\,dt = M^p\|f\|^p\,,$$

which proves the first part of the theorem. Suppose now that (i) and (iii) hold. Let $\delta > 0$, e an arbitrary set of measure δ, and e_1 a set of measure δ such that $\phi_r(x) \geq \phi(\delta)$ on e_1 and $\phi_r(x) \leq \phi(\delta)$ on the complement Ce_1 of e_1. Then we have

$$\int_e |h_r(t)|\,dt \leq \int_e dt \int_{e_1} |\phi_r(x)|\,|K|\,dx + \int_e \int_{Ce_1}$$

$$\leq M\int_{e_1} |\phi_r(x)|\,dx + \phi(\delta)\int_e dt \int_0^1 |K(x,t)|\,dx$$

$$\leq M\Phi(\delta) + M\delta\phi(\delta) \leq 2M\Phi(\delta)\,.$$

This shows that the norm of $h_r(t)$ in $M(\phi)$ does not exceed $2M$, and proves (ii).

REMARK. If the conditions of Theorem 9 are satisfied, then

$$G(t) = \int_0^1 K(x,t)g(x)\,dx \tag{5.9}$$

is a linear operator of norm $\leq 2M$ mapping $\Lambda^*(\phi,p)$ into itself.

We have in fact, for $g \in \Lambda^*(\phi,p)$ and $f \in \Lambda(\phi,p)$,

$$\int_0^1 G(t)f(t)\,dt = \int_0^1 g(x)\,dx \int_0^1 K(x,t)f(t)\,dt = \int_0^1 g(x)F(x)\,dx$$

$$\leq \|g\|_{\Lambda^*}\|F\|_\Lambda \leq M\|f\|_\Lambda\|g\|_{\Lambda^*}\,,$$

(the integrals evidently exist), and this shows that $G \in \Lambda^*$ and that $\|G\| \leq M\|g\|$.

Theorem 9 is akin to the "convexity theorem" of M. Riesz [12, p.198]. We mention for completeness that there is a generalization of this theorem, in which the different spaces L^p involved are replaced by the spaces $\Lambda(\phi,p)$ *with the same* ϕ. The proof, which follows closely the proof of M. Riesz's theorem in [12], is omitted.

5.3. *Moment problems.* We give an application of Theorem 9 to moment problems of the form

$$\mu_n = \int_0^1 x^n f(x)\,dx, \qquad n = 0, 1, 2, \cdots. \tag{5.10}$$

We shall write

$$\Phi_{n\nu} = \Phi\left(\frac{\nu+1}{n+1}\right) - \Phi\left(\frac{\nu}{n+1}\right), \qquad \Phi(x) = \int_0^x \phi\, dt,$$

$$\mu_{n\nu} = \binom{n}{\nu}\Delta^{n-\nu}\mu_\nu = \int_0^1 f(x) p_{n\nu}(x)\, dx,$$

$$p_{n\nu} = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}, \qquad \nu = 0, 1, \cdots, n,$$

and $\mu^*_{n\nu}$ for the decreasing rearrangement of the $|\mu_{n\nu}|$, $\nu = 0, 1, \cdots, n$. Moreover, we set

$$f_n(x) = (n+1)\mu_{n\nu} \qquad \text{for } \frac{\nu}{n+1} \le x < \frac{\nu+1}{n+1}, \tag{5.11}$$

and obtain

$$f_n(x) = \int_0^1 K_n(x,t) f(t)\, dt, \tag{5.12}$$

$$K_n(x,t) = (n+1)\, p_{n\nu}(t), \qquad \frac{\nu}{n+1} \le x < \frac{\nu+1}{n+1},$$

For the special case $\phi(x) = \alpha x^{\alpha-1}$ and $p = 1$, the following theorem (with another proof) has been given in [8].

THEOREM 10. *The sequence of real numbers μ_n is a moment sequence of a function of the space $\Lambda(\phi, p)$ or of $\Lambda^*(\phi, p)$ [for the case $\Lambda(\phi, 1)$, we assume $\phi(x) \longrightarrow \infty$ for $x \longrightarrow 0$] if and only if the norms of the functions* (5.11) *are uniformly bounded in this space.*

For the space $\Lambda(\phi, p)$, the condition is

$$\sum_{\nu=0}^{n} \Phi_{n\nu}\mu^{*p}_{n\nu} \le M(n+1)^{-p}, \tag{5.13}$$

and for $\Lambda^*(\phi, p)$, $p > 1$,

$$\mu^*_{n\nu} < \Phi_{n\nu} D_{n\nu}, \qquad \sum_{\nu=0}^{n} \Phi_{n\nu} D^q_{n\nu} \le M^q, \tag{5.14}$$

with some positive decreasing $D_{n\nu}$, $\nu = 0, 1, \cdots, n$.

Proof. If $f \in \Lambda(\phi, p)$, then condition (5.13) is satisfied by Theorem 9, because the kernel (5.12) satisfies (i) and (iii) with $M = 1$.

