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**1. Introduction.** Let $B$ be a given arcwise connected topological space and $b_0$ a basic point of $B$. Then we obtain a sequence of homotopy groups

$$\pi_1(B), \ \pi_2(B), \ \cdots, \ \pi_n(B), \ \cdots.$$

The fundamental group $\pi_1(B)$ is in general non-abelian and written multiplicatively. All higher homotopy groups $\pi_n(B)$, $n \geq 2$, are abelian and written additively. The group $\pi_1(B)$ operates on the left of every higher homotopy group $\pi_n(B)$, $n \geq 2$; that is to say, for every $w \in \pi_1(B)$ and every $a \in \pi_n(B)$, a unique element $wa \in \pi_n(B)$ is determined, and

$$w(a_1 + a_2) = wa_1 + wa_2, \quad w_1(w_2 a) = (w_1 w_2)a, \quad 1a = a.$$

For arbitrary elements $a \in \pi_m(B)$ and $b \in \pi_n(B)$, $m \geq 2$, $n \geq 2$, a Whitehead product $a \circ b$ is defined [10, p.411], which is an element of $\pi_{m+n-1}(B)$. The Whitehead product is known to be bilinear; namely,

$$(a_1 + a_2) \circ b = a_1 \circ b + a_2 \circ b, \quad a \circ (b_1 + b_2) = a \circ b_1 + a \circ b_2.$$

Roughly speaking, the realizability problem is whether these homotopy groups and mutual operations described above are otherwise completely arbitrary. It can be formulated precisely as follows. Let

$$\pi_1, \ \pi_2, \ \cdots, \ \pi_n, \ \cdots$$

be a given sequence of abstract groups. All groups except the first one are abelian and additive, while $\pi_1$ is written multiplicatively. There are given two kinds of operations between these groups. First, the group $\pi_1$ operates on the left of every group $\pi_n$ with $n \geq 2$. Secondly, for arbitrary elements $\alpha \in \pi_m$, $\beta \in \pi_n$, $m \geq 2$, $n \geq 2$, a bilinear product $\alpha \circ \beta$ is defined and is an element of the group $\pi_{m+n-1}$.

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The realizability problem is to construct an arcwise connected topological space $B$ and a basic point $b_0 \in B$ satisfying the following conditions:

(1.1) *There exists, for each integer* $n \geq 1$, *an isomorphism* $h_n: \pi_n(B) \approx \pi_n$ *of* $\pi_n(B)$ *onto* $\pi_n$.

(1.2) *For arbitrary elements* $w \in \pi_1(B)$ *and* $a \in \pi_n(B)$, $n \geq 2$, *we have* $h_n(wa) = h_1(w)h_n(a)$.

(1.3) *For arbitrary elements* $a \in \pi_m(B)$ *and* $b \in \pi_n(B)$, $m \geq 2$, $n \geq 2$, *we have* $h_{m+n-1}(a \circ b) = h_m(a) \circ h_n(b)$.

This general problem has not yet been solved. The first partial solution was given by J. H. C. Whitehead [12]. By means of an inductive construction based on his previous contributions, he succeeded to give an infinite polytope $B$ which satisfies the conditions (1.1) and (1.2). However, he gave no explicit information as to the Whitehead products of the higher homotopy groups of the space he constructed.

The object of the present work is to give a synthetic and algebraic construction of an arcwise connected topological space $B$ with a basic point $b_0$ and prove the following:

REALIZABILITY THEOREM. *There exists an arcwise connected topological space* $B$ *and a basic point* $b_0 \in B$ *satisfying the conditions* (1.1), (1.2), *and*

(1.4) *For arbitrary elements* $a \in \pi_m(B)$ *and* $b \in \pi_n(B)$, $m \geq 2$, $n \geq 2$, *we have* $a \circ b = 0$.

Our principal construction is motivated by the following observations:

(a) Let $\pi$ be a given group and $n$ a positive integer (we assume $\pi$ to be abelian if $n > 1$). Then we can construct an arcwise connected space $P(\pi, n)$ such that:

$$\pi_n(P(\pi,n)) \approx \pi, \quad \pi_i(P(\pi,n)) = 0 \qquad (i \neq n); \tag{1.5}$$

(1.6) If $n > 1$, there is a correspondence which associates with each endomorphism $h: \pi \longrightarrow \pi$ a continuous map $h^{\#}: P(\pi, n) \longrightarrow P(\pi, n)$ such that $(h_1 h_2)^{\#} = h_1^{\#} h_2^{\#}$, and $h^{\#}$ is the identity if $h$ is the identity.

(b) Let $\pi_2, \pi_3, \cdots$, be a sequence of abelian groups, and let $Y$ denote the topological product of all the spaces $P(\pi_n, n)$, $n = 2, 3, \cdots$. Then $Y$ is simply connected and $\pi_i(Y) \approx \pi_i$ for $i \geq 2$; moreover, all the Whitehead products in $Y$ vanish. This is a consequence of J. H. C. Whitehead [13, p. 289].

(c) Let $G$ be a group of homeomorphisms of $Y$, and let $X = P(\pi_1, 1)$ where $\pi_1$ is any given group. Let $\chi: \pi_1 \longrightarrow G$ be a homomorphism. Let $\tilde{X}$ denote the universal covering space of $X$. It is well known that $\tilde{X}$ is a bundle space over $X$ with discrete fiber $\pi_1$ and discrete structural group $\pi_1$. The homomorphism $\chi: \pi_1 \longrightarrow G$ induces a bundle space $B$ over $X$ with fiber $Y$ and structural group $G$ which is weakly associated with $\tilde{X}$. Then the operations of $\pi_1$ on $\pi_n = \pi_n(Y)$ are given by $w \longrightarrow \chi_*(w)$, where $\chi_*(w)$ is the automorphism of $\pi_n(Y)$ induced by the map $\chi(w): Y \longrightarrow Y$. By suitable choice of the homomorphism $\chi: \pi_1 \longrightarrow G$, the bundle space $B$ has the properties described in the Realizability Theorem.

As an application, we are able to show that Whitehead products of the higher homotopy groups of a given topological space are essential invariants of the space; that is, they are not completely determined by the homotopy groups and the operations of the fundamental group upon the higher homotopy groups.

**2. Semi-simplicial polytope.** First of all, let us recall the definition of semi-simplicial complexes of S. Eilenberg and J. A. Zilber [2] as what follows.

