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**TRANSLATION INVARIANT MEASURE OVER SEPARABLE  
HILBERT SPACE AND OTHER TRANSLATION SPACES**

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# TRANSLATION INVARIANT MEASURE OVER SEPARABLE HILBERT SPACE AND OTHER TRANSLATION SPACES

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1. **Introduction.** We consider the problem of defining a nontrivial, translation-invariant Borel measure over real separable Hilbert space. As noted by Loewner [4], this is not possible; but instead of relinquishing as he does the real number system for a non-Archimedean ordered field for the values of a “measure,” we shall consider several topological subspaces of Hilbert space arising frequently in analysis. These are locally compact; and using either the Kolmogoroff stochastic processes construction [2], or else following the Haar measure construction [1] or [5], we can get a nontrivial, essentially translation-invariant Borel measure. However, since the special subspaces considered are not groups under translation, and do not even contain a group germ, the usual Haar measure construction must be modified in a special fashion, and the precise translation invariance obtained is somewhat restrictive. Actually we carry through this modified Haar measure construction for the more general situation of a locally compact translation space, which is defined as an appropriate subspace of an Abelian topological group. The results are collected in a summary at the end.

2. **Formulation of the problem.** Let

$$\ell_2 = \left\{ x = \{x_n\} \mid \sum_{n=1}^{\infty} (x_n)^2 < +\infty, x_n \text{ real} \right\},$$

the square summable real sequences and thus the real separable Hilbert space prototype. Since  $\ell_2$  is a subset of  $R_\infty$ , the countably infinite Cartesian product of the real line  $(-\infty, \infty)$ , we have available on  $\ell_2$  as well as the  $\ell_2$  norm metric topology also the product topology defined relatively from  $R_\infty$ . Under these two topologies we shall consider the  $\ell_2$ -subsets

$$X = \{x \in \ell_2 \mid |x_n| \leq h(n) \text{ for all } n\},$$

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$$Y = \{x \in \ell_2 \mid \sum_{j=n}^{\infty} |x_j|^2 \leq f(n) \text{ for all } n\},$$

where  $f(n)$  and  $h(n)$  are specified functions defined over the integers  $n \geq 1$  with values real or  $+\infty$  having  $h(n) > 0$  and  $f(n) \geq f(n+1) > 0$ .

Let  $Z = X$  or  $Y$ ; we want to define the Borel class of subsets of  $Z$ . The open intervals of  $Z$  are defined relatively from the elementary open intervals of  $R_\infty$ , and so we can define  $\mathfrak{B}_1$  as the  $\sigma$ -algebra of subsets of  $Z$  generated by the open intervals,  $\mathfrak{B}_2$  as that generated by the product-topology open sets,  $\mathfrak{B}_3$  by the metric spheres, and  $\mathfrak{B}_4$  by the metricly open sets. Actually  $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}_3 = \mathfrak{B}_4$ , and will be denoted by  $\mathfrak{B}$  and called the class of Borel subsets of  $Z$ . To see this we note first by using the rationals that  $R_\infty$  and hence  $Z$  has a countable basis of open intervals, so  $\mathfrak{B}_1 = \mathfrak{B}_2$ . Similarly  $\mathfrak{B}_3 = \mathfrak{B}_4$ , since  $\ell_2$  and hence  $Z$  is a separable metric space and thus has a countable basis of spheres. Since any product-topology open set is clearly open metricly,  $\mathfrak{B}_2 \subseteq \mathfrak{B}_4$ . Now it is easy to see that any closed sphere

$$S = \{x \in Z \mid \|x - y\| \leq \rho\}$$

is actually closed in the product topology. Since any open sphere is a countable union of closed ones,  $\mathfrak{B}_3 \subseteq \mathfrak{B}_2$ . Thus  $\mathfrak{B}_3 = \mathfrak{B}_4$  makes  $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}_3 = \mathfrak{B}_4$ , as desired.

Define

$$[A + u] = \{x \in R_\infty \mid (x - u) \in A\}$$

for  $u \in R_\infty$  and for any subset  $A$  of  $R_\infty$ . We note that  $u \in Z$  and  $A \subseteq Z$  do not always make  $[A + u] \subseteq Z$  if  $Z \neq \ell_2$ . However, if  $A \in \mathfrak{B}$  and  $u \in R_\infty$  then  $[A + u] \cap Z \in \mathfrak{B}$ . For

$$\mathfrak{C} = \{A \mid [A + u] \cap Z \in \mathfrak{B}\}$$

is easily seen to be a  $\sigma$ -algebra containing the intervals of  $Z$ , so  $\mathfrak{B} = \mathfrak{B}_1 \subseteq \mathfrak{C}$ , which gives the result.

Our problem is to find a Borel measure  $\phi$ , that is, a nonnegative extended real set function defined and countably additive over  $\mathfrak{B}$ , which is nontrivial (Condition I) and translation-invariant (Condition II or II') according to a specified topology.

CONDITION I.  $\phi(Z) > 0$  and  $\phi(V) < +\infty$  for some nonempty  $V$  open in the specified topology;

CONDITION II.  $\phi([A + u]) = \phi(A)$  if  $A \in \mathfrak{B}$ ,  $u \in \ell_2$ , and  $[A + u] \subseteq Z$ ;

CONDITION II'. a)  $\phi([A + u]) = \phi(A)$  if  $A \in \mathfrak{B}$ ,  $u \in \ell_2$ ,  $A \subseteq V$ , where  $V$  and  $[V + u]$  are both open subsets of  $Z$ .

b)  $\phi([A + u] \cap Z) \leq \phi(A)$  if  $u \in \ell_2$  and both  $A$  and  $[A + u] \cap Z$  are open subsets of  $Z$ .

Condition II clearly implies II', and hence is a stronger requirement.

**3. Negative results.** We shall start with a few preliminary lemmas. First define

$$S(Z, x, \rho) = \{y \in Z \mid \|x - y\| < \rho\},$$

the  $\rho$ -radius open  $Z$ -sphere about  $x$ .

