

# Pacific Journal of Mathematics

## **A SHORT PROOF OF PILLAI'S THEOREM ON NORMAL NUMBERS**

JOHN E. MAXFIELD

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**1. Introduction.** The object of this paper is to give a short proof of the Pillai theorem [2] on normal numbers using the Niven-Zuckerman result [1] as a tool.

**DEFINITION 1.** A number  $\sigma$  is *simply normal* to the base  $r$  if, in the expansion to the base  $r$  of the fractional part of  $\sigma$ , we have  $\lim_{n \rightarrow \infty} n_c/n = 1/r$  for all  $c$ , where  $n_c$  is the number of occurrences of the digit  $c$  in the first  $n$  digits of  $\sigma$ .

**DEFINITION 2.** A number  $\sigma$  is *normal* to the base  $r$  if  $\sigma, r\sigma, r^2\sigma, \dots$  are each simply normal to all the bases  $r, r^2, r^3, \dots$ .

**THEOREM (Pillai).** *A necessary and sufficient condition that a number  $\sigma$  be normal to the base  $r$  is that it be simply normal to the bases  $r, r^2, r^3, \dots$ .*

**2. Proof.** The necessity of the condition follows from the definition of normality.

To prove sufficiency, assume that  $\sigma$  is simply normal to the bases  $r, r^2, \dots$ . Let  $A = (a_1 a_2 \dots a_v)$  be any fixed sequence of digits (to base  $r$ ), where  $v = hr - s$ ,  $h > 0$ ,  $0 \leq s < r$ ; and consider the occurrence of  $A$  in  $\sigma$ . Count the number of occurrences of  $A$  in the collection of sequences of length  $hr$ . There are  $s$  digits free after  $v$  of the  $hr$  digits are fixed. Thus there are  $(s + 1)r^s$  different occurrences of  $A$  in these sequences.

For any positive integer  $n$ , define  $f_n(A)$  to be the frequency of the occurrences of  $A$  in  $\sigma$  except in places where  $A$  will straddle the middle of sequences of length  $2h2^{n-1}r$  starting in places congruent to 1 (mod  $2h2^{n-1}r$ ), or where  $A$  will straddle the middle of sequences of length  $4h2^{n-1}r$  starting in places congruent to 1 (mod  $4h2^{n-1}r$ ), or  $\dots$ , or where  $A$  will straddle the middle of sequences of length  $2^s h2^{n-1}r$  starting in places congruent to 1 (mod  $2^s h2^{n-1}r$ ), and so on.

Certainly  $\lim_{n \rightarrow \infty} f_n(A)$ , if it exists, will be equal to  $f(A)$ , the frequency of  $A$  in  $\sigma$ .

We have

$$f_1(A) = \frac{(s + 1)r^s}{hr r^{hr}} = \frac{1}{r^v} - \frac{v - 1}{hr^v + 1},$$

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Received July 5, 1951.

*Pacific J. Math.* 2(1952), 23-24

since there are  $hr$  digits of  $\sigma$  to base  $r$  in each digit of  $\sigma$  to base  $r^{hr}$ , and  $\sigma$  is simply normal to the base  $r^{hr}$ . The number of occurrences of  $A$  straddling the middle of blocks of length  $2hr$  is  $(v-1)r^{2hr+s}$ . The frequency of these in  $\sigma$ , where the sequence of length  $2hr$  starts in a place congruent to 1 (mod  $2hr$ ), is

$$\frac{(v-1)r^{2hr+s}}{2hr r^{2hr}} = \frac{v-1}{2hr^{v+1}},$$

since there are  $2hr$  digits of  $\sigma$  to base  $r$  to each digit of  $\sigma$  to base  $r^{2hr}$ .

Thus

$$f_2(A) = \frac{1}{r^v} - \frac{v-1}{hr^{v+1}} + \frac{v-1}{2hr^{v+1}}.$$

Similarly,

$$f_3(A) = f_2(A) + \frac{v-1}{4hr^{v+1}} = \frac{1}{r^v} - \frac{v-1}{hr^{v+1}} + \frac{v-1}{hr^{v+1}} \left[ \frac{1}{2} + \frac{1}{4} \right]$$

and

$$f_n(A) = \frac{1}{r^v} - \frac{v-1}{hr^{v+1}} + \frac{v-1}{hr^{v+1}} \sum_{i=1}^{n-1} 1/2^i.$$

It follows that

$$\lim_{n \rightarrow \infty} f_n(A) = 1/r^v.$$

Accordingly, by the Niven-Zuckerman result [1], stating that a necessary and sufficient condition in order that a number  $\sigma$  be normal is that every fixed sequence of  $v$  digits occur in the expansion of  $\sigma$  with the frequency  $1/r^v$ , we see that  $\sigma$  is normal to the scale  $r$ .

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2. S. S. Pillai, *On normal numbers*, Proceedings of the Indian Acad. Sci., Section A, **12** (1940), 179-184.

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INSTITUTÉ FOR NUMERICAL ANALYSIS

Vari-Type Composition by

Elaine Barth

Delores Gilbertson

With the cooperation of

E. F. Beckenbach

E. G. Straus

Printed in the United States of America by  
Edwards Brothers, Inc., Ann Arbor, Michigan

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES  
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Vol. 2, No. 1

January, 1952

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