

Pacific Journal of Mathematics

**NOTE ON FOURIER ANALYSIS. XXXI. CESÀRO
SUMMABILITY OF FOURIER SERIES**

SHIGEKI YANO

NOTE ON FOURIER ANALYSIS XXXI:
CESÀRO SUMMABILITY OF FOURIER SERIES

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1. **Introduction.** M. Jacob [1] proved that if the series

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) n^{2\alpha} \quad (0 < \alpha < 1)$$

converges then the trigonometrical series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is summable $(C, -\alpha)$ almost everywhere. This result is equivalent to the proposition that if the series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is the Fourier series of a square-integrable function, then the series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) n^{-\alpha}$ is summable $(C, -\alpha)$ almost everywhere.

Considering the analogue of this theorem for integrable functions, we shall prove here the following:

THEOREM 1.¹ *If*

$$(1) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \frac{A_0(x)}{2} + \sum_{n=1}^{\infty} A_n(x)$$

is the Fourier series of an integrable function $f(x)$, then the series

$$(2) \quad a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) n^{-\alpha} \quad (0 < \alpha < 1)$$

and its conjugate series

$$(3) \quad \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) n^{-\alpha}$$

¹Prof. A. Zygmund has pointed out to the author that Theorem 1 can be obtained directly from known results.

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are summable $(C, -\alpha)$ almost everywhere.

2. Lemmas. For the proof we need the following lemmas.

LEMMA 1. *If*

$$(4) \quad H_n^{(\alpha)}(x) = 1 + \sum_{n=1}^{\infty} k^{-\alpha} \cos kx, \quad J_n^{(\alpha)}(x) = \sum_{n=1}^{\infty} k^{-\alpha} \sin kx$$

$(0 < \alpha < 1),$

then

$$(5) \quad |H_n^{(\alpha)}(x)| \leq A_\alpha x^{\alpha-1}, \quad |J_n^{(\alpha)}(x)| \leq B_\alpha x^{\alpha-1} \quad (0 < x \leq \pi),$$

where $A_\alpha, B_\alpha, \dots$ are constants which depend only on α and which may be different in different instances.

The proof of this lemma concerning the cosine series is given by Salem and Zygmund [2], and the part concerning the sine series can be proved similarly.

LEMMA 2. *If* $0 < \alpha < 1$ and

$$A_0^{(-\alpha)} = 1, \quad A_n^{(-\alpha)} = \binom{n-\alpha}{n} = \frac{(-\alpha+1)(-\alpha+2)\cdots(-\alpha+n)}{n!} \quad (n \geq 1),$$

then

$$(6) \quad \left| \sum_{k=0}^n A_{n-k}^{(-\alpha)} e^{ikx} \right| \leq C_\alpha |1 - e^{ix}|^{\alpha-1} \quad (n = 0, 1, 2, \dots ;$$

$0 \leq |x| \leq \pi).$

Lemma 2 is a well-known result of M. Riesz. (Indeed, the constant on the right side of (6) can be replaced by 2; however, the inequality in the above form can easily be derived from Lemma 1.)

Let us denote the $(C, -\alpha)$ -means of the series (2) and (3) by $N_n^{(\alpha)}(f; x)$ and $\bar{N}_n^{(\alpha)}(f; x)$, respectively. Then, setting $k^{-\alpha} = 1$ for $k = 0$, we have

$$N_n^{(\alpha)}(f; x) = \frac{1}{A_n^{(-\alpha)}} \sum_{k=0}^n A_{n-k}^{(-\alpha)} k^{-\alpha} A_k(x)$$

$$\begin{aligned}
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \frac{1}{A_n^{(-\alpha)}} \sum_{k=0}^n A_{n-k}^{(-\alpha)} \cos kt \, dt \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) N_n^{(\alpha)}(t) \, dt,
 \end{aligned}$$

where

$$(7) \quad N_n^{(\alpha)}(t) = \frac{1}{A_n^{(-\alpha)}} \sum_{k=0}^n A_{n-k}^{(-\alpha)} k^{-\alpha} \cos kt.$$