A *semi-simplicial complex* $K$ is a collection of elements $\{\sigma\}$ called *simplexes* together with two functions. The first function associates with each simplex $\sigma$ an integer $q \geq 0$ called the dimension of $\sigma$; we then say that $\sigma$ is a $q$-simplex. The second function associates with each $q$-simplex $\sigma\,(q > 0)$ of $K$ and with each $i\,(0 \leq i \leq q)$ a $(q-1)$-simplex $\sigma^{(i)}$ called the $i$-th *face* of $\sigma$, subject to the condition

$$[\sigma^{(j)}]^{(i)} = [\sigma^{(i)}]^{(j-1)} \tag{2.1}$$

for $q > 1$ and $i < j$. We may pass to lower dimensional faces of $\sigma$ by iteration. If $0 \leq i_1 < \cdots < i_n \leq q$ then we define inductively

$$\sigma^{(i_1, \cdots, i_n)} = [\sigma^{(i_2, \cdots, i_n)}]^{(i_1)}.$$

This is a $(q-n)$-simplex. If $0 \leq j_0 < \cdots < j_{q-n} \leq q$ is the set complementary to $\{i_1, \cdots, i_n\}$ then we also write

$$\sigma^{(i_1, \cdots, i_n)} = \sigma_{(j_0, \cdots, j_{q-n})}.$$

In particular, $\sigma_{(i)}$ for $0 \leq i \leq q$ is a 0-simplex called the $i$-th *vertex* of $\sigma$. We shall also refer to $\sigma_{(0)}$ as the *leading vertex* and $\sigma_{(0,1)}$ as the *leading edge*. For any two simplexes $\sigma$ and $\tau$ of $K$, we shall write

$$\tau < \sigma$$

if either $\tau = \sigma$ or $\tau = \sigma^{(i_1, \cdots, i_n)}$ for some set $(i_1, \cdots, i_n)$ of integers $0 \leq i_1 < \cdots < i_n \leq q$. A subcomplex $L$ of $K$ is a subcollection of simplexes of $K$ with the property that $\sigma \in L$ and $\tau < \sigma$ imply $\tau \in L$. Obviously, every semi-simplicial complex $K$ is a closure finite abstract complex [5, p.91] with its incidence numbers defined by means of the bounding relation

$$\partial\sigma = \sum_{i=0}^{q} (-1)^i \sigma^{(i)} .$$

Now, let $K$ be a given semi-simplicial complex. We shall construct a topological space $P(K)$, called the *semi-simplicial polytope associated with K.*

For every integer $q \geq 0$, to every $q$-simplex $\sigma$ of $K$ let us associate an open geometric $q$-cell $w_\sigma$, called the *open q-cell corresponding to* $\sigma$, which is the interior of some ordered geometric $q$-simplex $s_\sigma$; that is,

$$w_\sigma = \operatorname{Int} s_\sigma , \qquad s_\sigma = \langle v_0, \cdots , v_q \rangle .$$

If $s_\sigma$ is 0-dimensional, we define $\operatorname{Int} s_\sigma = s_\sigma$. We assume that no two of these open cells $\{w_\sigma | \sigma \in K\}$ have a point in common. Let each open cell $w_\sigma$ have the euclidean topology and the affine geometry of the geometric simplex $s_\sigma$.

Let $\sigma$ be an arbitrary $q$-simplex of $K$ and $s_\sigma$ be the ordered geometric $q$-simplex associated with $\sigma$ as above. We define the *closed q-cell* $\operatorname{Cl} w_\sigma$ as a set by taking

$$\operatorname{Cl} w_\sigma = \bigcup_{\tau < \sigma} w_\tau .$$

There is a natural transformation

$$\mu_\sigma : s_\sigma \longrightarrow \operatorname{Cl} w_\sigma$$

of $s_\sigma$ onto $\operatorname{Cl} w_\sigma$ defined as follows. For each $n$-dimensional face $(0 \leq n \leq q)$,

$$s' = \langle v_{j_0}, \cdots, v_{j_n} \rangle, \quad 0 \leq j_0 < \cdots < j_n \leq q ,$$

of $s_\sigma$, we define $\mu_\sigma$ on the interior $\operatorname{Int} s'$ of $s'$ to be the unique barycentric map of $\operatorname{Int} s'$ onto $w_\tau$, $\tau = \sigma_{(j_0, \cdots, j_n)}$, which preserves the order of vertices. Give

$\mathrm{Cl}\, w_\sigma$ the identification topology determined by $\mu_\sigma$; that is to say, a set $M \subset \mathrm{Cl}\, w_\sigma$ is called *open* if and only if its inverse image $\mu_\sigma^{-1}(M) \subset s_\sigma$ is open.

Let us denote by $P(K)$ the union of all open cells $w_\sigma$ corresponding to the simplexes $\sigma$ of $K$. We define a topology of $P(K)$ as follows: A set $M$ of $P(K)$ is said to be *open* if $M \cap \mathrm{Cl}\, w_\sigma$ is an open set of $\mathrm{Cl}\, w_\sigma$ for every closed cell $\mathrm{Cl}\, w_\sigma$. The topological space $P(K)$ thus obtained is the semi-simplicial polytope associated with $K$. It is a polyhedral realization of the semi-simplicial complex $K$.

We remark that, for each simplex $\sigma$ of $K$, the natural transformation

$$\mu_\sigma:\ s_\sigma \longrightarrow \mathrm{Cl}\, w_\sigma \subset P(K)$$

is a continuous map of $s_\sigma$ onto $\mathrm{Cl}\, w_\sigma$ and $\mu_\sigma | w_\sigma$ is the identity. Following J.H.C. Whitehead [11, p.221], we shall call it the *characteristic map* for the open cell $w_\sigma$ of $P(K)$.

Obviously, $P(K)$ is a *CW*-complex in the sense of J.H.C. Whitehead [11,p. 223]. Hence we have the following assertions.

(2.2) *$P(K)$ is a normal Hausdorff space.*

(2.3) *A transformation $f: P(K) \longrightarrow R$ of $P(K)$ into an arbitrary topological space $R$ is a continuous map if and only if the partial transformation $f | \mathrm{Cl}\, w_\sigma$ is continuous for each closed cell $\mathrm{Cl}\, w_\sigma$ of $P(K)$.*

3. **Simplicial maps.** We recall the definition of simplicial maps of semi-simplicial complexes [2, p.500]. A *simplicial map* $T: K_1 \longrightarrow K_2$ of a semi-simplicial complex $K_1$ into another such complex $K_2$ is a function which to each $q$-simplex $\sigma$ of $K_1$ assigns a $q$-simplex $\tau = T(\sigma)$ of $K_2$ in such a fashion that

$$\tau^{(i)} = T(\sigma^{(i)}) \qquad (i = 0, \cdots, q).$$

(3.1) *A simplicial map $T: K_1 \longrightarrow K_2$ induces a unique continuous map $f_T$: $P(K_1) \longrightarrow P(K_2)$, which maps $w_\sigma$ of $P(K_1)$ barycentrically onto $w_\tau$ of $P(K_2)$ with $\tau = T(\sigma)$.*

*Proof.* For each integer $q \geq 0$, let $K_1^q$ denote the $q$-dimensional skeleton of $K_1$; that is, $K_1^q$ is the set of simplexes in $K_1$ with dimensions not exceeding $q$. Then $P(K_1^q)$ can be chosen as a subpolytope of $P(K_1)$ and will be called the $q$-dimensional skeleton of $P(K_1)$. Define a map

$$\phi_0:\ P(K_1^0) \longrightarrow P(K_2)$$

as follows: For each simplex $\sigma$ in $K_1^0$, $\mathrm{Cl}\, w_\sigma = w_\sigma$ is a single point of $P(K_1^0)$. Since $\tau = T(\sigma)$ is of dimension zero, $\mathrm{Cl}\, w_\tau = w_\tau$ is a point of $P(K_2)$. Then $\phi_0$ is defined by taking $\phi_0(w_\sigma) = w_\tau$. It is clear that $\phi_0$ is uniquely determined by $T$. Since $P(K_1^0)$ is discrete, $\phi_0$ is a continuous map.

