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TWO THEOREMS ON METRIC SPACES

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1. Introduction. Let E be a metric space with distance function d. The space E is called *two-point homogeneous* if given any four points a, a', b, b' with $d(a,a') = d(b,b')$, there exists an isometry of E carrying a, a' to b, b', respectively. In a recent paper [7], the author has determined all the compact and connected two-point homogeneous spaces. It is the aim of the present note to discuss the noncompact case, and prove a conjecture of Busemann which can be regarded also as a sharpening of a theorem of Birkhoff [1]. The results concerning the noncompact two-point homogeneous spaces are not as satisfactory as the results for the compact case; we have to assume certain conditions on the metric.

By a segment in a metric space E, we shall mean an isometric image of a closed interval with the usual metric. A metric space will be said to have the property (L) if given a point p, there exists a neighborhood W of p so that each point x $(\neq p)$ of W can be joined to p by at most one segment in E. The following theorems will be proved:

THEOREM 1. *Let E be a finite-dimensional, finitely compact, convex metric space with property (L). If E is two-point homogeneous, then E is homeomorphic with a manifold.*

THEOREM 2. *Let E be a metric space with all the properties mentioned in Theorem* 1. *If, moreover,* $\dim E$ *is odd, then E is congruent either to the euclidean space, the hyperbolic space, the elliptic space, or the spherical space.*

Our Theorem 2 justifies the conjecture of Busemann [2, p. 233] that a two-point homogeneous three dimensional S.L. space [2, p. 78] is either elliptic, hyperbolic, or euclidean. It is to be noted that Theorem 2 no longer holds if $\dim E$ is even and greater than two. The complex elliptic spaces [7] and the hyperbolic Hermitian spaces[1] [2, p. 192] serve as counter examples.

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[1] These spaces were first introduced by H. Poincaré, and then discussed by G. Fubini and E. Study. Following E. Cartan, we call these spaces the hyperbolic Hermitian spaces.

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2. Preliminary results. Throughout this note, by a Busemann space [2, p. 11], we shall mean a finitely compact, convex metric space such that at each point p, there exists a neighborhood W with the following property: given any two points x, y of W and any $\epsilon > 0$, we can find a positive number $\delta < \epsilon$ for which a unique point z exists so that

$$d(x, y) + d(y, z) = d(x, z), \quad d(y, z) = \delta.$$

It can easily be verified that the class of all two-point homogeneous, finitely compact, convex metric space with the property (L) coincides with the class of all two-point homogeneous Busemann spaces. In the statements of our Theorems, we use the property (L) instead of Busemann's axioms merely because it is, geometrically, easier to visualize.

Let E be a Busemann space. We shall first see that each d-sphere[1] of sufficiently small radius is locally connected. In fact, let p be a point of E. We choose $\epsilon > 0$ so small that each point x with $0 < d(p, x) \leq \epsilon$ can be joined to p by one and only one segment. Let $K(p, \epsilon)$ be the d-sphere with center p and radius ϵ, and R the totality of points y with $0 < d(p, y) < \epsilon$. Then evidently R is an open set of E. Since E is convex, E must be locally connected. It follows then that R is locally connected.

For each point y of $K(p, \epsilon)$, we denote by $P_y(s)$ $(0 \leq s \leq \epsilon)$ the isometric representation of the segment joining p to y. Let J be the open interval $0 < s < \epsilon$. By our choice of ϵ, the mapping $h: K(p, \epsilon) \times J \longrightarrow R$ defined by $h(y, s) = P_y(s)$ is a one-to-one mapping of the topological product $K(p, \epsilon) \times J$ onto R. Moreover, from Busemann's results [2, I., §3] concerning the convergence of geodesics, we see immediately that h is bicontinuous. This tells us that $K(p, \epsilon) \times J$ and R are homeomorphic. Since R is locally connected, $K(p, \epsilon) \times J$, and hence $K(p, \epsilon)$, is locally connected.

3. Proof of Theorem 1. Let E be a metric space with all the properties mentioned in Theorem 1. From the above discussions, we know that for any point p of E, the d-sphere $K(p, \epsilon)$ with sufficiently small radius ϵ is locally connected. Let Γ be the group of all isometries of E, and Γ_p the totality of all those isometries which leave p invariant. In Γ, we introduce the topology as defined by van Dantzig and van der Waerden [4] (in fact, this is exactly the g-topology of R. Arens).