LEMMA 1. *For any real  $r > 0$  there exists no nonnegative, finitely additive set function  $\phi$  over the Borel subsets of*

$$Z = Y = \overline{S(\ell_2, 0, r)},$$

*satisfying II', (or thus II also), under the metric topology such that*

$$0 < \phi(S(\ell_2, 0, \rho)) < +\infty \quad \text{for } 0 < \rho \leq r.$$

*Proof.* Let

$${}_p x = \{{}_p x_j\} \in S(\ell_2, 0, r)$$

by defining  ${}_p x_j = 0$  if  $j \neq p$  and  ${}_p x_p = r/2$  for integer  $p \geq 1$ . Let

$$V_p = S\left(\ell_2, {}_p x, \frac{1}{4}r\right),$$

so that  $V_p \subseteq S(\ell_2, 0, r)$ ; and  $V_p \cap V_q = \emptyset$  for  $p \neq q$  follows from

$$\|y - y'\| \geq \|{}_p x - {}_q x\| - 2\left(\frac{1}{4}r\right) = \frac{\sqrt{2}-1}{2}r > 0$$

for  $y \in V_p$  and  $y' \in V_q$ . But II' under the metric topology makes

$$\phi(V_p) = \phi\left(S\left(\ell_2, 0, \frac{1}{4}r\right)\right) = b$$

with  $0 < b < +\infty$ . Thus

$$S(\mathcal{L}_2, 0, r) \supset \bigcup_{p=1}^N V_p,$$

and finite additivity of  $\phi$  yields the contradiction

$$0 < Nb = \sum_{p=1}^N \phi(V_p) \leq \phi(S(\mathcal{L}_2, 0, r)) < +\infty$$

for arbitrary integer  $N$ . Thus such  $\phi$  cannot exist.

LEMMA 2. *If*

$$0 < \inf_{n \geq 1} h(n) \text{ for } Z = X,$$

*or if*

$$0 < \inf_{n \geq 1} f(n) \text{ for } Z = Y,$$

*then for any  $x \in Z$  and  $\rho > 0$  there exists some  $z \in Z$  and  $\rho' > 0$  such that  $S(\mathcal{L}_2, z, \rho') \subseteq S(Z, x, \rho)$ .*

*Proof.* For the given  $x \in Z$  choose some  $N \geq 1$  so that

$$\sum_{j=N+1}^{\infty} (x_j)^2 \leq \left(\frac{1}{3}\rho\right)^2,$$

possible since  $x \in \mathcal{L}_2$ . Define

$$y' = (y_1, \dots, y_N) = P(y) \in E_N$$

as the projection of  $\mathcal{L}_2$  onto Euclidean  $N$  space  $E_N$ . Clearly  $P(Z)$  is a convex set with a nonvoid interior in  $E_N$  including the origin; so we can find an interior point  $z'$  on the line-segment from  $x' = P(x)$  to the origin so that

$$\sum_{n=1}^N (z_n - x_n)^2 < \left(\frac{1}{3}\rho\right)^2.$$

Define  $z \in \mathcal{L}_2$  so that  $z' = P(z)$  by taking  $z_n = 0$  for  $n \geq N+1$ . Thus

$$||x - z|| = \left[ \sum_{n=1}^N (z_n - x_n)^2 + \sum_{n=N+1}^{\infty} x_n^2 \right]^{1/2} < \frac{\sqrt{2}}{3} \rho.$$

Let

$$b_0 = \inf_{n \geq 1} h(n) > 0 \text{ for } Z = X,$$

or

$$b_0 = \left[ \inf_{n \geq 1} f(n) \right]^{1/2} > 0 \text{ for } Y.$$

Now if  $Z = X$ , by choosing  $\rho'' > 0$  so that  $\rho'' < b_0$  and

$$S(E_N, z', \rho'') \subseteq P(Z),$$

as we may since  $z' \in \text{int } P(Z)$ , we get

$$S(\mathcal{L}_2, z, \rho'') \subseteq Z.$$

If  $Z = Y$ , then  $z' \in \text{int } P(Z)$  makes

$$\sum_{j=n}^N (z_j)^2 < f(n)$$

for  $1 \leq n \leq N$ , so here we choose  $0 < \rho'' < b_0$  and

$$\rho'' < \min_{1 \leq n \leq N} \left( [f(n)]^{1/2} - \left[ \sum_{j=n}^N (z_j)^2 \right]^{1/2} \right).$$

Thus

$$\left[ \sum_{j=n}^{\infty} (y_j)^2 \right]^{1/2} \leq ||y - z|| + \left[ \sum_{j=n}^N (z_j)^2 \right]^{1/2} < f(n) \text{ for } 1 \leq n \leq N,$$

and

$$\left[ \sum_{j=n}^{\infty} (y_j)^2 \right]^{1/2} \leq ||y - z|| < b_0 \leq f(n) \text{ for } n \geq N + 1,$$

makes  $S(\ell_2, z, \rho'') \subseteq Y = Z$ .

Thus

$$\rho' = \min \left( \rho'', \frac{3 - \sqrt{2}}{3} \rho \right) > 0$$

yields

$$S(\ell_2, z, \rho') \subseteq Z \cap S(\ell_2, x, \rho) = S(Z, x, \rho)$$

as desired, since

$$\|y - z\| < \frac{3 - \sqrt{2}}{3} \rho$$

makes

$$\|x - y\| \leq \|y - z\| + \|x - z\| < \rho$$

because  $\|x - z\| < (\sqrt{2}/3)\rho$ .

THEOREM 3. *If*

$$0 < \liminf_{n \rightarrow \infty} h(n) \text{ with } Z = X,$$

*or if*

$$0 < \liminf_{n \rightarrow \infty} f(n) \text{ with } Z = Y,$$

*then there exists no Borel measure  $\phi$  on such  $Z$  which is nontrivial (I) and translation-invariant (II') under the norm-metric topology.*

*Proof.* Set

$$b_0 = \inf_{n \geq 1} h(n) \text{ if } Z = X,$$

or

$$b_0 = \left[ \inf_{n \geq 1} f(n) \right]^{1/2} \text{ if } Z = Y;$$

thus clearly  $b_0 > 0$  is required by hypothesis. Obviously

$$S(Z, 0, \rho) = S(\mathcal{L}_2, 0, \rho)$$

for  $0 < \rho \leq b_0$ , so the metricly open set

$$S(Z, x, \rho) = [S(\mathcal{L}_2, 0, \rho) + x] \cap Z = [S(Z, 0, \rho) + x] \cap Z$$

for such  $\rho$ . Hence if  $\phi$  exists, then  $\phi(S(Z, x, \rho)) \leq \phi(S(Z, 0, \rho))$  by Condition II' b) for  $x \in \mathcal{L}_2$ ,  $0 < \rho \leq b_0$ .