Now assume that there exists a unique continuous map.

$$\phi_{q-1}: \ P(K_1^{q-1}) \longrightarrow P(K_2)$$

which maps $w_\sigma$ of $P(K_1^{q-1})$ barycentrically onto $w_\tau$ of $P(K_2)$ with $\tau = T(\sigma)$ for a certain integer $q > 0$. We are going to construct a map

$$\phi_q: \ P(K_1^q) \longrightarrow P(K_2)$$

as follows: Let $\sigma$ be an arbitrary $q$-simplex of $K_1^q$ and $\tau = T(\sigma)$; $w_\sigma$ is the interior of the ordered geometric simplex $s_\sigma$, and $w_\tau$ that of $s_\tau$. Denote by $B_\sigma: s_\sigma \longrightarrow s_\tau$ the unique barycentric map of $s_\sigma$ onto $s_\tau$ preserving the order of vertices. Then $\phi_q$ is defined by taking

$$\phi_q(x) = \begin{cases} \phi_{q-1}(x) & (x \in P(K_1^{q-1})), \\ B_\sigma(x) & (x \in w_\sigma, \ \sigma \in K_1^q). \end{cases}$$

Now $\phi_q$ is uniquely determined by $T$ and maps $w_\sigma$ barycentrically onto $w_\tau$ with $\tau = T(\sigma)$ for each simplex $\sigma$ of $K_1^q$. To prove the continuity of $\phi_q$, it is sufficient to prove that of the partial map $\psi_\sigma = \phi_q \mid \mathrm{Cl}\, w_\sigma$ for each $q$-simplex $\sigma$ of $K_1^q$. By means of the property of $T$ and that of $\phi_n$ ($n \leq q$), it is easily seen that in the following diagram

$$\begin{array}{ccc} s_\sigma & \xrightarrow{\quad B_\sigma \quad} & s_\tau \\ \downarrow{\scriptstyle \mu_\sigma} & & \downarrow{\scriptstyle \mu_\tau} \\ \mathrm{Cl}\, w_\sigma & \xrightarrow{\quad \psi_\sigma \quad} & \mathrm{Cl}\, w_\tau \end{array}$$

commutativity holds; that is, $\mu_\tau B_\sigma = \psi_\sigma \mu_\tau$, where $\tau = T(\sigma)$ and $\mu_\sigma, \mu_\tau$ are characteristic maps. Let $U$ be an arbitrary open set of $\mathrm{Cl}\, w_\tau$ and $V = \psi_\sigma^{-1}(U)$ in $\mathrm{Cl}\, w_\sigma$. Since $\mu_\tau B_\sigma$ is continuous, $\mu_\sigma^{-1}(V) = (\mu_\tau B_\sigma)^{-1}(U)$ is an open set of $s_\sigma$. By the definition of the topology of $\mathrm{Cl}\, w_\sigma$, $V$ is open. Hence $\psi_\sigma$ is continuous. This proves the continuity of $\phi_q$. Hence we have completed the inductive construction of a sequence of continuous maps $\{\phi_q\}$, uniquely determined by $T$, such that

$$\phi_q | P(K_1^{q-1}) = \phi_{q-1}$$

for every $q > 0$, and $\phi_q$ maps $w_\sigma$ barycentrically onto $w_\tau$, $\tau = T(\sigma)$, for each simplex $\sigma \in K_1^q$.

The required continuous map $f_T : P(K_1) \longrightarrow P(K_2)$ is defined by taking

$$f_T | P(K_1^q) = \phi_q \qquad (q = 0, 1, \cdots).$$

This completes the proof of (3.1).

**4. The singular polytope $P(X)$.** Let $X$ be a given topological space. The singular complex $S(X)$ [2, p.502] is a typical semi-simplicial complex. The semi-simplicial polytope associated with $S(X)$ and constructed in §2 is essentially the *singular polytope* of J.B.Giever [4, p.182], which will be denoted simply by $P(X)$.

For the remainder of the present section, we shall assume that $X$ is arcwise connected and that $x_0 \in X$ is a given point. Following S. Eilenberg, we denote by $S_n(X)$ the subcomplex of $S(X)$ consisting of all singular simplexes $\sigma$ such that all faces of $\sigma$ of dimensions less than $n$ are collapsed at $x_0$. The associated semi-simplicial polytope of $S_n(X)$ can be chosen naturally as a subpolytope of $P(X)$ and will be denoted by $P_n(X)$.

Now let $M$ be a minimal subcomplex of $S(X)$ [2, p.502]. We can choose the associated polytope $P(M)$ as a subpolytope of $P(X)$. The following assertion is an immediate consequence of a corollary of Eilenberg and Zilber [2, p.503].

(4.1) *If the homotopy groups $\pi_i(X)$ vanish for each $i < n$, then $P(M)$ is a subpolytope of $P_n(X)$.*

Let $\Delta_q$ be a given ordered geometric $q$-simplex and let $I$ be the closed unit interval of real numbers. The topological product $\Delta_q \times I$ has a *standard triangulation* into ordered simplexes without the introduction of new vertices. By means of this standard triangulation and the arguments analogous to those used in §3 and those used by Eilenberg and Zilber [2, p.504], it is not difficult to construct a homotopy

$$\delta_t : P(X) \longrightarrow P(X) \qquad (0 \leq t \leq 1).$$

subject to the following conditions:

(i) $\delta_0$ is the identity map;

(ii) $\delta_1$ maps $P(X)$ into $P(M)$;

(iii) $\delta_t \mid P(M)$ is the identity map for all $t \in I$.

Then the main result of Eilenberg and Zilber [2] can be stated as follows.

(4.2) *The polytope $P(M)$ is a deformation retract of $P(X)$.*

Note that $\delta_t$ is not simplicial if $0 < t < 1$. The family of simplicial maps

$$f_{\phi_t} : P(X) \longrightarrow P(X) \qquad (0 \leq t \leq 1),$$

induced according to (3.1) by the family $\phi_t : S(X) \longrightarrow S(X)$ $(0 \leq t \leq 1)$, of Eilenberg and Zilber [2, p.504] is not continuous in $t$ because of our topology introduced in $P(X)$.