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[1] By a d-sphere we mean the totality of points equidistant from a fixed point with respect to the metric d. This should be distinguished from the $(n - 1)$-sphere which stands for the $(n - 1)$-dimensional topological sphere.

Then Γ_p forms a compact topological group [4]. Evidently, Γ_p is a transformation group of $K(p, \epsilon)$ in the sense of Montgomery and Zippin. From the two-point homogeneity, Γ_p is transitive on $K(p, \epsilon)$. Taking account of the finite dimensionality and local connectedness of $K(p, \epsilon)$ and the compactness of Γ_p, we can conclude [5] that Γ_p is a Lie group, and hence $K(p, \epsilon)$ is locally euclidean (here as well as in what follows, locally euclidean is always used in the topological sense). The set R, being homeomorphic with the topological product of $K(p, \epsilon)$ and the open interval J, must be locally euclidean as well. Hence our space E is locally euclidean at each point of R, and hence locally euclidean at all its points. Moreover, E is obviously separable and connected. It follows then that E is homeomorphic with a manifold.

4. The structure of d-spheres. Before proving Theorem 2, we find it convenient to establish some more properties of the d-spheres.

LEMMA. *Let E be a metric space satisfying all the conditions in Theorem 2. Then each d-sphere with sufficient small radius is homeomorphic with the* $(n-1)$-*dimensional topological sphere where* $\dim E = n$.

Proof. If $\dim E$ is equal to one, this is trivial. Now we shall assume that $n > 1$. Let p be a point of E, and ϵ so small that each point x with $0 < d(p,x) \leq \epsilon$ can be joined to p by one and only one segment. Set $K(p, \epsilon)$ to be the d-sphere with center p and radius ϵ, and

$$U = \{x \mid d(p, x) < \epsilon\} .$$

We shall show first that U is contractible to a point. Given each point y of $K(p,\epsilon)$, let us denote by $P_y(s)$ the isometric representation of the segment joining p to y. Then the pair (y, s), where $y \in K(p, \epsilon)$ and $0 \leq s < \epsilon$, can be regarded as polar coordinates of points in U. For any real number t with $0 \leq t \leq 1$, we define

$$\phi[t, P_y(s)] = P_y(ts).$$

We see immediately that ϕ is a well-defined mapping of the product $I \times U$, and

$$\phi[1, P_y(s)] = P_y(s), \quad \phi(t, p) = p, \quad \phi[0, P_y(s)] = p,$$

where I denotes the closed interval $\{t \mid 0 \leq t \leq 1\}$. The continuity of ϕ can easily be verified. Thus ϕ gives a contraction of U into the point p, and thus the homotopy group $\pi_i(U)$ vanishes for each i.

Now let us consider the set $R = U - p$. Since U is an n-dimensional open manifold and $n > 1$, the set R is connected and has the same homotopy group π_i as U for all dimensions i less than $n-1$. Thus $\pi_i(R) = 0$, $i = 1, 2, \cdots, n-2$. On the other hand, we have shown in §1 that R is homeomorphic with the topological product $K(p, \epsilon) \times J$, where J denotes an open interval. It follows then that $K(p, \epsilon)$ is connected and

$$\pi_i[K(p, \epsilon)] = 0, \qquad i = 1, 2, \cdots, n-2. \tag{1}$$

From the proof of Theorem 1, we know that $K(p, \epsilon)$ is a homogeneous space of a compact Lie group. Its connectedness and its simply-connectedness imply that it is an orientable manifold.

Since both $K(p, \epsilon)$ and J are manifolds, we have

$$\dim K(p, \epsilon) + \dim J = \dim R = \dim E = n ,$$

and hence $\dim K(p, \epsilon) = n - 1$. It follows immediately from (1) that $K(p, \epsilon)$ is a simply-connected homology sphere of even dimension $n - 1$. Therefore [6] $K(p, \epsilon)$ is a topological sphere. The lemma is proved.

