SOME INEQUALITIES IN CERTAIN NONORIENTABLE RIEMANNIAN MANIFOLDS

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1. Introduction. As is well known, the projective plane and the Moebius strip are nonorientable manifolds of dimension two. After introducing a Riemannian metric on each of them, we obtain two 2-dimensional nonorientable Riemannian manifolds. For convenience of reference, let us denote them by M_{p2}^2 and M_m^2, respectively. Each of these manifolds has an area A. Moreover, there exists a family of closed curves, which are not homotopic to zero, on each manifold; and hence the set of the lengths of all these closed curves in consideration has a positive greatest lower bound, a. The purpose of this paper is to investigate the relationship between these two geometrical constants, A and a. It is found that, in each case, there exists an inequality [1] connecting them, of the form

$$(1) \qquad\qquad A \geq ka^2,$$

k being a constant depending only on the conformal character of the Riemannian manifold. To establish such inequalities and to determine the corresponding best possible constants are the two central problems in this investigation.

For the time being, the projective plane is used in the following realization: it is given as the unit sphere with identification of diametrically opposite points. We assume further that the metric on M_{p2}^2 is given by

$$ds^2 = g(p) \ d\rho^2,$$

$d\rho^2$ being the line element of the unit sphere taken from the embedding Euclidean space; $g(p) \in C_\omega$, $g(p) > 0$ for any point p on the manifold. As for the Moebius strip, we assume that it is given by the strip

$$-\beta < \gamma < \beta,$$

with identification given by the fundamental group

$$x' = x + n(2\alpha),$$

$$(n = 0, \pm 1, \pm 2, \cdots).$$

$$y' = (-1)^n y$$

We assume further that the metric on M_m^2 is given by

$$ds^2 = g(x, y)(dx^2 + dy^2),$$

where $g(x,y) \in C_\omega$ and $g(x,y) > 0$. We shall see later that these assumptions are admissible in our cases.

The main idea of the method for solving these problems is to reduce the general metric, $g(p)d\rho^2$, to a simple and special one, $\bar{g}d\rho^2$, for which the *equality* in (1) holds, by an averaging process over a certain continuous group space; this enables us to handle our problems more easily. Let A_g, a_g, $A_{\bar{g}}$, $a_{\bar{g}}$ be the geometrical constants defined in terms of the original metric and the simplified metric respectively. Fortunately, this averaging process provides us a means of comparison between A_g and $A_{\bar{g}}$ and between a_g and $a_{\bar{g}}$; namely, we have

(2)

$$A_g \geq A_{\bar{g}},$$

$$a_g \leq a_{\bar{g}}.$$

A comparison of the equality yielded by the special metric mentioned above with the foregoing inequalities (2) gives us the desired result.

Take, for example, the manifold $M_{p^2}^2$. Each rotation of a 2-sphere about its center in the ordinary space is actually a conformal mapping of $M_{p^2}^2$ onto itself. All these rotations form a compact Lie group G. Averaging $[g(p)]^{1/2}$ over G by the Hurwitz integration,

$$\int_G [(g(p))^{1/2}]^\sigma \, \delta\sigma = h^{1/2}$$

where $\sigma \in G$, and where $\delta\sigma$ is the invariant volume element, we can easily show that h is a constant and that the simplified metric is an elliptic one; this produces the equality

(3) $$A_h = \frac{2}{\pi} a_h^2.$$

A combination of (3) with the following inequalities corresponding to (2),

$$A_g \geq A_h,$$

$$a_g \leq a_h,$$

shows that we have, in general,

$$A \geq \frac{2}{\pi} a^2.$$

The same method can be extended, with some restrictions, to the case of $M_{p^n}^n$, that is, the Riemannian manifold whose underlying topological space is an n-dimensional projective space.