Similarly

$$(8) \quad \bar{N}_n^{(\alpha)}(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \bar{N}_n^{(\alpha)}(t) \, dt,$$

where

$$(9) \quad \bar{N}_n^{(\alpha)}(t) = \frac{1}{A_n^{(-\alpha)}} \sum_{k=1}^n A_{n-k}^{(-\alpha)} k^{-\alpha} \sin kt.$$

LEMMA 3. For $0 < \alpha < 1$, we have

$$(10) \quad |N_n^{(\alpha)}(t)| \leq A_\alpha t^{\alpha-1}, \quad |\bar{N}_n^{(\alpha)}(t)| \leq B_\alpha t^{\alpha-1} \quad (n = 0, 1, 2, \dots; \quad 0 \leq t \leq \pi).$$

Proof. From (7), we have

$$\begin{aligned}
 N_n^{(\alpha)}(t) &= \frac{1}{A_n^{(-\alpha)}} \sum_{k=0}^{[n/2]} A_{n-k}^{(-\alpha)} k^{-\alpha} \cos kt + \frac{1}{A_n^{(-\alpha)}} \sum_{k=[n/2]+1}^n A_{n-k}^{(-\alpha)} k^{-\alpha} \cos kt \\
 (11) \quad &\equiv P_n + Q_n,
 \end{aligned}$$

say. Using Abel's transformation, we get

$$(12) \quad P_n = \frac{1}{A_n^{(-\alpha)}} \left\{ \sum_{k=0}^{[n/2]-1} A_{n-k}^{(-\alpha-1)} H_k^{(\alpha)}(t) + A_{n-[n/2]}^{(-\alpha)} H_{[n/2]}^{(\alpha)}(t) \right\}.$$

By Lemma 1 we obtain

$$\begin{aligned}
 (13) \quad |P_n| &\leq \frac{1}{A_n^{(-\alpha)}} \left\{ A_\alpha t^{\alpha-1} \sum_{k=0}^{[n/2]-1} |A_{n-k}^{(-\alpha-1)}| + A_\alpha t^{\alpha-1} A_{n-[n/2]}^{(-\alpha)} \right\} \\
 &\leq A_\alpha n^\alpha t^{\alpha-1} \left\{ \sum_{k=0}^n k^{-\alpha-1} + n^{-\alpha} \right\} \\
 &\leq A_\alpha t^{\alpha-1}.
 \end{aligned}$$

Now, using Abel's transformation again, we have

$$\begin{aligned}
 (14) \quad Q_n &= \frac{1}{A_n^{(-\alpha)}} \left\{ \sum_{k=[n/2]+1}^{n-1} \Delta k^{-\alpha} \sum_{j=[n/2]+1}^k A_{n-j}^{(-\alpha)} \cos jt \right. \\
 &\quad \left. + n^{-\alpha} \sum_{k=[n/2]+1}^n A_{n-k}^{(-\alpha)} \cos kt \right\},
 \end{aligned}$$

where

$$\Delta k^{-\alpha} = k^{-\alpha} - (k+1)^{-\alpha} = O(k^{-\alpha-1}).$$

Thus

$$\begin{aligned}
 (15) \quad |Q_n| &\leq \frac{A_\alpha}{A_n^{(-\alpha)}} \max_{[n/2]+1 \leq m \leq n} \left| \sum_{k=[n/2]+1}^m A_{n-k}^{(-\alpha)} \cos kt \right| \cdot \left\{ \sum_{k=[n/2]+1}^n k^{-\alpha-1} \right. \\
 &\quad \left. + n^{-\alpha} \right\} \\
 &\leq A_\alpha \max_{[n/2]+1 \leq m \leq n} \left| \sum_{k=[n/2]+1}^m A_{n-k}^{(-\alpha)} \cos kt \right|.
 \end{aligned}$$

Since

$$\sum_{k=0}^m A_{n-k}^{(-\alpha)} \cos kt = \Re \left\{ \sum_{k=0}^m A_{n-k}^{(-\alpha)} e^{-ikt} \right\} = \Re \left\{ e^{-int} \sum_{k=n-m}^n A_k^{(-\alpha)} e^{ikt} \right\},$$

we have, by Lemma 2,

$$(16) \quad \left| \sum_{k=0}^m A_{n-k}^{(-\alpha)} \cos kt \right| \leq A_\alpha |1 - e^{it}|^{\alpha-1} \leq C_\alpha t^{\alpha-1}.$$

Therefore, from (15) and (16),

$$(17) \quad |Q_n| \leq A_\alpha t^{\alpha-1}.$$

Combining (11), (13), and (17), we have the first inequality of (10). The second inequality can be proved similarly.