5. Proof of Theorem 2. Suppose E to be a metric space with all the properties mentioned in Theorem 2. If E is compact, then our Theorem 2 follows as a direct consequence of [7, Theorem VI]. Thus we can assume from now on that E is not compact. We shall first show that E is an open S. L. space in the sense of Busemann [2, p. 78]. To show this, it suffices [3, p.173] to establish that each geodesic is congruent to a euclidean line; for this, it suffices to demonstrate that given any two distinct points x, y and any $k > 0$, there exists a point z so that

$$d(x, y) + d(y, z) = d(x, z), \qquad d(y, z) = k .$$

In fact, since E is finitely compact and noncompact, E cannot be bounded. There exists then a sequence of points $p_0, p_1, p_2, \cdots$ with $d(p_0, p_i)$ tending to infinity. Thus we can choose i so large that $d(p_0, p_i) \geq d(x, y) + k$. Let τ be a segment joining p_0 to p_i. Evidently there exist three points x', y', z' in τ such that

$$d(x', y') + d(y', z') = d(x', z'), \qquad d(x', y') = d(x, y), \qquad d(y', z') = k .$$

From the two-point homogeneity of E, there is an isometry f of E carrying x', y' to x, y respectively. Then we can see immediately that the point $z = f(z')$ has all the required properties. Thus E is an open S. L. space.

Let $K(p, \epsilon)$ be the d-sphere with center p and radius ϵ, and Γ_p the group of all isometries of E which leave the point p invariant. From the above lemma, we know that $K(p, \epsilon)$ is an $(n-1)$-sphere and Γ_p a compact and transitive transformation group of $K(p, \epsilon)$. Moreover, it can easily be seen that Γ_p is effective on $K(p, \epsilon)$. In our further discussions, we shall rule out the trivial case where $\dim E = n = 1$. Thus $K(p, \epsilon)$ is connected, and the identity component Γ_p^0 of Γ_p forms a connected, compact, transitive, and effective transformation group of $K(p, \epsilon)$. Since $n-1$ is even, it follows [6] that Γ_p^0 is either isomorphic with the rotation group R_{n-1} or Cartan's exceptional group G_2. We shall discuss these two cases separately.

Case A. Suppose Γ_p^0 to be isomorphic with the group R_{n-1} of all rotations of the $(n-1)$-sphere. Let us represent $K(p, \epsilon)$ by the unit sphere in a certain n-dimensional euclidean space, and consider R_{n-1} not only as a topological group but also as a transformation group of $K(p, \epsilon)$ in the usual sense. It is well known that Γ_p^0 and R_{n-1} have the same topological type, that is, there exists a homeomorphism ϕ of $K(p, \epsilon)$ onto itself so that

$$R_{n-1} = \phi \Gamma_p^0 \phi^{-1} = \{\phi f \phi^{-1} \mid f \in \Gamma_p^0\}.$$

Since n is odd, given any point q of $K(p, \epsilon)$, there exists a rotation of period two which leaves fixed *only* q and its diametrically opposite point. It follows then that for each point q of $K(p, \epsilon)$, we can find a transformation f in Γ_p^0 such that (a) f is of period two, (b) f leaves q fixed, and (c) f has only two fixed points on $K(p, \epsilon)$. Now let g be any geodesic through p in E. It intersects $K(p, \epsilon)$ at two points, say q and q'. We consider the transformation f in Γ_p^0 having the above three properties (a), (b), and (c). Since f is an isometry leaving fixed p and q, it leaves the geodesic g pointwise invariant. Moreover, this isometry f cannot have any other fixed point, for otherwise f would have some other fixed points on $K(p, \epsilon)$ besides q and q'. Thus f is a reflection of E about g. Since p is an arbitrary point and g an arbitrary geodesic through p, there exists a reflection of E about each geodesic. From Schur's Theorem [2, p. 181], it follows that E is either hyperbolic or euclidean.

Case B. Suppose Γ_p^0 to be isomorphic with the exceptional group G_2. To discuss this case, we have to digress into a few properties of Cayley numbers. Let $1, e_i$ $(i = 1, 2, \cdots, 7)$ be the units of Cayley algebra. The multiplication rule is given by

$$e_i e_i = -1, \quad e_i e_j = -e_j e_i, \quad e_1 e_2 = e_3, \quad e_1 e_4 = e_5, \quad e_1 e_6 = e_7,$$

$$e_2 e_5 = e_7, \quad e_2 e_4 = -e_6, \quad e_3 e_4 = e_7, \quad e_3 e_5 = e_6,$$