In the case of M_m^2, let the rectangle

$$R: \quad \begin{cases} -\alpha \leq x < \alpha \\ -\beta < y < \beta \end{cases}$$

be its fundamental region, as will be explained in §3. There exists a one-parameter family of conformal mappings of M_m^2 onto itself,

$$x' = x + c,$$

$$y' = y,$$

c being real mod (4α). Averaging $[g(x+c,y)]^{1/2}$ over the interval $[0, 4\alpha]$ by the formula

$$\frac{1}{4\alpha} \int_0^{4\alpha} [g(x+c,y)]^{1/2} \, dc = [\bar{g}(y)]^{1/2}$$

we can see that $[\bar{g}(y)]^{1/2}$ is free of x and is an even function on account of the fact that the metric is invariant under the fundamental group Γ; that is,

$$g[x + n(2\alpha), (-1)^n y] = g(x,y).$$

A further consideration of the same problem with the metric $g(y)(dx^2 + dy^2)$, where $g(y)$ is positive and even, leads to a distinguished $g_0(y)$ such that $g_0(y)(dx^2 + dy^2)$ plays the same role as the elliptic metric in the case of $M_{p^2}^2$ or $M_{p^n}^n$; that is, $g_0(y)(dx^2 + dy^2)$ leads to the equality in (1).

2. **Riemannian manifold** $M_{p^2}^2$, **whose underlying topological space is a projective plane** P^2. To begin with, let us prove the following general lemma, which will often be used.

LEMMA 1. *Let M_i^2 ($i = 1, 2, \cdots, n$) be a set of n 2-dimensional Riemannian manifolds smooth of order 1 such that each M_i^2, with the same underlying topological space T^2, has a metric of the form*

$$ds_i^2 = g_i(p) ds^2,$$

where $g_i(p) > 0$, $g_i(p) \in C_0$ for $p \in M_i^2$; and ds^2 is a Riemannian metric which can be defined locally by

$$ds^2 = \sum_{j, k =1}^{2} g_{jk}(u_1, u_2) du^j du^k, \qquad\qquad g_{jk}(u_1, u_2) \in C_0.$$

Let $\bar{g}_n(p)$ be defined by the formula

$$\bar{g}_n(p) = \left\{ \frac{[g_1(p)]^{1/2} + \cdots + [g_n(p)]^{1/2}}{n} \right\}^2.$$

If the sets of lengths $S_i = \int_C [g_i(p)]^{1/2} ds$ ($i = 1, 2, \cdots, n$) of a family F of curves C on T^2 have the same nonnegative greatest lower bound,

$$a_{g_1} = a_{g_2} = \cdots = a_{g_n},$$

and the areas A_{g_i} of M_i^2 have the same value,

$$A_{g_1} = A_{g_2} = \cdots = A_{g_n},$$

then we have

(i)
$$A_{\bar{g}_n} \leq A_{g_1} = \cdots = A_{g_n}$$

and

(ii)
$$a_{\bar{g}_n} \geq a_{g_1} = \cdots = a_{g_n}.$$

Proof. By the definition of area and that of $\bar{g}_n$, we have

$$A_{\bar{g}_n} = \iint \bar{g}_n(p) d\omega = \iint \frac{[(g_1)^{1/2} + \cdots + (g_n)^{1/2}]^2}{n^2} d\omega,$$

where $d\omega$ is the area element, which can be expressed locally by the formula

$$d\omega = \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix}^{1/2} du^1 du^2 .$$

Making use of the inequality

$$(\alpha_1 + \cdots + \alpha_n)^2 \leq n(\alpha_1{}^2 + \cdots + \alpha_n{}^2),$$

we have

$$A_{\bar{g}_n} \leq \iint n \frac{(g_1 + \cdots + g_n)}{n^2} d\omega$$

$$\leq \frac{\iint g_1 d\omega + \cdots + \iint g_n d\omega}{n}$$

$$\leq \frac{A_{g_1} + \cdots + A_{g_n}}{n} .$$

By hypothesis, it follows that

$$A_{\bar{g}_n} \leq A_{g_1} = \cdots = A_{g_n} ,$$

which is (i).

The proof of (ii) follows from the definitions of the concepts concerned,

$$\int_C \bar{g}_n{}^{1/2} ds = \int_C \frac{g_1{}^{1/2} + \cdots + g_n{}^{1/2}}{n} ds = \frac{\int_C g_1{}^{1/2} ds + \cdots + \int_C g_n{}^{1/2} ds}{n}$$

$$\geq \frac{a_{g_1} + \cdots + a_{g_n}}{n} = a_{g_1} = \cdots = a_{g_n} ,$$

the line integrals being extended along any curve C of the family F. Hence

$$a_{\bar{g}_n} = \operatorname*{g.l.b.}_{c \,\in\, F} \left(\int_C \bar{g}_n{}^{1/2} ds \right) \geq a_{g_1} = \cdots = a_{g_n} .$$

We shall now prove the following theorem, which characterizes the relationship between the two geometrical constants A and a in $M^2_{p\,2}$.