Now set

$$b_1 = \inf \{ \text{all } \rho > 0 \text{ such that } \phi(S(Z, 0, \rho)) > 0 \},$$

so  $\phi(S(Z, 0, \rho)) > 0$  for  $\rho > b_1$ , and  $= 0$  for  $0 < \rho < b_1$  if  $b_1 > 0$ . Actually  $b_1 = 0$ . For if not set  $\delta = (\min b_0, b_1/2)$ ; then  $Z$ , being separable, is a countable union of spheres of radius  $\rho \leq \delta$ . But such spheres have

$$\phi(S(Z, x, \rho)) \leq \phi(S(Z, 0, \rho)) = 0,$$

implying  $\phi(Z) = 0$  by countable additivity, which contradicts Condition I. Thus  $b_1 = 0$  and  $\phi(S(Z, 0, \rho)) > 0$  for all  $\rho > 0$ .

We want to show that  $\phi(S(Z, 0, r)) < +\infty$  for some  $r > 0$ . By Condition I under the metric topology and Lemma 2 it is clear that there exists some  $r > 0$  and  $z \in Z$  such that

$$S(\mathcal{L}_2, z, r) \subseteq Z \text{ and } \phi(S(\mathcal{L}_2, z, r)) < +\infty.$$

Since  $S(\mathcal{L}_2, z, r) \subseteq Z$ , it is easily seen for either  $X = Z$  or  $Y = Z$  that we must have  $r \leq b_0$ , and hence

$$Z \supseteq S(\mathcal{L}_2, 0, r) = S(Z, 0, r).$$

Thus  $[S(Z, 0, r) + z] = S(\mathcal{L}_2, z, r)$ , an open subset of  $Z$ , so Condition II' a) makes

$$\phi(S(Z, 0, r)) = \phi(S(\mathcal{L}_2, z, r)) < +\infty.$$

Thus

$$0 < \phi(S(Z, 0, \rho)) < +\infty$$

with  $S(Z, 0, \rho) = S(\mathcal{L}_2, 0, \rho)$  for  $0 < \rho \leq r$  for some  $r$ ,  $0 < r < b_0$ , which is impossible by Lemma 1. Thus the stated  $\phi$  cannot exist.



We also easily get the following considerably weaker result for the product topology.

**THEOREM 4.** *If  $\{n \mid h(n) = +\infty\}$  is an infinite set, then there exists no Borel measure  $\phi$  on  $X$  which is nontrivial (I) and translation-invariant (II') under the product topology.*

*Proof.* Let  $V$  be any nonempty open interval of  $X$ . It is clear that by translating along each of the finite set of coordinates given in the definition of the interval  $V$ , we can find a finite or countable set of  ${}_p x \in \ell_2$  such that

$$[V + {}_p x] \subseteq X \text{ and } X = \bigcup_{p=1} [V + {}_p x].$$

Also Condition II'a) makes  $\phi(V + {}_p x) = \phi(V)$  if  $\phi$  exists. Thus  $\phi(X) > 0$  for nontriviality yields by countable additivity  $\phi(V) > 0$  for any open interval  $V \neq \emptyset$ .

Now Condition I under the product topology implies that some open interval  $V_0 \neq \emptyset$  has  $\phi(V_0) < +\infty$ , so  $0 < \phi(V_0) < +\infty$ . Since  $V_0$  is defined in terms of only a finite number of coordinates, and  $\{n \mid h(n) = +\infty\}$  is infinite, there must exist some  $p$  so that  $x \in V_0$  imposes no restriction on the  $p$ th coordinate of  $x$ . Let

$$W_0 = \{y \in V_0 \mid |y_p| < 1\},$$

a nonvoid open  $X$  interval, so  $\phi(W_0) > 0$ . Let  ${}_0 z_j = 0$  if  $j \neq p$ ,  ${}_0 z_p = 1$ , so clearly  $\{[W_0 + m {}_0 z]\}$  form a disjoint union of sets  $\subseteq V_0$  for different integer  $m$ , with

$$\phi([W_0 + m {}_0 z]) = \phi(W_0)$$

by Condition II'a). Thus

$$+\infty = \sum_{m=1}^{\infty} \phi(W_0) = \phi\left(\bigcup_{m=1}^{\infty} [W_0 + m {}_0 z]\right) \leq \phi(V_0) < +\infty,$$

which is a contradiction. Thus  $\phi$  cannot exist.

We remark that  $\ell_2 = X$  by taking  $h(n) \equiv +\infty$ , so Theorems 3 and 4 show that there exists no Borel measure  $\phi$  on  $\ell_2$  which is nontrivial and translation-invariant under either the norm metric or product topologies.

**4. Positive results via Kolmogoroff.** We want to give conditions under which an invariant measure does exist on  $X$  or  $Y$ , getting a converse of Theorem 3. For  $X$  we shall use the construction of Kolmogoroff [2, p.27] of a probability measure  $P$  over real product spaces, in our case  $R_\infty$ . Here we need a family  $Q$  of real set functions, each member  $Q_{n_1, \dots, n_k}$  being nonnegative and countably additive over the intervals of  $E_k$ , with coordinates indexed  $n_1, \dots, n_k$ , and having  $Q_{n_1, \dots, n_k}(E_k) = 1$ . The family  $Q$  is assumed to satisfy Kolmogoroff's two consistency conditions:

$$Q_{n_1, \dots, n_k}(-\infty, +\infty; a_2, b_2; \dots; a_k, b_k) = Q_{n_2, \dots, n_k}(a_2, b_2; \dots; a_k, b_k),$$

$$Q_{n_1, \dots, n_k}(a_1, b_1; \dots; a_k, b_k) = Q_{n'_1, \dots, n'_k}(a'_1, b'_1; \dots; a'_k, b'_k),$$

where  $n'_i = n_j$ ,  $a'_i = a_j$ ,  $b'_i = b_j$  for  $n'_1, \dots, n'_k$  a reordering of  $n_1, \dots, n_k$ . The resulting  $P$  has  $P(I) = Q(\tilde{I})$  if the interval  $I$  is the cylinder set by  $n_1, \dots, n_k$  of the interval  $\tilde{I}$  of  $E_k$ ,  $P$  being the Borel-Hopf extension [1, p.54] of  $Q$  from the intervals to the Borel sets.