together with the equalities obtained by cyclic permutation of the indices. Let

$$\Theta = \left\{ \sum_{i=1}^{7} x_i e_i \mid x_i = \text{real number}, \quad \sum_{i=1}^{7} (x_i)^2 = 1 \right\}$$

be the totality of all the Cayley numbers with vanishing real part and with norm equal to unity. Evidently, Θ forms a 6-sphere, and each automorphism of the Cayley algebra carries Θ into itself. We can regard therefore the group H of all automorphisms of Cayley algebra as a transformation group of Θ (the topology over H is defined in the usual manner). Now H acts effectively and transitively on Θ. Moreover, it is known that H is isomorphic with the exceptional group G_2.

For each $x = \sum_{i=1}^{7} x_i e_i$ in Θ, we shall denote the Cayley number $x_1 - \sum_{i=2}^{7} x_i e_i$ by x^*, and call it the *symmetric image* of x with respect to e_1. It is evident that

$$(1) \qquad (x^*)^* = x, \qquad x^* \begin{cases} = x, & \text{if } x = \pm e_1, \\ \neq x, & \text{otherwise.} \end{cases} \qquad x \in \Theta$$

Moreover, by a direct calculation, we can show that given any two Cayley numbers y, z in Θ, there exists an automorphism f in H such that

$$f(e_1) = e_1, \quad f(y) = y^*, \quad f(z) = z^*.$$

It is to be noted that this f depends on y and z. There is no automorphism of Cayley algebra which carries each x in Θ into its symmetric image x^*.

Now we can proceed to the proof of Theorem 2. Since Γ_p^0 is isomorphic with the exceptional group G_2, $K(p, \epsilon)$ must be six-dimensional [6]. It is known that each transitive transformation group of the 6-sphere which is isomorphic with the exceptional group G_2 has the same topological type as H.[1] Thus we can identify Θ and $K(p, \epsilon)$ in such a manner that Γ_p^0 and H coincide. Let x be a point of $K(p, \epsilon)$. It determines a ray $\overrightarrow{px}$, that is, the totality of points u of E for which either $d(x, u) + d(u, p) = d(x, p)$ or $d(u, x) + d(x, p) = d(u, p)$ [2, p. 76]. For each nonnegative number s, we denote by $P_x(s)$ the point u on the ray $\overrightarrow{px}$ with the property that

[1] This follows as a direct consequence of [6, Lemma 6].

$d(p, u) = s$. Since E is an open S. L. space, each point of E other than p can be represented in a unique way as $P_x(s)$, where $x \in K(p, \epsilon)$ and $s > 0$. Let y, z be any two points of $K(p, \epsilon)$, and let y^*, z^* be, respectively, their symmetric images with respect to e_1 [note that we have identified Θ with $K(p, \epsilon)$]. Then there exists a transformation f in Γ_p^0 such that $f(e_1) = e_1$, $f(y) = y^*$, $f(z) = z^*$. Since f is an isometry of E and leaves p fixed, we have, for any s, $s' \geq 0$, the relations

$$f[P_y(s)] = P_{y^*}(s), \qquad f[P_z(s')] = P_{z^*}(s').$$

This tells us that

$$d[P_y(s),\ P_z(s')] = d[P_{y^*}(s),\ P_{z^*}(s')] \qquad (s,\ s' \geq 0). \tag{2}$$

Now let us consider the mapping $h: E \longrightarrow E$ defined by $h[P_x(s)] = P_{x^*}(s)$, where $x \in K(p, \epsilon)$ and $s \geq 0$. Equality (2) tells us that this mapping h is an isometry of E. Moreover, from (1) we can see that h is of period two and that h has only two fixed points e_1 and $-e_1$ on $K(p, \epsilon)$. It follows then that h is a reflection of E about the geodesic joining p and e_1. However, our space E is two-point homogeneous so that there exists a reflection about every geodesic of E. From Schur's Theorem, we can conclude that E is either hyperbolic or euclidean. Theorem 2 is hereby proved.

6. Remarks. In all the arguments, we use only the weaker two-point homogeneity; that is, there exists a number $\delta > 0$ such that, for any four points x, x', y, y' with $d(x, x') = d(y, y') < \delta$. there exists an isometry of E carrying x, x' to y, y' respectively.

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