THEOREM 1. *Let M^2_{p2} be the Riemannian manifold whose underlying topological space is a projective plane and whose metric is locally defined by*

$$ds^2 = \sum_{i,k=1}^{2} g_{ik}(u_1, u_2)\, du^i du^k, \qquad g_{ik}(u_1, u_2) \in C_0 \, ;$$

let A be its area, and a the greatest lower bound of the lengths of all the closed curves not homotopic to zero on M^2_{p2}; then

$$A \geq \frac{2}{\pi}\, a^2 \, .$$

Moreover, $2/\pi$ is the best constant.

Proof. On account of the Weierstrass approximation theorem, it suffices to assume that the fundamental tensor $g_{ik}(u_1, u_2)$ is analytic. Then we can introduce, in the small, an isothermic coordinate system on M^2_{p2} so that the metric takes the form

$$g^*(v_1, v_2)\, (dv_1{}^2 + dv_2{}^2),$$

where $g^*(v_1, v_2) \in C$, $g^*(v_1, v_2) > 0$. We define the metric on the universal covering surface S^2 of the projective plane by a projection process. The Riemannian manifold M^2_{S2} thus obtained is actually a Riemannian surface. According to the uniformization theorem, we can map M^2_{S2} onto the unit 2-sphere manifold M^2_{U2}, and can arrange it in such a way that two diametrically opposite points of U^2 correspond to the same point of M^2_{p2}. The metric has then the form

$$ds^2 = g(p)d\rho^2 . \qquad g(p) > 0,\ g(p) \in C_\omega \text{ for } p \in M^2_{U2} \, ,$$

where $d\rho^2$ is the line element of the unit sphere U^2 taken from the embedding 3-dimensional Euclidean space.

We remark that the area A of M^2_{p2} is one half that of M^2_{U2}.

Let us consider all the rotations σ of the unit sphere U^2 about its center. All these rotations form a compact Lie group G. Applying the process of averaging over a compact Lie group, in this case the Hurwitz integration [2; 3, p.188], we have

$$\int_G \, [(g(p))^{1/2}]^\sigma \, \delta\sigma = [h(p)]^{1/2} \, .$$

We shall show that $[h(p)]^{1/2}$ is invariant with respect to all the left translations

$$\tau: \sigma \to \sigma' = \tau\sigma, \quad \tau, \sigma, \sigma' \in G,$$

and hence is a constant. In fact, let τ be any element of G; then, by definition,

$$[(h(p))^{1/2}]^\tau = \int_G [(g(p))^{1/2}]^{\tau\sigma} \, \delta\sigma = \int_G [(g(p))^{1/2}]^{\tau\sigma} \, \delta\tau\sigma,$$

since $\delta\sigma$ is invariant under all left translations. Therefore,

$$[h(\tau p)]^{1/2} = \int_G [(g(p))^{1/2}]^\lambda \, d\lambda = [h(p)]^{1/2}, \qquad \lambda = \tau\sigma \in G.$$

As the group G is transitive, $h^{1/2}$ is a constant.

Using $h d\rho^2$ instead of $g d\rho^2$ as the metric on the unit sphere U^2, we obtain a manifold with the spherical geometry. Preserving the metric $h d\rho^2$, and identifying the diametrical points on U^2, we get a manifold $_h M^2_{p2}$, with the elliptic geometry. The two geometrical constants A_h and a_h can actually be evaluated:

$$A_h = 2\pi h,$$

$$a_h = \pi h^{1/2}.$$

Hence

(4)
$$A_h = \frac{2}{\pi} a_h^2.$$

It is clear that if $g(p)$ is subjected to a transformation σ of G, the resulting metric $g^\sigma(p) d\rho^2$ is such that

(5)
$$a_{g^\sigma} = a_g$$

and

(6)
$$A_{g^\sigma} = A_g.$$

By approximating integrals by suitable sums and using Lemma 1, we easily obtain

(7)
$$a_h \geq a_g$$

and

(8) $$A_h \leq A_g .$$

Combining (4), (7), and (8), we have

$$A_g \geq A_h = \frac{2}{\pi} a_h^2 \geq \frac{2}{\pi} a_g^2 .$$

Dropping the unnecessary indices, we obtain the inequality

$$A \geq \frac{2}{\pi} a^2 .$$

That $2/\pi$ is the best constant is evident, since we already have shown that the equality sign actually is attained when the metric is elliptic.