THEOREM 5. *If*

$$\sum_{n=N+1}^{\infty} [h(n)]^2 < +\infty$$

*for some finite  $N$ , then for  $X$  the product and metric topologies coincide,  $X$  being locally compact; there exists a Borel measure  $\phi$  which is nontrivial (I) and translation-invariant (II) on  $X$ ; and such a measure is unique up to constant factors.*

*Proof.* The stated condition on  $h(n)$  makes the equivalence of the topologies over  $X$  obvious, as well as local compactness. Let  $X'$ ,  $\mathcal{E}'_2$ , and  $R'_\infty$  be defined like  $X$ ,  $\mathcal{E}_2$ , and  $R_\infty$ , except only with coordinates of  $n \geq N+1$ , so clearly

$$X = A_N \times X',$$

where  $A_N$  is an interval of  $E_N$ . Construct the Borel measure  $P^*$  on  $R'_\infty$  by the Kolmogoroff construction from

$$Q_{n_1, \dots, n_k}(a_1, b_1; \dots; a_k, b_k) = \prod_{j=1}^k \frac{1}{2h(n_j)} E(n_j, a_j, b_j),$$

where  $E(n, a, b)$  is the length, possibly zero, of the interval of intersection of  $[-h(n), h(n)]$  and  $[a, b]$ . This  $Q$ -function family has  $Q_{n_1, \dots, n_k}(E_k) = 1$ , has  $Q$  countably additive since it is a multiple of  $k$  dimensional Lebesgue measure, and satisfies Kolmogoroff's consistency conditions as needed.

Let

$$V_p = \{x \in R'_\infty \mid |x_p| > h(p)\}$$

open in  $R'_\infty$ ; clearly

$$P^*(V_p) = Q(\tilde{V}_p) = \frac{1}{2h(p)} [E(p, -\infty, -h(p)) + E(p, h(p), +\infty)] = 0.$$

Now

$$X' = \{x \in \mathcal{L}'_2 \mid |x_n| \leq h(n) \text{ for } n \geq N+1\};$$

and the given condition on  $h(n)$  makes it possible to replace  $\mathcal{L}'_2$  by  $R'_\infty$  in this formula, so that

$$X' = R'_\infty - \bigcup_{p=N+1}^{\infty} V_p,$$

which is in the Borel family  $\mathcal{B}^*$  of  $R'_\infty$ . Thus  $P^*(X') = P^*(R'_\infty) = 1$  follows from  $P^*(V_p) = 0$ , and  $X'$  is thick in  $R'_\infty$  (see [1, p. 74]). Hence  $P(A \cap X') = P^*(A)$  defines  $P$  uniquely over sets  $A \cap X'$ ,  $A \in \mathcal{B}^*$ , which form the Borel family  $\mathcal{B}$  of  $X'$ , so  $P$  is a Borel probability measure on  $X'$  with  $P(I \cap X') = Q(\tilde{I})$ .

Of  $\mu_N$  is  $N$ -dimensional Lebesgue measure,  $\phi = \mu_N \times P$  is a Borel measure on  $A_N \times X' = X$ . Also

$$\phi(X) = \mu_N(A_N) > 0,$$

and we obtain

$$\phi(B \times X') = \mu_N(B) < +\infty$$

for open bounded  $E_N$  intervals  $B \subseteq A_N$  by using  $P(X') = 1$ , and thus  $\phi$  is non-trivial ( $I$ ) on  $X$ .

We want to show  $\phi$  to be translation-invariant (II) on  $X$ . If  $W$  is any  $X$ -interval, then  $W = X \cap I$  with  $I$  an  $R_\infty$ -interval, and if  $u \in \ell_2$ , set

$$B_p = \{x \in R_\infty \mid |x_p| \leq h(p)\},$$

$$C_n = I \cap \left( \bigcap_{p=1}^n [B_p - u] \right) \cap X,$$

and

$$D_n = [I + u] \cap X \cap \left( \bigcap_{p=1}^n [B_p + u] \right),$$

so that

$$\phi(W \cap [X - u]) = \phi(I \cap [X - u] \cap X) = \lim_{n \rightarrow \infty} \phi(C_n)$$

and

$$\phi([W + u] \cap X) = \phi([I + u] \cap X \cap [X + u]) = \lim_{n \rightarrow \infty} \phi(D_n).$$

Now the first  $n$  coordinate edges of  $D_n$  are those of  $C_n$  translated by the corresponding  $u$  coordinates. Thus taking  $n >$  the greatest of the finite number of coordinate indices involved in  $I$ , from  $\phi = \mu_N \times P$  and  $P(X' \cap J) = Q(\tilde{J})$  we get  $\phi(C_n) = \phi(D_n)$ , both being the product of a normalization factor and the first  $n$  coordinate edge lengths. Thus we have

$$\phi(W \cap [X - u]) = \lim_{n \rightarrow \infty} \phi(C_n) = \lim_{n \rightarrow \infty} \phi(D_n) = \phi([W + u] \cap X),$$

as desired.

Now let  $[A + u] \subseteq X$  be given for some Borel subset  $A$  of  $X$ . If  $\{W_i\}$  is a countable disjoint  $X$ -interval family covering  $A$ , then also

$$A \subseteq \bigcup_i (W_i \cap [X - u]) \subseteq \bigcup_i W_i.$$

Since

$$\phi(A) = \inf_{A \subseteq \bigcup_i W_i} [\sum_i \phi(W_i)]$$

as the unique Borel-Hopf extension [1, p. 54] of  $\phi$  from the intervals to the Borel sets, we have

$$\begin{aligned}\phi(A) &= \inf_{A \subseteq \bigcup_i W_i} (\sum_i \phi(W_i \cap [X - u])) \\ &= \inf_{A \subseteq \bigcup_i W_i} (\sum_i \phi([W_i + u] \cap X)) \geq \phi([A + u])\end{aligned}$$

from

$$\phi(W_i \cap [X - u]) = \phi([W_i + u] \cap X).$$

Thus  $\phi(A) \geq \phi([A + u])$ , and symmetrically  $\phi([A + u]) \geq \phi(A)$ , so that  $\phi(A) = \phi([A + u])$  for Condition II of translation-invariance.

