Pacific Journal of Mathematics

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Vol. 2, No. 2 February 1952

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1. **Introduction.** Let A and B be $n \times n$ matrices whose elements lie in an infinite perfect[1] field F. Alfred Brauer [1] and W. V. Parker [2] have considered the question: "When do A and B have the same characteristic equation?" Their results have been sufficiency conditions with special forms of A and B. W. T. Reid [3] has considered a related problem.

The present paper is concerned with the following theorem that contains the results of Brauer and Parker as special cases.

THEOREM. *A necessary and sufficient condition for matrices A and B to have the same characteristic equation is that there exist a nonsingular matrix P (with elements in F) such that for $N = A - P^{-1}BP$:*

Every polynomial g in A and N, each term of which contains N at least once, is nilpotent.

We introduce a special canonical form in §2 and give the proof in §3.

2. **Canonical forms.** For any matrix A, there exists a nonsingular matrix P_1, with elements in F, such that

$$(2.1) \qquad P_1^{-1} A P_1 = A_1 \dotplus A_2 \dotplus \cdots \dotplus A_k ,$$

where the characteristic equation of A_i is $[p_i(x)]^{\alpha_i} = 0$, and $p_i(x)$ is an irreducible polynomial over F. Moreover, for each A_i we have the decomposition by the nonsingular matrix P_{2i} with elements in F:

$$(2.2) \qquad P_{2i}^{-1} A_i P_{2i} = A_{i1} \dotplus A_{i2} \dotplus \cdots \dotplus A_{ik_i} ,$$

in which each $A_{i\mu}$ is nonderogatory with characteristic equation $[p_i(x)]^{\alpha_{i\mu}} = 0$ and is of the form [4, p. 750]

[1] Every irreducible equation over F is separable.

Received June 21, 1951. The author gratefully acknowledges the encouragement and suggestions of Professor Lowell J. Paige of the University of California, Los Angeles, in the preparation of this paper.

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$$(2.3) \qquad A_{i\mu} \equiv \begin{pmatrix} C_i & I_i & 0 & \cdots & 0 \\ 0 & C_i & I_i & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & I_i \\ 0 & \cdots & & \cdots & C_i \end{pmatrix};$$

where C_i is the companion matrix of $p_i(x)$, and occurs $\alpha_{i\mu}$ times down the main diagonal; I_i is the identity matrix of order the degree of $p_i(x)$. Clearly

$$\sum_{\mu=1}^{k_i} \alpha_{i\mu} = \alpha_i.$$

Letting $P_2 = P_{21} \dotplus P_{22} \cdots \dotplus P_{2k}$ and $P = P_1 P_2$, we have a direct sum decomposition of A into matrices $A_{i\mu}$ of form (2.3). We shall indicate this by

$$(2.4) \qquad P^{-1} A P = \dotplus \sum_{i=1}^{k} \sum_{\mu=1}^{k_i} A_{i\mu}.$$

It should be pointed out that the existence of the canonical form (2.3) depends only on the perfectness of the field F.

3. **Proof of the theorem.** *Necessity.* Suppose A and B have the same characteristic equation

$$m(x) = \prod [p_i(x)]^{\alpha_i} = 0.$$

We may then find matrices P_a and P_b (see §2) such that

$$(3.1) \qquad P_a^{-1} A P_a = \dotplus \sum_{i=1}^{k} \sum_{\mu=1}^{k_i} A_{i\mu},$$

$$P_b^{-1} B P_b = \dotplus \sum_{i=1}^{k} \sum_{\mu=1}^{h_i} A_{i\mu}^{*};$$

where $A_{i\mu}$ and $A_{i\mu}^{*}$ (for the same subscript i) are of the form (2.3) and thus have the same blocks C_i on the main diagonal. Moreover $\dotplus \sum_{\mu=1}^{k_i} A_{i\mu}$ and $\dotplus \sum_{\mu=1}^{h_i} A_{i\mu}^{*}$ have the same order since A and B have the same characteristic equation.