The following assertion is a direct consequence of (4.2) and a theorem of J.B. Giever [4, Theorem VI].

(4.3) *The homotopy groups of $P(M)$ are isomorphic with those of $X$; that is,*

$$\pi_i(P(M)) \approx \pi_i(X) \qquad (i \geq 1).$$

5. **The polytope $P(\pi, n)$.** Throughout the present section, let $\pi$ be a (discrete) group and $n$ a positive integer. If $n = 1$, we make no assumption on $\pi$ and write it multiplicatively; otherwise, we assume $\pi$ abelian and written additively. Eilenberg and MacLane [3, p.517] define a semi-simplicial complex $K(\pi, n)$ which is very useful in the relations between homology and homotopy groups. A $q$-simplex $\sigma$ of $K(\pi, n)$ is a function $\sigma$ with values in $\pi$ defined over all sets of arguments $0 \leq a_0 \leq \cdots \leq a_n \leq q$ and subject to two specified conditions, namely the conditions (2.2) and (2.3) of [3]. We denote by $P(\pi, n)$ the semi-simplicial polytope associated with $K(\pi, n)$. Since $K(\pi, n)$ has only one 0-simplex, $P(\pi, n)$ is arcwise connected. We shall use the unique vertex $p_0$ of $P(\pi, n)$ as the base point for the homotopy groups.

THEOREM 1. *The homotopy groups of the polytope $P(\pi, n)$ are given below:*[1]

$$\pi_n(P(\pi, n)) \approx \pi;$$
$$\pi_i(P(\pi, n)) = 0 \qquad (i \neq n).$$

[1] This theorem is known to S. Eilenberg. He mentioned this fact in his address delivered before the Topology Conference of the International Congress of Mathematicians, 1950.

*Proof.* According to Realizability Theorem of J. H. C. Whitehead [12, p. 261], there exists an arcwise connected topological space $X$ with

$$\pi_n(X) \approx \pi, \quad \pi_i(X) = 0 \qquad (i \neq n). \tag{5.1}$$

Choose a minimal complex $M$ of the singular complex $S(X)$ [2, p. 502], and consider their associated polytopes $P(X)$ and $P(M) \subset P(X)$. It follows from (4.3) and (5.1) that

$$\pi_n(P(M)) \approx \pi, \quad \pi_i(P(M)) = 0 \qquad (i \neq n). \tag{5.2}$$

Since $\pi_i(X) = 0$ for each $i < n$, (4.1) tells us that $P(M)$ is a subpolytope of $P_n(X)$.

According to Eilenberg and MacLane [3, p. 517], there is a natural simplicial map

$$\kappa : M \longrightarrow K(\pi, n).$$

Since $\pi_i(X) = 0$ for each $i > n$, a result of Eilenberg and MacLane [3, p. 519] gives a simplicial map

$$\bar{\kappa} : K(\pi, n) \longrightarrow M$$

such that $\kappa\bar{\kappa}$ is the identity on $K(\pi, n)$. It follows also from the construction of $\kappa$ and $\bar{\kappa}$ given by Eilenberg and MacLane [3] that $\bar{\kappa}\kappa(\sigma) = \sigma$ for every $n$-simplex of $M$. Now let

$$f : P(M) \longrightarrow P(\pi, n), \qquad \bar{f} : P(\pi, n) \longrightarrow P(M)$$

be the continuous maps induced respectively by $\kappa$ and $\bar{\kappa}$ according to (3.1). Denoting the $n$-dimensional skeleton of $P(M)$ by $P^n(M)$, we obtain the result that $f\bar{f}$ is the identity map on $P(\pi, n)$, and $\bar{f}f \mid P^n(M)$ is that on $P^n(M)$.

Since $\pi_i(P(M)) = 0$ for each $i > n$, it follows from a standard obstruction method that $\bar{f}f : P(M) \longrightarrow P(M)$ is homotopic with the identity map on $P(M)$. Since $f\bar{f}$ is the identity map on $P(\pi, n)$, this proves that $P(M)$ and $P(\pi, n)$ are of the same homotopy type. Hence (5.2) implies Theorem 1.

Let $\pi_*$ be a subgroup of $\pi$. Then $K(\pi_*, n)$ is the subcomplex of $K(\pi, n)$ consisting of all simplexes $\sigma$ of $K(\pi, n)$ such that

$$\sigma(a_0, \cdots, a_n) \in \pi_*$$

for all sets of arguments $0 \leq a_0 \leq \cdots \leq a_n \leq \dim \sigma$. We can imbed $P(\pi_*, n)$ as a subpolytope of $P(\pi,n)$ in an obvious way. If $\pi_*$ is the subgroup consisting of a single element, then we use the notation $P_0(\pi,n)$ for this $P(\pi_*,n)$. It follows from Theorem 1 that

$$\pi_i(P_0(\pi,n)) = 0 \tag{5.3}$$

for all integers $i \geq 1$. Now (5.3) and the exactness of the homotopy sequence imply that the identity map

$$j: (P(\pi,n), p_0) \longrightarrow (P(\pi,n), P_0(\pi,n))$$

induces the onto isomorphisms:

$$j_*: \pi_i(P(\pi,n)) \approx \pi_i(P(\pi,n), P_0(\pi,n)) \qquad (i \geq 2). \tag{5.4}$$

For the remainder of the present section, we shall assume $n \geq 2$. There is a natural homomorphism

$$k_*: \pi \longrightarrow \pi_n(P(\pi,n), P_0(\pi,n))$$

described as follows. For an arbitrary element $\alpha$ of $\pi$, there is one and only one $n$-simplex $\sigma$ of $K(\pi,n)$ such that $\sigma(0, \cdots, n) = \alpha$. The open $n$-cell $w_\sigma$ of $P(\pi,n)$ is the interior of a geometric $n$-simplex $s_\sigma$ with ordered vertices. The order of the vertices determines an orientation of the pair $(s_\sigma, \partial s_\sigma)$. The characteristic map

$$\mu_\sigma: s_\sigma \longrightarrow \mathrm{Cl}\, w_\sigma$$

carries the pair $(s_\sigma, \partial s_\sigma)$ into the pair $(P(\pi,n), P_0(\pi,n))$ and maps each vertex of $s_\sigma$ into $p_0$. Hence $\mu_\sigma$ determines an element $[\mu_\sigma]$ of the group $\pi_n(P(\pi,n), P_0(\pi,n))$. The homomorphism $k_*$ is defined by setting $k_*(\alpha) = [\mu_\sigma]$.

By a careful examination of the proof of Theorem 1, it is not difficult to see that the homomorphism $j_*^{-1}k_*$ is an isomorphism of $\pi$ onto $\pi_n(P(\pi,n))$; that is,

$$\lambda_n = j_*^{-1}k_*: \pi \approx \pi_n(P(\pi,n)). \tag{5.5}$$

Hence $k_*$ is also an isomorphism onto.