Finally for the uniqueness of  $\phi$  it is easy to see by division of intervals into large numbers of equal subintervals that any nontrivial, translation-invariant  $\psi$  will have  $\psi(I)$ ,  $I$  being an interval of  $X$ , proportional to the length of each of the edges of  $I$ . By our definition of  $\mu_N$  and  $Q$ , this makes  $\psi(I) = K\phi(I)$ , with  $0 < K < +\infty$  and  $K$  independent of  $I$ . The extension to all Borel sets thus gives  $\psi(A) = K\phi(A)$ ,  $A \in \mathfrak{B}$ , as desired.

**5. Haar measure and translation spaces.** For the space  $Y$  our positive result is a complete converse of Theorem 3. We shall get the result by considering a considerably more general situation. Let the Hausdorff space  $R$  be an Abelian topological group, and as before define

$$[A + u] = \{x \in R \mid (x - u) \in A\}$$

under  $R$ -group addition for  $A \subseteq R$  and  $u \in R$ . Consider a fixed closed subset  $Z$  of  $R$ , which becomes a Hausdorff space under the relative topology from  $R$ , but not in general a group under  $R$ -group addition. Such a space containing the zero of  $R$  is said to be a translation space if it satisfies the following condition:

i) If  $V$  is any open subset of  $Z$  containing zero, then  $Z$  is covered by the open interiors in  $Z$  of the sets of the collection  $\{Z \cap [V + u] \mid u \in R\}$ .

LEMMA 6.  $X$  is a translation space for  $R = \mathcal{L}_2$  under the metric topology.

*Proof.* Let  $V$  be the given neighborhood of zero, so that we have some small  $\rho > 0$  with  $S(Z, 0, \rho) \subseteq V$ . Then for any given  $z \in Z = X$  we will find

$u \in Z$  and  $\rho' > 0$  so that

$$S(Z, z, \rho') \subseteq Z \cap [S(Z, 0, \rho) + u] \subseteq Z \cap [V + u],$$

which makes  $z \in \text{int}(Z \cap [V + u])$  for Condition i). First since the given  $z \in \ell_2$ , we can find finite  $N$  so that

$$\left( \sum_{n=N+1}^{\infty} z_n^2 \right)^{1/2} < \frac{1}{2} \rho,$$

and then define  $u \in Z = X$  by  $u_n = z_n$  for  $1 \leq n \leq N$  and  $u_n = 0$  for  $n > N$ . Then set

$$\rho' = \min \left( \frac{1}{2} \rho, h(n) \text{ for } n = 1, 2, \dots, N \right) > 0,$$

so any  $x \in S(Z, z, \rho')$  has

$$\|x - u\| \leq \|x - z\| + \|z - u\| < \rho' + \frac{1}{2} \rho \leq \rho.$$

Any such  $x$  also has

$$|x_n - u_n| = |x_n - z_n| < \rho' \leq h(n)$$

for  $1 \leq n \leq N$ , and

$$|x_n - u_n| = |x_n| \leq h(n)$$

for  $n > N$ , so that  $x \in [S(Z, 0, \rho) + u]$ . Thus

$$S(Z, z, \rho') \subseteq Z \cap [S(Z, 0, \rho) + u],$$

as desired.

LEMMA 7.  $Y$  is a translation space for  $R = \ell_2$  under the metric topology.

*Proof.* If  $V$  is the given neighborhood of zero in  $Z = Y$ , we can find  $\rho > 0$  with  $\rho^2 < f(1)$  and  $S(Z, 0, \rho) \subseteq V$ . Now either  $\rho^2 \leq f(n)$  for all  $n$ , or else by the definition of  $Y$  there is a unique finite  $N$  with

$$f(N) \geq \rho^2 > f(N+1).$$

In the first case for the given  $z \in Z$  we take  $u = z$ , and since now  $S(Y, 0, \rho) = S(\ell_2, 0, \rho)$  by  $\rho^2 \leq f(n)$ , we have

$$S(Z, z, \rho) = Z \cap [S(\ell_2, 0, \rho) + u] \subseteq Z \cap [V + u]$$

for  $z \in \text{int}(Z \cap [V + u])$  as desired for Condition i).

In the second case for the given  $z \in Z = Y$  we define  $u \in Z$  by  $u_n = z_n$  for  $1 \leq n \leq N$ , and  $u_n = 0$  for  $n > N$ . In this case also we have

$$S(Z, u, \rho) = Z \cap [S(Z, 0, \rho) + u].$$

For the left side clearly includes the right side, while if  $y \in S(Z, u, \rho)$ , then for  $1 \leq n \leq N$  we have

$$\sum_{j=n}^{\infty} (y_j - u_j)^2 \leq \sum_{j=1}^{\infty} (y_j - u_j)^2 < \rho^2 \leq f(n).$$

For  $n > N$  we have

$$\sum_{j=n}^{\infty} (y_j - u_j)^2 = \sum_{j=n}^{\infty} y_j^2 \leq f(n),$$

so that

$$y \in Z \cap [S(Z, 0, \rho) + u],$$

and hence

$$S(Z, u, \rho) \subseteq Z \cap [S(Z, 0, \rho) + u]$$

for equality. Finally since  $z \in S(Z, u, \rho)$  by

$$\|z - u\| = \left( \sum_{j=N+1}^{\infty} z_j^2 \right)^{1/2} \leq \sqrt{f(N+1)} < \rho,$$

we have

$$z \in S(Z, u, \rho) \subseteq Z \cap [V + u],$$

so that

$$z \in \text{int}(Z \cap [V + u]),$$

$S(Z, u, \rho)$  being open, for Condition i).

Thus  $X$  and  $Y$  are special translation spaces, so the result we shall obtain for translation spaces applies to them. For the general translation space  $Z$  we define the Borel class  $\mathfrak{B}$  as the  $\sigma$ -algebra generated by the open subsets of  $Z$ , given by the relative topology from  $R$ . For a Borel measure  $\phi$  defined over  $\mathfrak{B}$  we note that Condition I of nontriviality and  $\Pi'$  of translation-invariance still make perfect sense in this more general context, if  $u \in \ell_2$  in  $\Pi'$  is replaced by  $u \in R$ . We shall now establish that a locally compact translation space does possess something like a Haar measure, that is a nontrivial, translation-invariant, regular Borel measure. First we need a few more lemmas.