Clearly $\dotplus \sum_{\mu=1}^{k_i} A_{i\mu}$ is contained in the algebra of all $\alpha_i \times \alpha_i$ matrices, with elements in the field $F(C_i)$, whose elements below the main diagonal are zero. Moreover,

$$N_i = \dot{+} \sum_{\mu=1}^{k_i} A_{i\mu} - \dot{+} \sum_{\mu=1}^{h_i} A_{i\mu}^*$$

is in the radical of this algebra since all elements on or below the main diagonal are zero. Thus $g(\dot{+} \sum_{\mu=1}^{k_i} A_{i\mu}, N_i)$, for g satisfying the conditions of the theorem, is a radical element and thus nilpotent. Hence, letting

$$N^1 = N_1 \dot{+} N_2 \dot{+} \cdots \dot{+} N_k = P_a^{-1} A P_a - P_b^{-1} B P_b,$$

we see that $g(P_a^{-1} A P_a, N^1)$ is nilpotent. Finally, letting

$$P = P_b P_a^{-1} \quad \text{and} \quad N = P_a N^1 P_a^{-1} = A - P^{-1} B P,$$

we have the result that

$$P_a\, g(P_a^{-1} A P_a, N^1) P_a^{-1} = g(A, N) \tag{3.2}$$

is nilpotent. This completes the proof of the necessity.

Sufficiency. Assume that a P exists such that every polynomial g, satisfying the conditions of the theorem, is nilpotent. Define

$$A_\theta = A - \theta N \qquad (N = A - P^{-1} B P),$$

$m_\theta(\lambda) = |\lambda I - A_\theta| \equiv \lambda^n + a_1(\theta)\lambda^{n-1} + \cdots + a_{n-1}(\theta)\lambda + a_n(\theta)$; where θ is an indeterminate and $a_i(\theta)$ $(i = 1, 2, \cdots, n)$ are polynomials in θ with coefficients in F.

Clearly, $m_0(\lambda) = 0$ and $m_1(\lambda) = 0$ are the characteristic equations of $A_0 \equiv A$ and $A_1 \equiv P^{-1} B P$, respectively.

If we now let θ assume values from F we have

$$m_0(A_\theta) = m_0(A) + h_\theta(A, N) = h_\theta(A, N);$$

moreover $h_\theta(A, N)$ contains N in each term and is nilpotent by hypothesis.

The characteristic roots of $m_0(A_\theta)$ are $m_0(\alpha_\theta^i)$ $(i = 1, \cdots, n)$, where the α_θ^i are the characteristic roots of A_θ. Since $m_0(A_\theta)$ is nilpotent we must have

$$m_0(\alpha_\theta^i) = 0 \qquad (i = 1, \cdots, n). \tag{3.3}$$

From (3.3) it is clearly seen that there can be only a finite number of different characteristic equations $m_\theta(\lambda) = 0$, since all the characteristic roots of A_θ are roots of $m_0(\lambda) = 0$. Since F is assumed to be infinite, this implies that $a_i(\theta)$ is a constant independent of θ. Thus $m_0(\lambda) \equiv m_1(\lambda)$, and the proof of the sufficiency is complete.

References

1. Alfred Brauer, *On the characteristic equations of certain matrices*, Bull. Amer. Math. Soc. 53 (1947), 605-607.

2. W. V. Parker, *On matrices whose characteristic equations are identical*, Proc. Amer. Math. Soc. **1** (1950), 464-466.

3. W. T. Reid, *A note on the characteristic polynomials of certain matrices*, Proc. Amer. Math. Soc. **1** (1950), 584-585.

4. John Williamson, *The idempotent and nilpotent elements of a matrix*, Amer. J. Math. 58 (1936), 747-758.

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Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES

Pacific Journal of Mathematics
Vol. 2, No. 2 February, 1952