3. Proof of Theorem 1. We proceed now to the proof of Theorem 1. From (7), we have

$$|N_n^{(\alpha)}(f; x)| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+t)| |N_n^{(\alpha)}(t)| dt.$$

By virtue of Lemma 3, we get

$$|N_n^{(\alpha)}(f; x)| \leq A_\alpha \int_{-\pi}^{\pi} |f(x+t)| |t|^{\alpha-1} dt,$$

whence

$$(18) \quad \begin{aligned} \int_{-\pi}^{\pi} \sup_n |N_n^{(\alpha)}(f; x)| dx &\leq A_\alpha \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} |f(x+t)| |t|^{\alpha-1} dt \\ &\leq A_\alpha \int_{-\pi}^{\pi} |t|^{\alpha-1} dt \int_{-\pi}^{\pi} |f(x+t)| dx \\ &\leq A_\alpha \int_{-\pi}^{\pi} |f(x)| dx. \end{aligned}$$

Similarly we have

$$(19) \quad \int_{-\pi}^{\pi} \sup_n |\bar{N}_n^{(\alpha)}(f; x)| dx \leq B_\alpha \int_{-\pi}^{\pi} |f(x)| dx.$$

From these maximal inequalities we can easily deduce the conclusions of Theorem 1. For example, we can proceed as follows. Let $\epsilon > 0$ be arbitrary, and let $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is a trigonometrical polynomial and $\int_{-\pi}^{\pi} |f_2(x)| dx < \epsilon/2A_\alpha$, A_α being the constant which appears in the right

side of (18). By $f^*(x)$ we denote the function defined by $\sum_{n=0}^{\infty} A_n(x)n^{-\alpha}$ (this series converges almost everywhere); then

$$(20) \quad f^*(x) = f_1^*(x) + f_2^*(x),$$

where $f_1^*(x)$ and $f_2^*(x)$ are determined by $f_1(x)$ and $f_2(x)$, respectively, in the same way as $f^*(x)$ is determined by $f(x)$.

From (18) we have

$$(21) \quad \int_{-\pi}^{\pi} \sup |N_n^{(\alpha)}(f_2; x)| dx \leq A_\alpha \int_{-\pi}^{\pi} |f_2(x)| dx < \epsilon/2,$$

and a fortiori,

$$(22) \quad \int_{-\pi}^{\pi} |f_2^*(x)| dx < \epsilon/2.$$

From

$$(23) \quad N_n^{(\alpha)}(f; x) - f^*(x) = N_n^{(\alpha)}(f_1; x) - f_1^*(x) + N_n^{(\alpha)}(f_2; x) - f_2^*(x)$$

we get

$$(24) \quad \limsup_{n \rightarrow \infty} |N_n^{(\alpha)}(f; x) - f^*(x)| \leq \sup_n |N_n^{(\alpha)}(f_2; x)| + |f_2^*(x)|,$$

whence, by (21) and (22),

$$(25) \quad \int_{-\pi}^{\pi} \limsup_{n \rightarrow \infty} |N_n^{(\alpha)}(f; x) - f^*(x)| dx < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore it follows that $N_n^{(\alpha)}(f, x) \rightarrow f^*(x)$ almost everywhere; this implies the validity of the first part of Theorem 1. The proof for conjugate series is analogous.

4. On multiple Fourier series. Theorem 1 can be extended to multiple Fourier series. For simplicity we shall state the result for the case of double Fourier series.