Now let $h: \pi \longrightarrow \pi$ be a given endomorphism of the group $\pi$. Then $h$ induces a simplicial map

$$\eta: K(\pi,n) \longrightarrow K(\pi,n)$$

described as follows: For each simplex $\sigma \in K(\pi,n)$, $\eta(\sigma)$ is the simplex of $K(\pi,n)$ such that $\dim \eta(\sigma) = \dim \sigma$ and

$$\eta(\sigma)(a_0, \cdots, a_n) = h(\sigma(a_0, \cdots, a_n))$$

for every set of arguments $0 \leq a_0 \leq \cdots \leq a_n \leq \dim \sigma$. Let us denote by

$$h^{\#} = f_\eta : P(\pi,n) \longrightarrow P(\pi,n)$$

the continuous map induced by the simplicial map $\eta$ according to (3.1). The following properties of the correspondence $h \longrightarrow h^{\#}$ are immediate.

(5.6) *For any two endomorphisms* $h_1, h_2 : \pi \longrightarrow \pi$ *of the group* $\pi$, *we have* $(h_1 h_2)^{\#} = h_1^{\#} h_2^{\#}$.

(5.7) *If* $h : \pi \longrightarrow \pi$ *is the identity endomorphism of the group* $\pi$, *then* $h^{\#}$ *is the identity map of* $P(\pi, n)$.

Since $h^{\#}(p_0) = p_0$, $h^{\#}$ induces a homomorphism

$$h_* : \pi_n(P(\pi,n)) \longrightarrow \pi_n(P(\pi,n)) .$$

THEOREM 2. *In the following rectangle of homomorphisms*

$$\begin{array}{ccc} \pi & \xrightarrow{\lambda_n} & \pi_n(P(\pi,n)) \\ \downarrow h & & \downarrow h_* \\ \pi & \xrightarrow{\lambda_n} & \pi_n(P(\pi,n)) \end{array}$$

*the commutativity relation* $h_* \lambda_n = \lambda_n h$ *holds.*

*Proof.* It is obvious that the partial map $h^{\#}|P_0(\pi,n)$ coincides with the identity map on $P_0(\pi,n)$. Hence $h^{\#}$ induces an endomorphism $h_0$ of the relative homotopy group $\pi_n(P(\pi,n), P_0(\pi,n))$. Since $\lambda_n = j_*^{-1} k_*$, the above rectangle can be decomposed into the following two:

$$\begin{array}{ccccc} \pi & \xrightarrow{k_*} & \pi_n(P(\pi,n), P_0(\pi,n)) & \xleftarrow{j_*} & \pi_n(P(\pi,n)) \\ \downarrow h & & \downarrow h_0 & & \downarrow h_* \\ \pi & \xrightarrow{k_*} & \pi_n(P(\pi,n), P_0(\pi,n)) & \xleftarrow{j_*} & \pi_n(P(\pi,n)) . \end{array}$$

The commutativity of the left rectangle, $k_* h = h_0 k_*$, is a direct consequence of the definitions of $k_*$ and $h^{\#}$. The commutativity of the right rectangle, $h_0 j_* = j_* h_*$, is a property of the induced homomorphisms of the homotopy sequence. Since $j_*$ is an isomorphism onto, we have $j_*^{-1} h_0 = h_* j_*^{-1}$. Hence we obtain

$$h_*\lambda_n = h_*j_*^{-1}k_* = j_*^{-1}h_0k_* = j_*^{-1}k_*h = \lambda_n h .$$

This completes the proof.

(5.8) COROLLARY. *If we identify the groups $\pi$ and $\pi_n(P(\pi,n))$ by means of the isomorphism $\lambda_n$, then the endomorphisms $h$ and $h_*$ coincide.*

**6. Existence of the space $B$.** Throughout the present section and the following one, let

$$\{\pi_n\} = \pi_1, \quad \pi_2, \cdots, \quad \pi_n, \cdots$$

be a given sequence of groups, where $\pi_1$ is a (multiplicative) group and $\pi_n$ $(n > 1)$ is an (additive) abelian group admitting $\pi_1$ as a given group of left operators; that is, for every $\xi \in \pi_1$ and every $\alpha \in \pi_n$, the element $\xi\alpha \in \pi_n$ is defined and

$$\xi(\alpha + \beta) = \xi\alpha + \xi\beta, \quad \xi(\eta\alpha) = (\xi\eta)\alpha, \quad 1\alpha = \alpha .$$

For each integer $n \geq 1$, let $P_n = P(\pi_n, n)$ denote the polytope associated with the complex $K(\pi_n, n)$. We shall use the following notations:

$$X = P_1, \qquad Y = P_2 \times P_3 \times \cdots \times P_n \times \cdots .$$

Both $X$ and $Y$ are arcwise connected Hausdorff spaces. Let $\theta_n : Y \longrightarrow P_n$ $(n = 2, 3, \cdots)$ denote the projection of $Y$ onto the factor space $P_n$. The following properties of the space $Y$ are immediate consequences of results in a note due to J.H.C. Whitehead, [13, p. 289]:

(6.1) *$Y$ is 1-connected; that is, $\pi_1(Y) = 0$.*

(6.2) *$\pi_n(Y) \approx \pi_n$ for every integer $n \geq 2$.*

(6.3) *The Whitehead products in $Y$ are all trivial; that is, for any two elements $a \in \pi_m(Y)$ and $b \in \pi_n(Y)$, we have $a \circ b = 0$.*

Each element $\xi \in \pi_1$ determines, for every $n = 2, 3, \cdots$, an automorphism

$$\xi_n : \pi_n \approx \pi_n$$

defined by $\xi_n(\alpha) = \xi\alpha \in \pi_n$ for any $\alpha \in \pi_n$. According to §5, $\xi_n$ induces a homeomorphism

$$\xi_n^{\#} : P_n \longrightarrow P_n \qquad (n = 2, 3, \cdots)$$

of $P_n$ onto $P_n$ . Define a homeomorphism

$$\xi^{\#} : Y \longrightarrow Y \qquad (\xi \in \pi_1)$$

of $Y$ onto itself by taking

$$\theta_n \xi^{\#}(y) = \xi_n^{\#} \theta_n(y) \qquad (y \in Y;\ n = 2, 3, \cdots). \tag{6.4}$$

The association $\xi \longrightarrow \xi^{\#}$ clearly determines a homomorphism

$$\rho : \pi_1 \longrightarrow \mathrm{Hom}\,(Y)$$

of the discrete group $\pi_1$ into the discrete group Hom $(Y)$ of all homeomorphisms of $Y$ onto itself. Let

$$G = \rho(\pi_1) \subset \mathrm{Hom}\,(Y)\ ;$$

then $G$ is a *topological transformation group* of $Y$ and is isomorphic with the quotient group of $\pi_1$ over the kernel of the homomorphism $\rho$.