A slight generalization of Theorem 1, referring to certain special Riemannian metrics on the n-dimensional projective space P_n, can be proved in a similar fashion, using Hölder's inequality

$$(a_1 b_1 + \cdots + a_m b_m) \leq (a_1^p + \cdots + a_m^p)^{1/p} (b_1^q + \cdots + b_m^q)^{1/q} ,$$

where $a_i, b_i \geq 0$ and $p, q > 1$ such that $1/p + 1/q = 1$. The generalized theorem reads as follows:

THEOREM 2. *Let $M_{p^n}^n$ be the Riemannian manifold whose underlying topological space is an n-dimensional projective space P^n, which we suppose represented by the unit n-sphere U^n of the $(n+1)$-dimensional Euclidean space with identification of diametrically opposite points p and p_d, and whose metric can be represented in the form*

$$ds^2 = g(p) d\rho^2 ,$$

where $g(p) > 0$, $g(p) \in C_0$, $g(p) = g(p_d)$ for $p \in M_{p^n}^n$, and $d\rho^2$ is the line-element of the n-sphere, taken from the embedding Euclidean space; let V be its volume, and a the greatest lower bound of the lengths of all the closed curves which are not homotopic to zero on $M_{p^n}^n$; then

$$V \geq \frac{\pi^{\frac{1-n}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} a^n = k_n a^n .$$

Further, the above k_n is the best constant.

The proof of Theorem 2 is left to the reader.

3. Riemannian manifold M_m^2, whose underlying topological space is a Moebius strip. Let M_m^2 be the Riemannian manifold whose underlying topological space is a Moebius strip and whose metric is locally defined by

$$ds^2 = \sum_{i; k=1}^{2} g_{ik}(u_1, u_2)\, du^i du^k, \qquad\qquad g_{ik} \in C_0.$$

By Weierstrass' approximation theorem, it suffices to assume that $g_{ik} \in C_\omega$. After introduction of the isothermic coordinate system, the metric takes the form

$$ds^2 = g^*(u, v)(du^2 + dv^2),$$

where $g^*(u, v) \in C_\omega$ and $g^*(u, v) > 0$. We define the metric $d\bar{s}^2$ on the universal covering surface of the Moebius strip by a projection process: $ds^2 = d\bar{s}^2$; that is, the metric is invariant under the fundamental group of the Moebius strip. The covering manifold of M_m^2 thus obtained is actually a simply connected Riemann surface. According to the uniformization theorem, we can map it conformally onto a strip

$$S: \begin{cases} -\beta < y < \beta \\ -\infty < x < \infty \end{cases}$$

of the (x, y)–plane. The fundamental group Γ appears then in the form:

$$\sigma: \begin{cases} x' = x + n(2\alpha), \\ y' = (-1)^n y \end{cases} \qquad (n = 0, \pm1, \pm2, \cdots).$$

The given manifold M_m^2 is mapped isogonally onto the fundamental region

$$R: \begin{cases} -\alpha \leq x < \alpha \\ -\beta < y < \beta \end{cases}$$

with a metric of the form

$$ds^2 = g(x, y)(dx^2 + dy^2),$$

where $g(x,y) > 0$ and $g(x,y) \in C_\omega$. Moreover,

$$g(x + n(2\alpha),(-1)^n y) = g(x,y).$$

We are now in a position to prove the following theorem, which connects the two geometrical constants A and a in M_m^2.

THEOREM 3. *Let M_m^2 be the Riemannian manifold whose underlying topological space is a Moebius strip and whose metric is locally defined by*

$$ds^2 = \sum_{i,\,k=1}^{2} g_{ik}(u_1,u_2)\,du^i\,du^k, \qquad\qquad g_{ik} \in C_0 \ ;$$

let A be its area, and a the greatest lower bound of the lengths of all the closed curves which are not homotopic to zero on M_m^2; then we have

$$A \geq \frac{2}{\pi} \cdot \frac{e^{\beta\pi/a} - 1}{e^{\beta\pi/a} + 1} \cdot a^2 = k_{\alpha\beta}\,a^2$$

where 2α and 2β are the Euclidean lengths of the sides of the fundamental region R of the Moebius strip. Moreover, the above constant $k_{\alpha\beta}$ is best for a given ratio β/α.