Let $f(x, y)$ be an integrable function periodic with period 2π in each variable, and let its Fourier series be

$$(26) \quad f(x, y) \sim \sum_{m, n=0}^{\infty} A_{mn}(x, y),$$

where

$$\begin{aligned}
 A_{00}(x, y) &= \frac{1}{4} a_{00}, \quad A_{0n}(x, y) = \frac{1}{2} (a_{0n} \cos ny + b_{0n} \sin ny), \\
 A_{m0}(x, y) &= \frac{1}{2} (a_{m0} \cos mx + b_{m0} \sin mx), \\
 (27) \quad A_{mn}(x, y) &= a_{mn} \cos mx \cos ny + b_{mn} \cos mx \sin ny \\
 &\quad + c_{mn} \sin mx \cos ny + d_{mn} \sin mx \sin ny,
 \end{aligned}$$

and

$$(28) \quad a_{mn} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) \cos mx \cos ny \, dx \, dy \quad (m, n = 0, 1, 2, \dots),$$

and similarly for b_{mn} , c_{mn} and d_{mn} .

We shall say that the double series $\sum_{m,n=0}^{\infty} A_{mn}$ is summable (C, α, β) if the (C, α, β) -means

$$\frac{1}{A_m^{(\alpha)} A_n^{(\beta)}} \sum_{j=0}^m \sum_{k=0}^n A_{m-j}^{(\alpha)} A_{n-k}^{(\beta)} A_{jk}$$

of $\sum A_{mn}$ converge.

Under these definitions we have:

THEOREM 2. *If $f(x, y)$ is integrable, then the series*

$$(29) \quad \sum_{m,n=0}^{\infty} A_{mn}(x, y) m^{-\alpha} n^{-\beta} \quad (0 < \alpha < 1, 0 < \beta < 1)$$

is summable $(C, -\alpha, -\beta)$ almost everywhere. A similar result holds for its conjugate series.

Proof. The $(C, -\alpha, -\beta)$ -means $N_{mn}^{(\alpha, \beta)}(f; x, y)$ of the series (29) can be written as follows:

$$(30) \quad N_{mn}^{(\alpha, \beta)}(f; x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+u, y+v) N_m^{(\alpha)}(u) N_n^{(\beta)}(v) \, du \, dv,$$

where $N_n^{(\alpha)}(t)$ is the same as in (7). Following the line of proof of Theorem 1, we obtain the result.

5. On the capacity of sets. Generalizing a result of A. Beurling, Salem and Zygmund [2] proved the following theorem:

If the series

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) n^{-\alpha} \quad (0 < \alpha < 1)$$

converges, then the trigonometrical series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is convergent except possibly on a set of $(1 - \alpha)$ -capacity zero.

We shall here prove the L_1 - analogue of this theorem.

THEOREM 3. *Under the same assumption as in Theorem 1, the set E of the points where the series (2) is not summable $(C, -\alpha)$ is of $(1 - \alpha)$ - capacity zero. A similar result hold for the conjugate series (3).*

REMARKS. (a) Theorem 1 is, of course, implied by Theorem 2. (b) For the notion of capacity and other definitions, the reader is referred to the paper [2] of Salem and Zygmund.

To prove Theorem 2 we need the following lemmas.

LEMMA 4. *If*

$$H^{(\alpha)}(x) = \sum_{n=0}^{\infty} n^{-\alpha} \cos nx, \quad J^{(\alpha)}(x) = \sum_{n=1}^{\infty} n^{-\alpha} \sin nx \quad (0 < \alpha < 1),$$

then

$$H^{(\alpha)}(x) \sim x^{\alpha-1}, \quad J^{(\alpha)}(x) \sim x^{\alpha-1} \quad (x \rightarrow +0).$$

This is known (see, for example, [3; p. 116]).

LEMMA 5. *If a set E is of positive α -capacity ($0 < \alpha < 1$), then there exists a positive distribution μ concentrated on E such that if the Fourier-Stieltjes series of $\mu(x)$ is denoted by*

$$d\mu(x) \sim \frac{1}{2\pi} + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx),$$

then the series

$$\sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx) n^{-1+\alpha}$$

and its conjugate series

$$\sum_{n=1}^{\infty} (\alpha_n \sin nx - \beta_n \cos nx) n^{-1+\alpha}$$

are the Fourier series of bounded functions.