Remembering the isomorphism $\lambda_1 : \pi_1 \approx \pi_1(X)$ defined by (5.5), we shall call

$$\chi = \rho \lambda_1^{-1} : \pi_1(X) \longrightarrow G$$

the *characteristic homomorphism.*

Now let us consider the universal covering space $\tilde{X}$ of $X$. It is well known that $\tilde{X}$ is a bundle space over $X$ with discrete fiber $\pi_1$ and structural group $\pi_1$ . Then the characteristic homomorphism $\chi$ induces a *weakly associated* bundle space $B$ over $X$ with $Y$ as fiber and $G$ as structural group [7]. The bundle space $B$ is uniquely determined up to an equivalence in the sense of fiber bundles. In the following sections, we shall give an explicit construction of the bundle space $B$.

**7. Barycentric subdivisions of semi-simplicial polytopes.** Let $K$ be a given semi-simplicial complex and $P(K)$ its associated polytope. We are going to define baryeentric subdivisions of $P(K)$.

For each simplex $\sigma \in K$, let us denote by $s'_\sigma$ the *barycentric first derived* [6, p. 3] of the ordered geometric simplex $s_\sigma$ associated with $\sigma$.

Since the characteristic map

$$\mu_\sigma : \ s_\sigma \longrightarrow \mathrm{Cl}\, w_\sigma$$

reduces to the identity map if it is restricted within the interior $w_\sigma$ of $s_\sigma$, $\mu_\sigma$ induces a simplicial subdivision of $w_\sigma$ into $\mu_\sigma(\text{Int } s'_\sigma)$, named the *barycentric first derived* $w'_\sigma$ of $w_\sigma$, which is a finite set of open geometric simplexes. If we replace each open cell $w_\sigma$ of $P(K)$ by its barycentric first derived $w'_\sigma$, we obtain a subdivision of $P(K)$, called the *first barycentric subdivision* $P'(K)$ of $P(K)$.

More generally, let us denote by $s_\sigma^{(n)}$ the *barycentric n-th derived* [6, p.3] of $s_\sigma$. Then the characteristic map $\mu_\sigma$ induces a simplicial subdivision of $w_\sigma$ into $\mu_\sigma(\text{Int } s_\sigma^{(n)})$, called the *barycentric n-th derived* $w_\sigma^{(n)}$ of $w_\sigma$. If we replace each open cell $w_\sigma$ by its barycentric $n$-th derived $w_\sigma^{(n)}$, we obtain the *n-th barycentric subdivision* $P^{(n)}(K)$ of $P(K)$. It is clear that the characteristic map $\mu_\sigma$ carries each open simplex of $s_\sigma^{(n)}$ barycentrically onto some open simplex of $P^{(n)}(K)$.

Let $v$ be an arbitrary vertex (that is, an open 0-simplex) of $P^{(n)}(K)$, where $n \geq 1$. The *star* of $v$, denoted by $\text{St}(v)$, is defined to be the union of all open simplexes $\xi$ of $P^{(n)}(K)$ such that $\text{Cl}\,\xi$ contains $v$. The following assertion can easily be proved.

(7.1) *The star of each vertex of* $P^{(n)}(K)$ $(n \geq 1)$ *is contractible* (*in itself*) *to a point.*

By a *simplicial polytope* $P$, we understand the union of a collection of closed geometric simplexes $\{s_\alpha\}$, where the index $\alpha$ runs over a certain abstract set $A$, such that (i) every face of an arbitrary simplex $s_\alpha$ of the collection belongs to the collection and (ii) the intersection $s_\alpha \cap s_\beta$ of any two simplexes of the collection is either vacuous or a face on both of them, with the topology defined as follows: A set $M \subset P$ is said to be open if and only if, for each closed geometric simplex $s_\alpha$ of the collection, $M \cap s_\alpha$ is an open set $s_\alpha$ in its euclidean topology. Simplicial polytopes are called topological polyhedra by J.H.C. Whitehead [9, p.316]. The following assertion can easily be proved.

(7.2) *For each* $n \geq 2$, $P^{(n)}(K)$ *is a simplicial polytope.*

**8. Explicit construction of the bundle.** Let us return to the notations of §6. The vertices of the first barycentric subdivision of $X$ are barycenters $\{x_\sigma\}$ of the open cells $\{w_\sigma\}$ of $X$, where $\sigma$ runs over all simplexes of the semi-simplicial complex $K = K(\pi_1, 1)$. In particular, we shall denote by $x_0$ the vertex which corresponds to the unique 0-simplex of $K$. Hence $x_0$ is the basic point of the fundamental group $\pi_1(X)$ of $X$. Let $V_\sigma$ denote the star of the vertex $x_\sigma$ in the first barycentric subdivision $X$. Then we obtain an open covering $\Omega = \{V_\sigma\}$ of $X$, indexed by $\sigma \in K$. According to (7.1), each member $V_\sigma$ of the covering $\Omega$ is contractible (in itself) to the point $x_\sigma$.

For each $\sigma \in K$, let $s_\sigma = < v_0, \cdots, v_q >$ be the associated geometric simplex. Denote by $c_\sigma : I \longrightarrow s_\sigma$ the unique linear map such that $c_\sigma(0) = v_0$ and $c_\sigma(1) = x_\sigma$. Define a path

$$C_\sigma : I \longrightarrow X$$

joining $x_0$ to $x_\sigma$ by taking $C_\sigma = \mu_\sigma c_\sigma$, where $\mu_\sigma : s_\sigma \longrightarrow \mathrm{Cl} w_\sigma$ is the characteristic map.

Let $\sigma$ and $\tau$ be any two simplexes of $K$ such that $V_\sigma \cap V_\tau$ is nonvoid. Take a point $x \in V_\sigma \cap V_\tau$. Choose a path $D : I \longrightarrow V_\sigma$ joining $x_\sigma$ to $x$ and a path $E : I \longrightarrow V_\tau$ joining $x_\tau$ to $x$. Then the closed path $C_\tau E D^{-1} C_\sigma^{-1}$ represents an element $\xi_{\tau\sigma}$ of $\pi_1(X)$ which clearly does not depend on the choice of the point $x$ and the paths $D$ and $E$. Let $\sigma, \tau, \theta$ be any three simplexes of $K$ such that $V_\sigma \cap V_\tau \cap V_\theta$ is nonvoid; then it is easy to see that $\xi_{\theta\tau}\xi_{\tau\sigma} = \xi_{\theta\sigma}$. Call

$$g_{\tau\sigma} = \chi(\xi_{\tau\sigma}) \in G .$$

Then, as constant maps on $V_\sigma \cap V_\tau$ into $G$, the collection $\{g_{\tau\sigma}\}$ together with the covering $\Omega = \{V_\sigma\}$ form[1] a *system of coordinate transformations in $X$ with values in $G$* [7, §3.1].