Proof. Let us consider the continuous group

$$H: \begin{cases} x' = x + c \\[2mm] y' = y\,, \end{cases}$$

c being real (mod 4α); H consists of conformal transformations of the Moebius strip onto itself. It is evident that every two points which are equivalent under Γ remain equivalent under Γ after being operated on by elements of H. Defining the mean value, $[\bar{g}(y)]^{1/2}$, of $[g(x+c,y)]^{1/2}$ by the formula

$$\frac{1}{4\alpha} \int_0^{4\alpha} [g(x+c,y)]^{1/2}\,dc = [\bar{g}(y)]^{1/2},$$

we can prove, by a method similar to that in the former cases, that

(9)
$$\bar{A} \leq A\,,$$
$$\bar{a} \geq a\,,$$

where $\bar{A}$ and $\bar{a}$ have the same meaning as A and a except that we use the metric $\bar{g}(y)(dx^2 + dy^2)$ instead of the general one. Moreover, from the invariance of $g(x, y)$ under the group Γ, it follows immediately that

$$\bar{g}(y) = \bar{g}(-y).$$

We now shall consider the same problem with the simple metric

$$ds^2 = g(y)(dx^2 + dy^2),$$

where $g(-y) = g(y) > 0$. Noting (9), we see that if the best inequality is found for such a $g(y)$, it is also found for all $g(x, y)$, and hence our problem is solved.

We are now going to determine a special positive, even, and for nonnegative y monotonically decreasing function $g(y)$ such that a family F^* of closed geodesic lines through the origin and not homotopic to zero on M_m^2 can be defined in terms of it.

Let us first establish a differential equation for such $g(y)$. Putting

$$ds = [g(y)(1 + x'^2)]^{1/2} \, dy \qquad\qquad (x' = \frac{dx}{dy}),$$

we know that the equation for the extremals is

$$\frac{d}{dy} \left\{ x' \left[\frac{g(y)}{1 + x'^2} \right]^{1/2} \right\} = 0.$$

Solving this equation, we have

$$x = \int_0^y \frac{c \, d\eta}{[g(\eta) - c^2]^{1/2}} + k.$$

Since the geodesics under consideration have to go through the origin, the constant k has to be zero, and hence the equation becomes

$$x = \int_0^y \frac{c \, d\eta}{[g(\eta) - c^2]^{1/2}}.$$

The condition that the geodesics of the family be closed and not homotopic to zero requires that

$$\frac{dy}{dx} \bigg|_{(\alpha, \tau)} = 0$$

for $-\beta < \tau < \beta$; that is,

$$[g(\tau)]^{1/2} = c$$

for $-\beta < \tau < \beta$. Hence we have

$$(10) \qquad \alpha = \int_0^\tau \left[\frac{g(\tau)}{g(\eta) - g(\tau)} \right]^{1/2} d\eta, \qquad\qquad 0 \le \tau < \beta.$$

For simplicity, let us normalize $g(y)$ so that $g(0) = 1$. Since $g(y)$ is supposed to be monotonically decreasing for nonnegative y, we can put

$$1 - g(\eta) = t,$$

$$1 - g(\tau) = \omega,$$

and have

$$\omega - t = g(\eta) - g(\tau)$$

$$d\eta = - \frac{dt}{g'(\eta)} \; .$$

Then equation (10) takes the form

$$(11) \qquad \frac{-\alpha}{[1-\omega]^{1/2}} = \int_0^\omega \frac{1}{g'(\eta)} \cdot \frac{dt}{[\omega - t]^{1/2}} \; .$$

This is an Abel integral equation. According to the formula (cf. [1, p.484]) for the solution of such an equation,

$$f(x) = \int_0^x \frac{y(t)\,dt}{(x-t)^{1/2}} \, ,$$

we have

$$(12) \qquad y(t) = \frac{1}{\pi} \left[\frac{f(0)}{t^{1/2}} + \int_0^t \frac{f'(z)\,dz}{(t-z)^{1/2}} \right],$$

and, in our case,

$$\frac{1}{g'(\eta)} = - \frac{\alpha}{\pi} \left[\frac{1}{t^{1/2}} + \frac{1}{2} \int_0^t \frac{dz}{(1-z)^{3/2}(t-z)^{1/2}} \right]$$

$$= - \frac{\alpha}{\pi} \left[\frac{1}{t^{1/2}} + \frac{t^{1/2}}{1-t} \right] = - \frac{\alpha}{\pi} \cdot \frac{1}{(1-t) \, t^{1/2}} = - \frac{\alpha}{\pi} \cdot \frac{1}{g(\eta)[1-g(\eta)]^{1/2}}$$