Lemma 5 is due to Salem and Zygmund [2].

To prove Theorem 3, let us assume that the set E is of positive $(1 - \alpha)$ -capacity. Then by Lemma 5 we can find a positive distribution μ concentrated on E such that if

$$d\mu(x) \sim \frac{1}{2\pi} + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx),$$

then the series $\sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx) n^{-\alpha}$ is the Fourier series of a bounded function.

Since it is easy to verify by Lemmas 1 and 4 that

$$(31) \quad |N_n^{(\alpha)}(t)| \leq A t^{\alpha-1} \leq A_\alpha H^{(\alpha)}(t) + B_\alpha,$$

we have, by (17),

$$(32) \quad \begin{aligned} |N_n^{(\alpha)}(f; x)| &\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)| |N_n^{(\alpha)}(x-t)| dt \\ &\leq A_\alpha \int_{-\pi}^{\pi} |f(t)| H^{(\alpha)}(x-t) dt + B_\alpha \int_{-\pi}^{\pi} |f(t)| dt. \end{aligned}$$

Integrating both sides of (22), we have

$$\begin{aligned}
 & \int_0^{2\pi} \sup_n |N_n^{(\alpha)}(f; x)| d\mu(x) \\
 (33) \quad & \leq A_\alpha \int_0^{2\pi} d\mu(x) \int_{-\pi}^{\pi} |f(t)| H^{(\alpha)}(x-t) dt + B_\alpha \int_{-\pi}^{\pi} |f(t)| dt \\
 & \leq A_\alpha \int_{-\pi}^{\pi} |f(t)| dt \int_0^{2\pi} H^{(\alpha)}(x-t) d\mu(x) + B_\alpha \int_{-\pi}^{\pi} |f(t)| dt.
 \end{aligned}$$

By a well-known theorem, the Fourier series of

$$\frac{1}{\pi} \int_0^{2\pi} H^{(\alpha)}(x-t) d\mu(x)$$

is

$$1 + \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt) n^{-\alpha},$$

which is the Fourier series of a bounded function. Hence for almost all t we have

$$\left| \int_0^{2\pi} H^{(\alpha)}(x-t) d\mu(x) \right| \leq M(\mu),$$

$M(\mu)$ being a constant depending only on the distribution μ . Therefore, from (33), we have

$$\begin{aligned}
 & \int_0^{2\pi} \sup_n |N_n^{(\alpha)}(f; x)| d\mu(x) \leq A_\alpha M(\mu) \int_{-\pi}^{\pi} |f(t)| dt + B_\alpha \int_{-\pi}^{\pi} |f(t)| dt \\
 (34) \quad & \leq C_\alpha(\mu) \int_{-\pi}^{\pi} |f(x)| dx,
 \end{aligned}$$

where $C_\alpha(\mu)$ is a constant depending only on α and on the distribution μ .

Using this maximal inequality (34) and following the proof of Theorem 1, we see² that

²We must first prove that the set of points where the series $\sum_{n=0}^{\infty} A_n(x) n^{-\alpha}$ does not converge is of μ -measure zero. This can be done by the maximal inequality

$$\int_0^{2\pi} \sup_n \left| \sum_{k=0}^n A_k(x) k^{-\alpha} \right| d\mu(x) \leq A(\mu) \int_{-\pi}^{\pi} |f(x)| dx,$$

which is a simple consequence of (34), by the same argument as in [3, p. 254].

$$\int_0^{2\pi} \limsup_{n \rightarrow \infty} |N_n^{(\alpha)}(f, x) - f^*(x)| d\mu(x) = 0,$$

which implies that E is of μ -measure zero, contrary to the hypothesis that μ is concentrated on E , ($\int_E d\mu(x) = 1$).

For the conjugate series (3) an analogous argument can be applied, and Theorem 3 is proved completely.

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