LEMMA 8. *If  $V \subseteq W$  are both open subsets of the translation space  $Z$  and if  $[W + u] \cap Z$  is open in  $Z$  for some  $u \in R$ , then so also is  $[V + u] \cap Z$ .*

*Proof.* Since  $Z$  is a translation space, it is closed in  $R$ , so  $Z - W$  and  $Z - V$  are both closed in  $R$  as well as in  $Z$ . Since open and closed subsets of the topological group  $R$  remain such under translation,  $B = [(Z - W) + u] \cap Z$  and  $C = [(Z - V) + u] \cap Z$  are both closed in  $R$ , and hence in  $Z$ . Defining  $A = (R - [Z + u]) \cap Z$ , we have

$$A \cup B = Z - ([W + u] \cap Z),$$

known closed in  $Z$ , so that  $\overline{A - A} \subseteq B$  must follow. We obtain  $B \subseteq C$  from  $V \subseteq W$ , and this makes  $\overline{A - A} \subseteq C$ ; thus  $Z - ([V + u] \cap Z) = A \cup C$  is closed in  $Z$ , or  $[V + u] \cap Z$  is open, as desired.

Let  $[B + C] = \{x + y \mid x \in B \text{ and } y \in C\}$  and  $B^- = \{x \mid -x \in B\}$  for the following lemma.

LEMMA 9. *If the translation space  $Z$  has compact subsets  $B$  and  $C$  with  $B \cap C = \phi$ , then there exists some  $Z$ -neighborhood  $V$  of zero so that*

$$[B + V^-] \cap [C + V^-] = \phi.$$

*Moreover, both  $[V + z] \cap B \neq \phi$  and  $[V + z] \cap C \neq \phi$  are not simultaneously possible for any  $z \in R$ .*

*Proof.* Since  $B$  and  $C$  are compact subsets of  $Z$ , they are also such of the topological group  $R$ . Thus there exists an  $R$ -neighborhood  $W$  of zero so that



$$[B + W^-] \cap [C + W^-] = \phi.$$

Hence  $V = Z \cap W$ , so  $V^- \subseteq W^-$ , gives the first result. If  $[V + z] \cap B \neq \phi$  and  $[V + z] \cap C \neq \phi$ , then  $z \in [B + V^-] \cap [C + V^-] = \phi$ , a contradiction, which gives the last.

Following Halmos [1, p. 252], if  $B$  and  $C$  are subsets of the translation space  $Z$ , we let  $(C:B)$  denote the least cardinal (thus  $\aleph_0$  or an integer  $\geq 0$ ) of sets  $P$  of  $z \in R$  such that

$$C \subseteq \bigcup_{z \in P} [B + z].$$

LEMMA 10. *If  $C$  is a compact subset of the translation space  $Z$  and  $V$  is an open  $Z$ -subset containing zero, then  $(C:V) < +\infty$ .*

*Proof.* By Condition i) we have

$$C \subseteq \bigcup_{u \in R} \text{int } (Z \cap [V + u]),$$

an open covering of compact  $C$ . Thus there exists a finite set  $A$  of such  $u$  with

$$C \subseteq \bigcup_{u \in A} \text{int } (Z \cap [V + u]) \subseteq \bigcup_{u \in A} [V + u],$$

and hence

$$(C:V) \leq (\text{card } A) < +\infty.$$

This lemma is the only place where Condition i) is used to get our following main result on the existence of a Haar measure.

THEOREM 11. *If  $Z$  is a locally compact translation space, then there exists a regular Borel measure  $\phi$  on  $Z$  which is nontrivial (I) and translation-invariant ( $\Pi'$ ).*

*Proof.* Since  $Z$  is locally compact, it possesses a neighborhood  $V_1$  of zero such that  $\bar{V}_1$  is compact, so  $0 < (\bar{V}_1:V) < +\infty$  for any other  $Z$ -neighborhood  $V$  of zero, by Lemma 10. Also clearly

$$(C:V) \leq (C:\bar{V}_1) (\bar{V}_1:V) \leq (C:V_1) (\bar{V}_1:V),$$

so we may define

$$\lambda_v(C) = (\bar{V}_1 : V)^{-1} (C : V)$$

and have

$$0 \leq \lambda_v(C) \leq (C : V_1) < +\infty$$

for any compact subset  $C$  of  $Z$  and any  $Z$ -neighborhood  $V$  of zero. Following Halmos [1, pp. 254-256], we construct a content  $\lambda$  from  $\lambda_v$ . Let  $\Omega$  be the Cartesian product of the bounded closed intervals  $[0, (C : V_1)]$  over all compact subsets  $C$  of  $Z$ ;  $\Omega$  is compact by Tychonoff's theorem, and each  $\lambda_v \in \Omega$ . Setting

$$\Lambda(V) = \{ \lambda_w \mid W \subseteq V, W \text{ a } Z\text{-neighborhood of zero} \},$$

we see that  $\Omega$  contains by compactness some  $\lambda \in \bigcap_v \overline{\Lambda(V)}$ , the intersection being over all  $Z$ -neighborhoods  $V$  of zero. As in [1], this function  $\lambda(C)$  defined over compact  $Z$ -subsets  $C$  is a content; that is, for subsets  $B$ ,  $C$ , and  $D$  compact we have

$$0 \leq \lambda(C) \leq \lambda(B) < +\infty$$

if  $C \subseteq B$ , and

$$\lambda(C \cup D) \leq \lambda(C) + \lambda(D)$$

with equality if  $C \cap D = \emptyset$  by use of Lemma 9. Also  $\lambda(\bar{V}_1) = 1$  since  $\lambda_v(\bar{V}_1) = 1$  for any  $V$ . For translation invariance we note that if  $[C + z] \subseteq Z$  for a compact  $Z$ -subset  $C$  and  $z \in R$ , then  $[C + z]$  is also compact, since translation by  $z$  is a homeomorphism of  $R$  onto  $R$ ;  $([C + z] : V) = (C : V)$ , obviously; and thus  $\lambda_v([C + z]) = \lambda_v(C)$  for any neighborhood  $V$  makes  $\lambda([C + z]) = \lambda(C)$ .

Let  $W$  be any subset of  $Z$ , define the inner content

$$\lambda_*(W) = \sup \lambda(C)$$

over compact  $C \subseteq W$ , and for any subset  $E$  define

$$\phi(E) = \inf \lambda_*(W)$$

over open  $Z$  subsets  $W \supseteq E$ . Restricting  $\phi$  to  $\mathfrak{B}$ , we see that  $\phi$  is a regular Borel measure on  $Z$ ;  $\phi$  is nontrivial (I) by

$$\phi(Z) \geq \phi(\bar{V}_1) \geq \lambda(\bar{V}_1) = 1 \text{ and } \phi(V_1) \leq \lambda(\bar{V}_1) = 1,$$

(see [1, 53 C and E, p. 234]).