The construction of N.E.Steenrod [7, §3.2] gives a fiber bundle (with coordinate system)

$$F = \{B, X, p, Y, G, V_\sigma, \phi_\sigma\}$$

with base space $X$, fiber $Y$, group $G$, and the coordinate transformations $\{g_{\tau\sigma}\}$. To clarify the precise situation, we shall briefly describe the entities of $F$ as follows.

Let us regard the indexing complex $K$ as a topological space with the discrete topology; that is to say, every simplex $\sigma$ of $K$ is considered as a point which is an open set of $K$. Let $T$ be the subset of $X \times Y \times K$ consisting of these triples $(x, y, \sigma)$ such that $x \in V_\sigma$. Define in $T$ an equivalence relation:

$$(x,y,\sigma) \sim (x', y', \tau) \quad \text{if} \quad x = x', \quad g_{\tau\sigma} \cdot y = y'.$$

[1] The author has the advantage of reading the book [7] in manuscript. The system $\{g_{\tau\sigma}\}$ of coordinate transformations constructed here is essentially a particular case of that constructed by N.E.Steenrod [7, Sec.13.8]. The sketch given here is to clarify the precise situation.

Define the *bundle space* $B$ to be the totality of these equivalence classes in $T$. Let

$$\omega: T \longrightarrow B$$

assign to each $(x, y, \sigma)$ its equivalence class $[x, y, \sigma]$. Give $B$ the identification topology determined by $\omega$; namely, a set $U$ in $B$ is called open if $\omega^{-1}(U)$ is an open set of $T$. Then $B$ is a Hausdorff space and $\omega$ a continuous open map. The *projection*

$$p : B \longrightarrow X$$

is defined by $p([x, y, \sigma]) = x$. The *coordinate functions*

$$\phi_\sigma: V_\sigma \times Y \longrightarrow p^{-1}(V_\sigma) \qquad (\sigma \in K)$$

are defined by $\phi_\sigma(x, y) = [x, y, \sigma]$ for each $x \in V_\sigma$ and $y \in Y$ with $\sigma$ running over $K$.

Let $y_0$ denote the point of $Y$ such that $\theta_n(y_0)$ is the unique vertex of $P_n$ for each $n = 2, 3, \cdots$. Denote by 0 the unique 0-simplex of $K$ and call $b_0 = [x_0, y_0, 0] \in B$. Then we have $p(b_0) = x_0$ and $\phi_0(x_0, y_0) = b_0$. We shall understand that $x_0, y_0, b_0$ are respectively the basic points of the various homotopy groups of the spaces $X, Y, B$ studied in the next section.

**9. The homotopy groups of the bundle space $B$.** In the present section, we shall study in details the homotopy groups of the bundle space $B$, constructed in the foregoing section, and their mutual operations. The realizability theorem, stated in the introduction, follows as an immediate consequence of these investigations.

First of all, let us recall the (*exact*) homotopy sequence [7, §17.3],

$$\cdots \xrightarrow{p^*_{n+1}} \pi_{n+1}(X) \xrightarrow{\Delta_{n+1}} \pi_n(Y) \xrightarrow{i^*_n} \pi_n(B) \xrightarrow{p^*_n} \pi_n(X) \xrightarrow{\Delta_n} \cdots$$
$$\cdots \xrightarrow{p^*_2} \pi_2(X) \xrightarrow{\Delta_2} \pi_1(Y) \xrightarrow{i^*_1} \pi_1(B) \xrightarrow{p^*_1} \pi_1(X),$$

of the fiber bundle $F = \{B, X, p, Y, G, V_\sigma, \phi_\sigma\}$, with $x_0, y_0, b_0$ as the basic points of the homotopy groups of the spaces $X, Y, B$, respectively. Here, the homomorphisms $p^*_n$ $(n \geq 1)$ are those induced by the projection $p : B \longrightarrow X$, and $i^*_n$ $(n \geq 1)$ those induced by the map $i: Y \longrightarrow B$ defined by

$$i(y) = \phi_0(x_0, y_0) = [x_0, y_0, 0] \qquad (y \in Y). \tag{9.1}$$

If we identify $Y$ with $p^{-1}(x_0)$, then $i$ is the injection of $p^{-1}(x_0)$ into $B$.

THEOREM 3. *For each integer $n \geq 1$, there is a natural isomorphism $h_n: \pi_n(B) \approx \pi_n$ of $\pi_n(B)$ onto $\pi_n$.*

*Proof.* First, let us prove that $p_1^*$ is an isomorphism onto. By means of the arcwise connectedness of the fiber $p^{-1}(x_0)$, it can be shown by a standard argument that $p_1^*$ maps $\pi_1(B)$ onto $\pi_1(X)$. According to (6.1), we have $\pi_1(Y) = 0$. An application of the exactness of the homotopy sequence gives that the kernel of $p_1^*$ is $i_1^*(\pi_1(Y)) = 0$. Hence $p_1^*$ is an isomorphism onto. We define

$$h_1 = \lambda_1^{-1} p_1^* : \pi_1(B) \approx \pi_1 , \tag{9.2}$$

where $\lambda_1 : \pi_1 \approx \pi_1(X)$ is the isomorphism defined by (5.5) for $n = 1$.

Next let $n \geq 2$. Since $X = P_1$, we have

$$\pi_{n+1}(X) = 0 = \pi_n(X) .$$

Then it follows from the exactness of the homotopy sequence that $i_n^*$ is an isomorphism onto. The projection $\theta_n : Y \longrightarrow P_n$ induces an isomorphism onto:

$$\theta_n^* : \pi_n(Y) \approx \pi_n(P_n) .$$

We define

$$h_n = \lambda_n^{-1} \theta_n^* i_n^{*-1} : \pi_n(B) \approx \pi_n \qquad (n \geq 2), \tag{9.3}$$

where $\lambda_n : \pi_n \approx \pi_n(P_n)$ is the isomorphism defined by (5.5). This completes the proof of the theorem.

According to S. Eilenberg [1], the fundamental group $\pi_1(B)$ operates on the left of $\pi_n(B)$ for each $n \geq 2$. Let $w \in \pi_1(B)$ and $a \in \pi_n(B)$ be arbitrarily given elements. Choose a path $C : I \longrightarrow B$ with $C(0) = b_0 = C(1)$ which represents $w$, and a map $f: I^n \longrightarrow B$ with $f(\partial I^n) = b_0$ which represents $a$, where $I$ denotes the closed unit interval of real numbers and $I^n$ the closed unit $n$-cube of the euclidean $n$-space with $\partial I^n$ denoting its boundary. Let $f_t: I^n \longrightarrow B$ $(0 \leq t \leq 1)$ be any homotopy such that $f_1 = f$ and $f_t(\partial I^n) = C(t)$ for all $0 \leq t \leq 1$; then the element $wa \in \pi_n(B)$ is represented by the map $f_0$. We remind that $\pi_1$ operates on the left $\pi_n$ for each $n \geq 2$.