Thus we have established the differential equation for $g(y)$,

$$(13) \qquad \frac{dg}{dy} = - \frac{\pi}{\alpha} \, g \, (1-g)^{1/2} .$$

The general solution of (13) is found to be

$$\log \frac{1 + (1-g)^{1/2}}{1 - (1-g)^{1/2}} = \frac{\pi}{\alpha} \, y + k .$$

When $y = 0$, we have $[1-g(0)]^{1/2} = 0$ and hence $k = 0$. Therefore,

$$\log \frac{1 + (1-g)^{1/2}}{1 - (1-g)^{1/2}} = \frac{\pi}{\alpha} \, y .$$

An explicit expression for $g(y)$ is as follows:

$$(14) \qquad g(y) = \frac{4 e^{\pi y / \alpha}}{(1 + e^{\pi y / \alpha})^2}$$

for $-\beta < y < \beta$.

From the explicit expression (14) for $g(y)$, every property of $g(y)$ we assumed at the beginning is verified. It is a positive, even, and monotonic decreasing function for $y \geq 0$. Moreover, $g(y) \to 0$ as $y \to \infty$ and $g'(0) = 0$. Such a $g(y)$, with those properties just mentioned and defining the family F^* of the closed geodesics through $(0, 0)$ and not homotopic to zero, is distinguished. Let us denote it by $g_0(y)$; that is, $g_0(y)$ is defined by either (13) or (14).

We are now in a position to establish the inequality in question for a positive, even, analytic function $g(y)$. By the definition of a, we have

$$2 \int_0^\tau [g(y) \cdot (1 + x'^2)]^{1/2} \, dy = \varphi(\tau) \geq a$$

for a closed curve, not homotopic to zero, on M_m^2. By taking this curve as one of F^*, we obtain

$$(1 + x'^2)^{1/2} = \left[\frac{g_0(y)}{g_0(y) - g_0(\tau)} \right]^{1/2} .$$

Hence,

$$(15) \qquad \int_0^\tau \left[\frac{g(y) g_0(y)}{g_0(y) - g_0(\tau)} \right]^{1/2} dy = \frac{\varphi(\tau)}{2} \geq \frac{a}{2} .$$

This equation can be put into a more suitable form by setting

$$1 - g_0(y) = t,$$

$$1 - g_0(\tau) = \omega.$$

Then we have

$$\omega - t = g_0(y) - g_0(\tau),$$

$$dy = - \frac{dt}{g_0'(y)} ,$$

$$\tau = g_0^{-1}(1 - \omega) .$$

Equation (15) then takes the form

$$(16) \qquad \int_0^\omega \left[\frac{g(y) g_0(y)}{\omega - t} \right]^{1/2} \frac{dt}{g_0'(y)} = \frac{\varphi}{2} [g_0^{-1}(1 - \omega)].$$

In formula (12), we have

$$f(z) = - \frac{\varphi}{2} [g_0^{-1}(1 - z)]$$

$$f(0) = - \frac{\varphi(0)}{2}$$

$$f'(z) = \frac{1}{2} \cdot \frac{\varphi'(g_0^{-1}(1 - z))}{g_0'(g_0^{-1}(1 - z))} .$$

Solving (16), we obtain

$$\frac{[g(y) g_0(y)]^{1/2}}{g_0'(y)} = \frac{1}{\pi} \left[- \frac{\varphi(0)}{2 t^{1/2}} + \frac{1}{2} \int_0^t \frac{\varphi'(g_0^{-1}(1 - z))}{g_0'(g_0^{-1}(1 - z))} \cdot \frac{dz}{(t - z)^{1/2}} \right].$$

Putting

$$g_0^{-1}(1 - z) = u,$$

$$g_0^{-1}(1 - t) = y,$$

we have

$$\frac{[g(y)\,g_0(y)]^{1/2}}{g_0'(y)} = - \frac{\varphi(0)}{2\pi\,[1 - g_0(y)]^{1/2}} - \frac{1}{2\pi} \int_0^y \frac{\varphi'(u)\,du}{[g_0(u) - g_0(y)]^{1/2}},$$

or

$$[g(y)\,g_0(y)]^{1/2} = - \frac{\varphi(0)\,g_0'(y)}{2\pi[1 - g_0(y)]^{1/2}} - \frac{1}{2\pi}\,g_0'(y) \int_0^y \frac{\varphi'(u)\,du}{[g_0(u) - g_0(y)]^{1/2}}.$$