It remains only to show that  $\phi$  is translation-invariant ( $\Pi'$ ). First

$$\lambda_*([W + z]) = \lambda_*(W)$$

for  $z \in R$  and any  $Z$ -subset  $W$  having  $[W + z] \subseteq Z$ . For then compact  $C \subseteq W$  has  $[C + z] \subseteq Z$  and compact, so  $\lambda([C + z]) = \lambda(C)$  and thus  $\lambda_*([W + z]) \geq \lambda_*(W)$ . The opposite inequality follows symmetrically to give the result, since any compact  $C' \subseteq [W + z]$  has  $C = [C' - z]$  compact with

$$C \subseteq W \subseteq Z \text{ and } \lambda(C) = \lambda(C').$$

Now if  $V$  is an open  $Z$ -subset then  $\phi(V) = \lambda_*(V)$  since  $\lambda_*$  is monotone. Thus If  $V$  and  $[V + u] \cap Z$  are both open in  $Z$ , and  $u \in R$ , then  $W \subseteq V$  and  $[W + u] = [V + u] \cap Z$ , where  $W = ([V + u] \cap Z) - u$  so that

$$\phi([V + u] \cap Z) = \lambda_*([W + u]) = \lambda_*(W) \leq \lambda_*(V) = \phi(V)$$

for part b) of Condition  $\Pi'$ .

For part a), assume  $A \in \mathfrak{B}$ ,  $u \in R$ , and  $A \subseteq V_0$ , where  $V_0$  and  $[V_0 + u]$  are both open  $Z$ -subsets. Then for any open  $Z$ -subset  $V \supseteq A$ , Lemma 8 with  $V' = V \cap V_0$  and  $W' = V_0$  both open makes  $[V \cap V_0 + u]$  open also, and we note that

$$[A + u] \subseteq [V \cap V_0 + u] \subseteq [V_0 + u] \subseteq Z.$$

Hence

$$\lambda_*([V \cap V_0 + u]) = \lambda_*(V \cap V_0)$$

makes

$$\begin{aligned} \phi(A) &= \inf_{\text{open } V \supseteq A} \lambda_*(V) = \inf_{\text{open } V \supseteq A} \lambda_*(V \cap V_0) \\ &= \inf_{\text{open } V \supseteq A} \lambda_*([V \cap V_0 + u]) \geq \inf_{\text{open } W \supseteq [A+u]} \lambda_*(W) = \phi([A + u]). \end{aligned}$$

Symmetrically,  $\phi([A + u]) \geq \phi(A)$  gives  $\phi([A + u]) = \phi(A)$  for our result.

Presumably results similar to Theorem 11 are true for similar subspaces of non-Abelian topological groups. We have considered only the Abelian case for simplicity and because the interesting examples in analysis are Abelian.

COROLLARY 12. *If*

$$\liminf_{n \rightarrow \infty} f(n) = 0,$$

*then the space  $Y$  is locally compact under coincident metric and product topologies, and  $Y$  possesses a regular Borel measure nontrivial (I) and translation invariant (II') under this topology.*

*Proof.* The coincidence of the topologies and local compactness of  $Y$  is trivial from  $f(n) \downarrow 0$ ; and Lemma 7 and Theorem 11 give the rest.

**6. Another translation space example.** In addition to  $X$  and  $Y$ , we want to give another example of a translation space, still with  $R = \ell_2$ . Let

$$Z_1 = \left\{ x \in \ell_2 \mid \sum_{n=1}^{\infty} n^{2r} (x_n)^2 \leq M \right\}$$

for some fixed real  $r > 0$  and  $M > 0$ , so that clearly  $Z_1$  is actually compact. Such a space would arise by using Fourier analysis on  $L_2$ -function-spaces in which the  $r$ th derivative was subjected to a fixed bound in norm. We shall now show that  $Z_1$  is a translation space, though our proof seems unnecessarily long.

LEMMA 13. *If  $u \in Z_1$  has  $u_n = 0$  for  $n > N$  for some finite  $N$ , and*

$$\rho N^r \leq \frac{1}{2} \left\{ \sum_{n=1}^N n^{2r} (u_n)^2 \right\}^{1/2}$$

*for some  $\rho > 0$ , then*

$$Z_1 \cap [S(Z_1, 0, \rho) + u] = S(Z_1, u, \rho)$$

*open in  $Z_1$ .*

*Proof.* We only need to show that

$$S(Z_1, u, \rho) \subseteq Z_1 \cap [S(Z_1, 0, \rho) + u],$$

the opposite inclusion being obvious. Consider any  $z \in S(Z_1, u, \rho)$ ; we need only show  $(z - u) \in Z_1$ . Here  $\|z - u\| < \rho$ , so

$$\sum_{n=1}^N n^{2r} (z_n - u_n)^2 < N^{2r} \rho^2,$$

and thus from

$$\rho N^r \leq \frac{1}{2} \left\{ \sum_{n=1}^N n^{2r} (u_n)^2 \right\}^{1/2}$$

we obtain, by Minkowski's inequality,

$$0 < \left\{ \sum_{n=1}^N n^{2r} (u_n)^2 \right\}^{1/2} - \rho N^r < \left\{ \sum_{n=1}^N n^{2r} (z_n)^2 \right\}^{1/2}.$$

Thus  $u_n = 0$  for  $n > N$  and  $z \in Z_1$  yields

$$\begin{aligned} \sum_{n=1}^{\infty} n^{2r} (z_n - u_n)^2 &= \sum_{n=1}^N n^{2r} (z_n - u_n)^2 + \sum_{n=N+1}^{\infty} n^{2r} (z_n)^2 \\ &< \rho^2 N^{2r} + \sum_{n=1}^{\infty} n^{2r} (z_n)^2 - \sum_{n=1}^N n^{2r} (z_n)^2 \\ &< \rho^2 N^{2r} + M - \left( \left\{ \sum_{n=1}^N n^{2r} (u_n)^2 \right\}^{1/2} - \rho N^r \right)^2 \\ &= M - \left( \left\{ \sum_{n=1}^N n^{2r} (u_n)^2 \right\}^{1/2} - 2\rho N^r \right) \left\{ \sum_{n=1}^N n^{2r} (u_n)^2 \right\}^{1/2} \leq M. \end{aligned}$$

Thus we have shown that

$$\sum_{n=1}^{\infty} n^{2r} (z_n - u_n)^2 < M,$$

so  $(z - u) \in Z_1$  as desired.

