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1. **Introduction.** If X is a subset of a topological space X^*, and there exists a mapping (continuous function) $\rho: X^* \longrightarrow X$ such that $\rho(x) = x$ for $x \in X$, then X is called a *retract* of X^*, and ρ is called a *retraction.* If U is a neighborhood of X in X^* (that is, a set open in $X^*$and containing X), and there exists a retraction $\rho': U \longrightarrow X$, then X is called a *neighborhood retract* of X^*. If X is a separable metric space such that every homeomorphic image of X as a closed subset of a separable metric space M is a neighborhood retract of M, then X is called an *absolute neighborhood retract* or an *ANR*. It is with such spaces that we shall be principally concerned. They are of particular interest because of their usefulness in homotopy theory, and also because every locally-finite polyhedron is an *ANR* [5].

Section 2 is concerned with two theorems of point-set topological nature. For many other theorems along this line, see [3, §10].

In §3, we are concerned with spaces of mappings into *ANR*'s, and with homotopy problems involving *ANR*'s; in particular, we obtain certain restrictions on the cardinality of collections of homotopy classes of mappings into *ANR*'s. For closely related results, see [4], especially the corollary to Theorem 5.

2. **Theorems in point-set-topology.** The following theorem is a slight generalization of a standard result.

THEOREM 1. *If a subset X_0 of an ANR X is a neighborhood retract of X, then X_0 is an ANR.*

This generalization consists in not requiring that X_0 be closed. For the more restricted theorem and its proof (which in actuality establishes Theorem 1), see [3, §10.1].

An interesting corollary of Theorem 1 is that *an open subset of an ANR is an ANR.*

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We now prove a theorem which shows that the study of ANR's may—for many purposes—be reduced to the study of connected ANR's.

THEOREM 2. *A topological space X is an ANR if and only if X has at most countably many components, each of which is open in X, and each of which is itself an ANR.*

Proof of Theorem 2. Suppose X is an ANR. Then it is well known that X is locally connected, and hence has open components. Since X must be perfectly separable, this implies that there are at most countably many components; and that they are themselves ANR's follows from the corollary to Theorem 1. This proves the necessity.

Now suppose $X = \bigcup C_i$, where each C_i is an open component of X, and also an ANR, and suppose that X is a closed subset of a separable metric space M. For definiteness we assume that the set of C_i's is denumerably infinite; the proof must be simplified slightly if the union is finite.

Since C_1 and $(C_2 \cup C_3 \cup \cdots)$ are disjoint closed subsets of M, we can pick open subsets U_1 and U_1^* of M with $C_1 \subset U_1$ and $(C_2 \cup C_3 \cup \cdots) \subset U_1^*$, and with $U_1 \cap U_1^* = \phi$ (the null set). Then obviously $U_1 \cap (C_2 \cup C_3 \cup \cdots) = \phi$ (where $\overline{U}_1$ is the closure of U_1).

Having picked $U_1, U_2, \cdots, U_{n-1}$ such that each U_i is open, contains C_i, is disjoint from the other U_i, and has a closure disjoint from C_j for all $j \neq i$; we see that the sets

$$\bigcup_{1}^{n-1} \overline{U}_i \cup \bigcup_{n+1}^{\infty} C_i \equiv K_n$$

and C_n are disjoint closed sets. Hence we may choose disjoint open sets $U_n \supset C_n$ and $U_n^* \supset K_n$.

Then we easily see that by induction we may pick open sets U_i $(i = 1, 2, \cdots)$ so that for each i we have $C_i \subset U_i$, and so that for $i \neq j$ we have $U_i \cap U_j = \phi$.

Since each C_i is a closed ANR subset of M, we may pick open sets V_i $(i = 1, 2, \cdots)$ such that each V_i contains C_i, and retractions $\rho_i : V_i \longrightarrow C_i$, $i = 1, 2, \cdots$.

Define

$$\rho : \bigcup_{1}^{\infty} (U_i \cap V_i) \longrightarrow X$$

by setting $\rho(x) = \rho_i(x)$ for $x \in U_i \cap V_i$. It is clear that ρ retracts a neighborhood of X onto X, as required.

3. **Results in homotopy theory.** Our next theorem has a corollary concerning homotopy groups and the homotopy classification problem.

THEOREM 3 : *Let X be a compact topological space, and let Y be an ANR. Then the space Y^X of continuous functions on X into Y has open, arcwise-connected components.*

If $X_0 \subset X$, and $y_0 \in Y$, then the space $Y^X\{X_0, y_0\}$ of functions in Y^X which carry X_0 to y_0 also has open, arcwise-connected components.