THEOREM 4. *For arbitrary elements $w \in \pi_1(B)$ and $a \in \pi_n(B)$ $(n \geq 2)$ we have*

$$h_n(wa) = h_1(w)h_n(a).$$

*Proof.* Choose a path $C: I \longrightarrow X$ with $C(0) = x_0 = C(1)$ which represents the element $p_1^*(w)$ of the fundamental group $\pi_1(X)$. Denote by $Y_0$ the fiber over $x_0$; that is, $Y_0 = p^{-1}(x_0) = i(Y)$. Let $j: Y_0 \longrightarrow Y$ denote the inverse of $i$. According to N.E.Steenrod [7, §3.1], there is a homotopy $H_t: Y_0 \longrightarrow B$ $(0 \leq t \leq 1)$ such that $H_1$ is the identity and $pH_t(Y_0) = C(t)$ for each $0 \leq t \leq 1$. More precisely, we choose $H_t$ to be a *translation of* $Y_0$ *along* $C^{-1}$ *into itself* [7, §13.1]. Call

$$\xi = h_1(w) = \lambda_1^{-1}p_1^*(w) \in \pi_1.$$

It follows from Steenrod's proof in his construction [7, §13.8] of the system $\{g_{\tau\sigma}\}$ of the coordinate transformations of the fiber bundle $F$ that the homeomorphism

$$\psi = jH_0 i : Y \longrightarrow Y$$

is in $G$, and

$$\psi = \chi p_1^*(w) = \rho h_1(w) = \rho(\xi) = \xi^{\#},$$

where $\rho: \pi_1 \longrightarrow \mathrm{Hom}(Y)$ is the homomorphism which maps $\xi$ into $\xi^{\#}$ defined by (7.4). Hence $H_0$ is the homeomorphism $i\xi^{\#}j$ of $Y_0$ onto itself and maps $b_0$ into itself. Define a path $\tilde{C}: I \longrightarrow B$ by taking $\tilde{C}(t) = H_t(b_0)$ for each $t \in I$. Since $p\tilde{C} = C$, and $p_1^*$ is an isomorphism, $\tilde{C}$ represents the element $w$ of $\pi_1(B)$.

Choose a map $g: I^n \longrightarrow Y$ with $g(\partial I^n) = y_0$ which represents the element $i_n^{*-1}(a)$ of $\pi_n(Y)$. Then the map $f = ig: I^n \longrightarrow B$ is a representative of the element $a \in \pi_n(B)$ and maps $I^n$ into $Y_0$. Define a homotopy $f_t: I^n \longrightarrow B$ $(0 \leq t \leq 1)$ by taking $f_t = H_t f$ for each $0 \leq t \leq 1$. Then we have $f_1 = f$ and $f_t(\partial I^n) = \tilde{C}(t)$ for every $0 \leq t \leq 1$. Hence, by definition, the map $f_0 = H_0 f = i\xi^{\#}g$ represents the element $wa \in \pi_n(B)$. It follows that $\xi^{\#}g$ is a representative of the element $i_n^{*-1}(wa)$ of $\pi_n(Y)$.

The homeomorphism $\xi_n^{\#}: P_n \longrightarrow P_n$ induces an automorphism

$$\xi_*: \pi_n(P_n) \approx \pi_n(P_n).$$

By (7.4), we have $\theta_n\xi^{\#}g = \xi_n^{\#}\theta_n g$. Hence

$$\theta_n^* i_n^{*-1}(wa) = \xi_*\theta_n^* i_n^{*-1}(a).$$

According to Theorem 2, we have $\lambda_n^{-1}\xi_* = \xi\lambda_n^{-1}$. So we deduce that

$$h_n(wa) = \lambda_n^{-1}\theta_n^* i_n^{*-1}(wa) = \lambda_n^{-1}\xi_*\theta_n^* i_n^{*-1}(a)$$

$$= \xi \lambda_n^{-1} \theta_n^* i_n^{*-1}(a) = \xi h_n(a) = h_1(w) h_n(a).$$

This completes the proof.

For arbitrarily given elements $a \in \pi_m(B)$ and $b \in \pi_n(B)$ $(m \geq 2, n \geq 2)$, let us choose representative maps $f: I^m \longrightarrow B$ and $g: I^n \longrightarrow B$ with $f(\partial I^m) = b_0 = g(\partial I^n)$. Since $I^{m+n} = I^m \times I^n$, we have

$$\partial I^{m+n} = (I^m \times \partial I^n) \cup (\partial I^m \times I^n). \tag{9.4}$$

The Whitehead product [10, p. 411] of the elements $a$ and $b$ is an element $a \circ b$ of $\pi_{m+n-1}(B)$ determined by the map $h: \partial I^{m+n} \longrightarrow B$ which is defined as follows:

$$h(x,y) = \begin{cases} f(x) & (x \in I^m,\ y \in \partial I^n), \\ g(y) & (x \in \partial I^m,\ y \in I^n). \end{cases} \tag{9.5}$$

THEOREM 5. *For arbitrary elements $a \in \pi_m(B)$ and $b \in \pi_n(B)$ $(m \geq 2, n \geq 2)$, we have $a \circ b = 0$.*

*Proof.* Let $\alpha = i_m^{*-1}(a) \in \pi_m(Y)$ and $\beta = i_n^{*-1}(b) \in \pi_n(Y)$. Then we have $i_{m+n-1}^*(\alpha \circ \beta) = a \circ b$. Hence Theorem 5 is an immediate consequence of (6.3).

According to Theorems 3-5, our bundle space $B$ constructed in §8 satisfies all the conditions in the Realizability Theorem stated in the introduction. This completes the proof of the Realizability Theorem.

**10. An application.** Take an even sphere $S^{2r}$ and let

$$\pi_n = \pi_n(S^{2r}) \qquad (n = 1, 2, \cdots).$$

The foregoing construction gives an arcwise connected topological space $B$ with

$$\pi_n(B) \approx \pi_n(S^{2r}) \qquad (n = 1, 2, \cdots).$$

Since $\pi_1(B) = 0 = \pi_1(S^{2r})$, the operations of the fundamental groups on the higher homotopy groups are all trivial for both $B$ and $S^{2r}$. However, the Whitehead products of the higher homotopy groups are essentially different for the spaces $B$ and $S^{2r}$. In fact, if $e$ is a generator of the group $\pi_{2r}(S^{2r})$, then the Whitehead product $e \circ e$ is nonzero because it has Hopf invariant $\pm 2$ [8, p. 205]; but all the Whitehead products for the space $B$ are zero. This proves that the Whitehead products of a topological space are essential invariants of the space and that they are not determined by the homotopy groups together with the operations of the fundamental group upon the higher homotopy groups.

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