Integrating from 0 to β, we get

$$\int_0^\beta [g(y)\,g_0(y)]^{1/2}\,dy = - \frac{\varphi(0)}{2\pi} \int_0^\beta \frac{g_0'(y)}{[1 - g_0(y)]^{1/2}}\,dy$$

$$- \frac{1}{2\pi} \int_0^\beta g_0'(y) \left\{ \int_0^y \frac{\varphi'(u)\,du}{[g_0(u) - g_0(y)]^{1/2}} \right\} dy$$

$$= \frac{\varphi(0)}{\pi} [1 - g_0(\beta)]^{1/2}$$

$$- \frac{1}{2\pi} \int_0^\beta \varphi'(u) \left\{ \int_u^\beta \frac{g'(y)\,dy}{[g_0(u) - g_0(y)]^{1/2}} \right\} du$$

$$= \frac{\varphi(0)}{\pi} [1 - g_0(\beta)]^{1/2}$$

$$+ \frac{1}{\pi} \int_0^\beta \varphi'(u)[g_0(u) - g_0(\beta)]^{1/2}\,du$$

$$= - \frac{1}{\pi} \int_0^\beta \varphi(u) \left\{ \frac{d}{du} [g_0(u) - g_0(\beta)]^{1/2} \right\} du$$

$$= \frac{1}{\pi} \int_0^\beta \varphi(u) \left\{ - \frac{d}{du} [g_0(u) - g_0(\beta)]^{1/2} \right\} du.$$

We remark that $-d[g_0(u) - g_0(\beta)]^{1/2}/du \geq 0$. Squaring and applying Schwarz's inequality, we have

$$\left\{ \int_0^\beta [g(y) \ g_0(y)]^{1/2} \ dy \right\}^2$$

$$= \frac{1}{\pi^2} \left\{ \int_0^\beta \varphi(u) \left(- \frac{d}{du} [g_0(u) - g_0(\beta)]^{1/2} \right) du \right\}^2,$$

$$\int_0^\beta g(y) \, dy \cdot \int_0^\beta g_0(y) \, dy \geq \frac{a^2}{\pi^2} [1 - g_0(\beta)].$$

The equality sign holds when $g = g_0$ due to a converse part of the theorem on Schwarz's inequality. Then we have

(17)
$$\int_0^\beta g(y) \, dy \geq \frac{a^2}{\pi^2} \cdot \frac{1 - g_0(\beta)}{\int_0^\beta g_0(y) \, dy} .$$

From (14), we can easily compute

(18)
$$1 - g_0(\beta) = \left(\frac{e^{\pi\beta/\alpha} - 1}{e^{\pi\beta/\alpha} + 1} \right)^2 = \frac{\pi^2}{4} k_{\alpha\beta}^2$$

(19)
$$\int_0^\beta g_0(y) \, dy = \frac{2\alpha}{\pi} \cdot \frac{e^{\pi\beta/\alpha} - 1}{e^{\pi\beta/\alpha} + 1} = k_{\alpha\beta} \alpha .$$

Combining (17), (18), and (19), we obtain

(20)
$$\int_0^\beta g(y) \, dy \geq k_{\alpha\beta} \frac{a^2}{4\alpha} .$$

We remark that the equality sign in (20) holds for $g = g_0$. In fact, in this case, it can easily be proved that $a = 2\alpha$ by using the Weierstrass theory of extremal fields. From (19), we have

$$(21) \qquad \int_0^\beta g_0(y)\, dy = \alpha\, k_{\alpha\beta} = k_{\alpha\beta} \frac{a^2}{4\alpha}.$$

Combining the definition of area and (20), we obtain

$$A = 4\alpha \int_0^\beta g(y)\, dy \geq k_{\alpha\beta}\, a^2.$$

By (21), we know that

$$A = k_{\alpha\beta}\, a^2$$

for $g = g_0$. This shows that our $k_{\alpha\beta}$ is the best constant.

3. **Added in proof.** In a course on Riemannian Geometry given at Syracuse University in 1949, Professor C. Loewner proved the inequality $A \geq 3^{1/2}\, a^2/_2$ for the case of M_t^2, the Riemannian manifold whose underlying topological space torus. The present investigation originates from this idea and has a similar method of treatment.

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