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NELSON DUNFORD

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Introduction

In attempting to extend elementary divisor theory to the case of a linear operator on a complex Banach space one is naturally led to a consideration of the various equivalent definitions of the multiplicity $\nu(\lambda)$ of a complex number λ as a root of the minimal equation of a finite matrix T. Of the numerous equivalent definitions of this integer we have found only one which seems to have some virtue when applied to the infinite dimensional case. That one is as follows: $\nu(\lambda)$ is the smallest positive integer or zero for which

$$|\xi-\lambda|^{\nu(\lambda)}\ |(\xi-T)^{-1}|$$

is bounded for ξ near λ. Thus the rate of growth of the resolvent

$$T(\xi)=(\xi-T)^{-1}$$

for ξ near λ determines $\nu(\lambda)$. In this paper we consider the problem of determining conditions on the rate of growth and the mean rate of growth of the resolvent which are necessary and sufficient for a complete reduction of a linear operator on a complex Banach space. What is to be meant by a "complete" reduction? There are several apparent meanings that might be given to the notion of the resolution of the identity for an operator, all reducing to the classical one in the case of a finite matrix. For example, are we to require that E_σ be defined for all Borel sets σ or for σ in some sufficiently large subalgebra; should it be countably or just finitely additive; should it be bounded or not? All problems are legitimate and in this paper we have chosen the most restrictive of all the obvious interpretations. Consequently the conditions found on $T(\xi)$ are restrictive and the corresponding class of operators is small. On the other hand, such operators have many important properties not shared by operators outside this class. Other meanings for the notion of resolution of the identity will be

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considered in another report.

Before stating what is to be meant by a resolution of the identity for T, let us recall that if T is a linear operator in the finite dimensional linear vector space X over the field of complex numbers, and $\Pi_i(\lambda-\lambda_i)^{\nu_i}$ (λ_i distinct) is its minimal polynomial, then there are projections E_{λ_i} with

$$E_{\lambda_i}X = [x|(T-\lambda_i)^{\nu_i}x = 0]$$

and such that

$$I = E_{\lambda_1} + \cdots + E_{\lambda_k}.$$

If, for a Borel set σ in the complex plane, E_σ is defined to be the sum of those E_{λ_i} for which $\lambda_i \in \sigma$, then E_σ is a resolution of the identity for T in the sense that it has the properties (i) below:

(i)
$E_\sigma E_\delta = E_{\sigma\delta}$, $E_{\sigma'} = I - E_\sigma$, $TE_\sigma = E_\sigma T$

$E_\sigma x$ is completely additive in σ, $x \in X$

the spectrum of T when considered as an operator in $E_\sigma X$ is contained in $\overline{\sigma}$, the closure of σ.

If, for a given linear operator T in a complex Banach space, there exists a family E_σ (σ a Borel set) of operators in X satisfying (i), then E_σ is called a *resolution of the identity for* T. Such operators will be called *spectral operators*. If T is a spectral operator its resolution of the identity is unique, and operators $f(T)$ corresponding to scalar functions analytic and single valued on the spectrum $\sigma(T)$ are given by the formula

(ii)
$$f(T) = \sum_{n=0}^{\infty} \int_{\sigma(T)} \frac{f^{(n)}(\lambda)}{n!} (T-\lambda)^n \, dE_\lambda ,$$

where the integral exists as a Riemann integral in the uniform topology of operators and the series is convergent in the uniform topology of operators.

The main problem is, however, to determine when T is a spectral operator. We have endeavored to state conditions on the rate of growth and the mean rate of growth of the resolvent

$$T(\xi) = (\xi - T)^{-1}$$

which are sufficient and in some cases necessary and sufficient for the existence of a resolution of the identity. In order to do this, we have had to restrict ourselves to the case where the spectrum $\sigma(T)$ lies in a sufficiently smooth Jordan curve. To describe briefly in this introduction the nature of the results obtained in this direction, suppose that T has its spectrum in the interval $[0, 1]$. The underlying assumption is then that for each $\lambda \in [0, 1]$ there is a positive integer $\nu(\lambda)$ and a positive number $M(\lambda)$ such that

(iii) $$|\mu^{\nu(\lambda)}\, T(\lambda + i\mu)| \leq M(\lambda), \quad 0 < |\mu| < 1.$$

This alone is far from sufficient to ensure that T is a spectral operator, even in case

$$\nu(\lambda) = M(\lambda) = 1.$$

An obvious necessary condition may be stated in terms of the following notion of residue. Let C be a rectifiable Jordan curve contained in the set where $x^* T(\xi)x$ is analytic. Let σ be the set of all singularities of $x^* T(\xi)x$ which are inside C. Then

$$(x^*, x)_\sigma = \frac{1}{2\pi i} \int_C x^* T(\xi)x\, d\xi$$

is called a residue of $x^*T(\xi)x$. It is clear that if T has a resolution of the identity then

$$(x^*, x)_\sigma = x^* E_\sigma x,$$

and hence

(iv) $$|(x^*, x)_\sigma| \leq K|x^*|\ |x|, \quad x \in X,\ x^* \in X^*.$$

Conditions (iii) and (iv) are very nearly sufficient to ensure that T is a spectral operator. In reflexive spaces they are sufficient. In general though there are operators satisfying (iii) and (iv) with

$$\nu(\lambda) = M(\lambda) = 1$$

and not possessing a resolution of the identity. A final condition which in the case of a weakly complete space X makes the set of (iii), (iv), (v) sufficient for the existence of a resolution of the identity is the following. Let M_λ, N_λ be zeros and the range of $(T - \lambda)^{\nu(\lambda)}$, respectively. The condition is:

(v) For every λ in a set dense in $[0, 1]$, $M_\lambda + N_\lambda$ is dense in X.