Conversely, let $\|f_n\|_\Lambda \leq M$. Since

$$\int_e |f_n(x)|\, dx \leq \phi(\delta)^{-1} \int_0^\delta \phi(x)\, |f_n(x)|\, dx \leq M\phi(\delta)^{-1}, \qquad \text{meas } e = \delta,$$

it follows in case $p = 1$ that the integrals $\int_e |f_n|\, dx$ are uniformly absolutely continuous and uniformly bounded. In case $p > 1$, this follows by Hölder's inequality. We deduce that for a certain subsequence $f_{n_k}(x)$, the integrals $\int_e f_{n_k}(x)\, dx$ converge for any $e = (0, x)$ with x rational; hence they converge for any measurable set $e \subset (0, 1)$. We then have

$$\lim_{k\to\infty} \int_e f_{n_k}(x)\, dx = \int_e f(x)\, dx, \tag{5.15}$$

with some $f \in L$. Then also

$$\int_0^1 f_{n_k}\psi\, dx \longrightarrow \int_0^1 f\psi\, dx \tag{5.16}$$

for any bounded ψ. For any such ψ we have, by (3.2),

$$\left|\int_0^1 f\psi\, dx\right| \leq \lim \left|\int_0^1 f_{n_k}\psi\, dx\right| \leq M\, \|\psi\|_{\Lambda^*};$$

hence this must be true for any ψ in Λ^*. Thus by §3, it follows that $f \in \Lambda(\phi, p)$.

We remark also that it follows easily from (5.16) that we have

$$\int_0^1 f_{n_k}\psi_k\, dx \longrightarrow \int_0^1 f\psi\, dx, \tag{5.17}$$

if the sequence $\psi_k(x)$ is uniformly convergent towards a bounded function $\psi(x)$.

Now let P be the vector space of all polynomials

$$\psi(x) = a_0 + a_1 x + \cdots + a_m x^m$$

with usual addition and scalar multiplication. On P we define an additive and homogeneous functional F by

$$F(\psi) = a_0\mu_0 + a_1\mu_1 + \cdots + a_m\mu_m .$$

Let

$$B_n^\psi(x) = \sum_{\nu=0}^{n} \psi\left(\frac{\nu}{n}\right) p_{n\nu}(x)$$

be the Bernstein polynomial of order n of $\psi(x)$; then it is known [10] that

$$B_n^\psi(x) = a_0^{(n)} + a_1^{(n)}x + \cdots + a_m^{(n)}x^m ,$$

and that $a_i^{(n)} \longrightarrow a_i$ for $n \longrightarrow \infty$. Hence $F(B_n^\psi) \longrightarrow F(\psi)$. In particular, let $\psi(x) = x^m$. We have

$$(5.18) \qquad F(B_n^\psi) = \sum_{\nu=0}^{n} \left(\frac{\nu}{n}\right)^m F(p_{n\nu}) = \sum_{\nu=0}^{n} \left(\frac{\nu}{n}\right)^m \mu_{n\nu}$$

$$= \int_0^1 f_n(x) g_n(x)\, dx ,$$

where $\psi_n(x)$ is equal to $(\nu/n)^m$ in the interval $[\nu/(n+1), (\nu+1)/(n+1)]$. As $\psi_n(x) \longrightarrow \psi(x)$ uniformly, we deduce from (5.18) and (5.17) that

$$\int_0^1 f(x)x^m\, dx = \lim F(B_n) = F(\psi) = \mu_m , \qquad m = 0, 1, \cdots .$$

Since $f \in \Lambda(\phi, p)$, this proves that the condition is sufficient in case of the space Λ. The proof for the space $\Lambda^*(\phi, p)$, which is similar, is omitted.

References

1. R. Bellman, *A note on a theorem of Hardy on Fourier constants*, Bull. Amer. Math. Soc. 50 (1944), 741-744.

2. G. Birkhoff, *Lattice theory*, 2nd ed., New York, 1948.

3. G. H. Hardy, *The arithmetic mean of a Fourier constant*, Messenger of Math. 68 (1929), 50-52.

4. ———, *Divergent series*, Oxford, 1949.

5. G. H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, England, 1934.

6. L. Kantorovitch, *Lineare halbgeordnete Räume*, Mat. Sbornik (N.S.) 2 (44) (1937), 121-168.

7. G. G. Lorentz, *A problem of plane measure*, Amer. J. Math. **71** (1949), 417-426.

8. ———, *Some new functional spaces*, Ann. of Math. (2) **51** (1950), 37-55.

9. ———, *Multiply subadditive functions*, to appear in the Canadian J. Math.

10. ———, *Bernstein polynomials*, to appear as a book at the University of Toronto Press.

11. G. M. Petersen, paper not yet published.

12. A. Zygmund, *Trigonometric series*, Warszawa, 1935.

UNIVERSITY OF TORONTO

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. F. Beckenbach, at the address given above.

Authors are entitled to receive 100 free reprints of their published papers and may obtain additional copies at cost.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $8.00; single issues, $2.50. Special price to individual faculty members of supporting institutions and to members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES

Pacific Journal of Mathematics
Vol. 1, No. 3 BadMonth, 1951