**THEOREM 14.**  $Z_1$  satisfies Condition i), and hence is a compact translation space possessing a Haar measure in the sense of Theorem 11.

*Proof.* We merely need to verify Condition i) for  $Z_1$ . Thus given any open  $Z_1$ -subset  $V$  containing zero and any  $z \in Z_1$ , we shall find some  $u \in Z_1$  and  $\rho > 0$  so that  $S(Z_1, 0, \rho) \subseteq V$  and

$$z \in Z_1 \cap [S(Z_1, 0, \rho) + u] = S(Z_1, u, \rho)$$

open in  $Z_1$ , which makes  $z \in \text{int} (Z_1 \cap [V + u])$ , as desired. Here we need consider only  $z \neq 0$ , since  $u = 0$  makes  $0 \in V = \text{int} (Z_1 \cap [V + u])$  for the result if  $z = 0$ . Since  $z \neq 0$ , we may choose  $N$  sufficiently large so that

$$\beta = \left( \sum_{n=1}^{\infty} n^{2r} (z_n)^2 \right)^{-1} \left( \sum_{n=N+1}^{\infty} n^{2r} (z_n)^2 \right)$$

has  $0 \leq \beta < 1/5$ , and so that

$$\frac{\sqrt{M}}{2N^r} < \rho_1$$

for some  $\rho_1$  such that  $S(Z_1, 0, \rho_1) \subseteq V$ . Let

$$\rho = \frac{1}{2N^r} \left( \sum_{n=1}^N N^{2r} (z_n)^2 \right)^{1/2},$$

so

$$\rho \leq \frac{\sqrt{M}}{2N^r} < \rho_1 \quad \text{and} \quad S(Z_1, 0, \rho) \subseteq V.$$

Define  $u \in Z_1$  by  $u_n = z_n$  for  $1 \leq n \leq N$  and  $u_n = 0$  for  $n > N$ . By Lemma 13, we have

$$Z_1 \cap [S(Z_1, 0, \rho) + u] = S(Z_1, u, \rho)$$

open in  $Z_1$ . Finally to complete the proof we have  $z \in S(Z_1, u, \rho)$ , for

$$\begin{aligned} \|z - u\|^2 &= \sum_{n=N+1}^{\infty} (z_n)^2 \leq \frac{1}{N^{2r}} \left( \sum_{n=N+1}^{\infty} n^{2r} (z_n)^2 \right) \\ &= \frac{\beta}{N^{2r}} \left( \sum_{n=1}^{\infty} n^{2r} (z_n)^2 \right) < \frac{1}{N^{2r}} \left( \frac{1-\beta}{4} \right) \left( \sum_{n=1}^{\infty} n^{2r} (z_n)^2 \right) \\ &= \frac{1}{(2N^r)^2} \left( \sum_{n=1}^N n^{2r} (z_n)^2 \right) = \rho^2, \end{aligned}$$

or  $\|z - u\| < \rho$ , as desired, since  $\beta < (1 - \beta)/4$  from  $0 \leq \beta < 1/5$ .

**7. Summary of results.** We have discussed here the translation spaces

$$X = \{x \in \ell_2 \mid |x_n| \leq h(n)\}$$

and

$$Y = \{x \in \ell_2 \mid \sum_{j=n}^{\infty} x_j^2 \leq f(n)\},$$

and also

$$Z_1 = \{x \in \ell_2 \mid \sum_{n=1}^{\infty} n^{2r} (x_n)^2 \leq M\}$$

in § 6, all being subspaces of real separable Hilbert space. For  $X$  under the metric topology we have found (Theorem 3) that there exists no nontrivial, translation-invariant ( $\Pi$  or  $\Pi'$ ) Borel measure if

$$\liminf_{n \rightarrow \infty} h(n) > 0;$$

under the product topology we have the same conclusion if  $h(n) = +\infty$  infinitely often (Theorem 4). If

$$\sum_{n=1}^{\infty} [h(n)]^2 < +\infty,$$

which is equivalent to local compactness, then under the metric topology  $X$  has a nontrivial, translation-invariant ( $\Pi$ ) Borel measure which is unique up to constant factors (Theorem 5). For  $Y$  under the metric topology

$$\liminf_{n \rightarrow \infty} f(n) = 0,$$

or thus  $f(n) \downarrow 0$ , is equivalent to local compactness, and necessary and sufficient for the existence of a nontrivial, translation-invariant ( $\Pi'$ ) Borel measure (Theorem 3 and Corollary 12). Also we found (Theorem 12) that any locally compact translation space possesses a nontrivial, translation-invariant ( $\Pi'$ ) Borel measure; thus so does  $Z_1$  (Theorem 14).

It is clear from the foregoing results that local compactness is in general

the crucial condition for the existence of a nontrivial, translation-invariant Borel measure. This is well known for topological groups [5, p.144], and conjectured for spaces with a group germ (a neighborhood of zero in which group addition is always possible). However, it is to be noted that neither  $X$  nor  $Y$ , when locally compact, nor  $Z_1$  has a group germ. Thus our results seem to be new, and the concept of a translation-space a fruitful one. In fact the idea of a group germ cannot lead to anything here; for it is not difficult to see that any convex metric subspace of  $\ell_2$ , which is locally compact and contains a group germ under  $\ell_2$ -vector-addition, must be finite dimensional, hence a subspace of  $E_N$  and thus trivial. In connection with local compactness it should be noted that our results are not complete for  $X$ ; here if  $\sum^\infty [h(n)]^2 = +\infty$  the space is not locally compact under the metric topology and presumably no nontrivial, invariant Borel measure exists. We could only show this if

$$\liminf_{n \rightarrow \infty} h(n) > 0,$$

which assumes more.

The construction of an invariant measure on subspaces of real separable Hilbert space suggests an attempt to carry over vector analysis from  $E_N$ . In particular, in a later paper the author investigates the relationship between  $\ell_2$ -vector-differentiation [6, p.72] and Fourier transforms over  $X$ . Here  $X$  is a modification of Jessen's torus space [3] and can be made into a group, so standard Fourier theory applies [7 or 5].

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