(We give the function spaces the topology of uniform convergence, which for compact X's coincides with the more general compact-open topology.)

In the case that X is metric, a known theorem tells us that Y^X is itself an *ANR*, whence the first result follows easily. However, the second result is of perhaps greater interest, because of its bearing on homotopy groups (see below). We shall prove the second result; the proof of the first proceeds similarly, but is slightly easier.

Proof of Theorem 3. Let d be a metric on Y. The topology on our function space is obtained by taking

$$d^*(f, g) = \sup_{x \in X} d(f(x), g(x))$$

as a metric for functions f and g. We first show that *if $f \in Y^X(X_0, y_0)$ then there exists an $\epsilon > 0$ such that if $d^*(f, g) < \epsilon$, then g can be joined to f by an arc in $Y^X\{X_0, y_0\}$.*

By Wojdyslawski's imbedding theorem, we may take Y to be a closed subset of a convex subset Z of a Banach space S. Since Y is an *ANR*, there exists a retraction $\rho : U \longrightarrow Y$ of a neighborhood U of Y in Z onto Y. Since $f(X)$ is compact and disjoint from the closed subset $Z - U$ of Z, we may choose $\epsilon > 0$ such that the ϵ-neighborhood of $f(X)$ in Z is contained in U.

Now suppose $d^*(f, g) < \epsilon$. Define

$$H(x, t) = (1-t) f(x) + tg(x) \qquad ((t, x) \in X \times I).$$

It is clear that each $H(x, t) \in U$, so we may define

$$h(x, t) = \rho(H(x, t)) \qquad ((t, x) \in X \times I).$$

Plainly $h : X \times I \longrightarrow Y$ is continuous. Hence by a theorem of R.H. Fox [2], the function $\rho^* : I \longrightarrow Y^X$ defined by

$$\rho^*(t)(x) = h(x, t) \qquad (t \in I, x \in I).$$

is also continuous. One sees easily that $\rho^*(0) = f$, $\rho^*(1) = g$, and that the image of ρ^* is actually in $Y^X\{X_0, y_0\}$.

This proves the statement made at the beginning of the proof. This statement makes it obvious that the arc-components are open. One shows from this statement that the arc-components coincide with the components, in exactly the same way that one shows that a connected open subset of Euclidean space is arcwise connected.

A result of Fox [2] implies that if X is either locally compact and regular, or locally separable (that is, satisfies the first axiom of countability), and the function spaces are given the compact-open topology, then the arcs in Y^X and in $Y^X\{X_0, y_0\}$ are essentially identical with the homotopies of functions in these spaces. It follows from Theorem 3 and the equivalence of the topologies for a compact X that *if X is compact and either regular or locally separable, and Y is an ANR, then Y^X and $Y^X\{X_0, y_0\}$ have open components, and these components coincide with the homotopy classes of the function spaces.*

Arens [1] has given a theorem which - specialized to the cases with which we are dealing - states that if X has a basis of cardinal number a, and Y has a basis of cardinal number b, then Y^X (and hence $Y^X\{X_0, y_0\}$ as well) has a basis whose cardinal number does not exceed both a and b. Eliminating trivial cases by assuming that X does not have finite basis, we see that each of our function spaces has a basis of cardinal number not greater than a. This easily implies that the collections of open components of our function spaces cannot have cardinalities greater than a; with the conclusion obtained by use of Fox's theorem, this establishes the following result.

THEOREM 4: *Let the topological space X be compact, and either regular or locally separable. Suppose that X has a basis of (infinite) cardinal number a. Let Y be an ANR. Then the collection of homotopy classes of functions in Y^X, and also the collection of homotopy classes of functions in $Y^X\{X_0, y_0\}$, each has cardinality not greater than a.*

The following theorem is an immediate corollary to the second conclusion of of the above theorem.

THEOREM 5: *A homotopy group of an ANR is at most countable.*

The necessity for strong hypotheses in Theorems 3, 4, and 5 is indicated by examples we have found which show that (*i*) an arcwise-connected, locally-connected, and compact subset of the plane may have an uncountable fundamental group; (*ii*) the homotopy classes of the space of mappings of the 1-sphere into an arcwise-connected and locally-connected subset of 3-space need not coincide with the components of this function space; (*iii*) the homotopy classes of the space of mappings of a separable metric, arcwise-connected, locally-contractible, compact space into itself need not be open in this space of mappings.

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