In case $\nu(\lambda) = 1$, the condition (iv) may be stated in the equivalent form:

$$\text{(iv)}' \qquad \text{l.u.b.}_{0<\mu<1} \int_0^1 |x^*\{T(\lambda + i\mu) - T(\lambda - i\mu)\}x|\, d\lambda < \infty.$$

Unless $\nu(\lambda) = 1$ the condition (iv)′ is more restrictive than (iv). However there is a condition analogous to (iv)′ which is equivalent to (iv). It may be stated in terms of a decomposition of the resolvent. It turns out that for a spectral T there are two operators $U(\xi)$ and $V(\xi)$ such that

$$T(\xi) = U(\xi) + V(\xi)$$

and such that $x^*V(\xi)x$ is the derivative of a single valued analytic function at every point ξ where $x^*T(\xi)x$ is analytic, and $U(\xi)$ satisfies the condition (iv)′. The condition (iv) may be replaced by:

(iv)″ The resolvent $T(\xi)$ has a decomposition as described above.

In any one of the following situations the conditions (iii) and (iv) (or (iv)′ or (iv)″) are sufficient for the existence of a resolution of the identity since *in these cases* (v) *will automatically be satisfied*:

(a) The union of the resolvent set and the continuous spectrum is dense on $[0, 1]$.

(b) There is no interval of positive length consisting entirely of points in the point spectrum of the adjoint.

(c) X is reflexive.

(d) T is completely continuous.

Let $d(\xi)$ be the distance from ξ to the spectrum $\sigma(T)$; then a condition more restrictive than (iii) is

$$\text{(iii)}' \qquad |d^m(\xi)\, T(\xi)| \leq M, \text{ near } \sigma(T).$$

This condition is necessary and sufficient for the simplification of (ii) to

$$\text{(ii)}' \qquad f(T) = \sum_{n=0}^{m-1} \int_{\sigma(T)} \frac{f^{(n)}(\lambda)}{n!} (T - \lambda)^n\, dE_\lambda .$$

Thus, in a weakly complete space, (iii)′, (iv), (v) imply that T is a spectral

operator satisfying (ii)′. In a reflexive space, (iii)′ and (iv) are equivalent to the statement that T is a spectral operator satisfying (ii)′.

In case X is not weakly complete, the above statements remain valid providing the notion of the resolution of the identity is weakened in the following manner. Instead of requiring that E_σ be defined for all Borel sets, we demand that it be defined and countably additive on the Boolean algebra determined by the real intervals. This enables one to define the integral occurring in (ii)′. Thus in this extended sense we may say that for an arbitrary complex Banach space the conditions (iii)′, (iv), (v) imply that T is a spectral operator satisfying (ii)′.

Although this is the second in a series of articles on spectral theory, not much knowledge of the contents of the first [1] paper is assumed or used. We collect here the terminology, notation, and results from that paper that are used in the present one. An *admissible* domain is an open set bounded by a finite number of rectifiable Jordan curves. It is called a *T-admissible* domain in case its boundary is contained in the resolvent set $\rho(T)$ of T. The class of complex valued functions analytic and single valued on some T-admissible domain containing the spectrum $\sigma(T)$ is denoted by $F(T)$ or $F(\sigma(T))$. For $f \in F(T)$, the operator $f(T)$ is defined by the formula

$$f(T) = \frac{1}{2\pi i} \int_C f(\lambda) T(\lambda)\, d\lambda,$$

where C is the boundary of some T-admissible domain containing the spectrum of T. The mapping, given by the above formula, of the algebra of analytic functions into an algebra of operators is a homomorphism which assigns the operators I, T to the functions 1, λ, respectively.

I. Operators with nondense spectra and preliminary lemmas

In this section we consider an operator T whose spectrum $\sigma(T)$ is nondense in the complex plane. Two conditions concerning the singularities of the analytic function $(\xi - T)^{-1}x$ are introduced (these are 1.7 and 1.14 below). As we show later, these are necessary conditions for the existence of a resolution of the identity regardless of the operator T or the character of the space X. The main purpose of §1 is to show how near these two conditions come to being sufficient. Later, in §2, we shall determine the meaning of these two conditions in terms of the rate of growth and the mean rate of growth of the resolvent $T(\xi)$ for ξ near the spectrum. The basic assumption for §1 is then:

1.1. ASSUMPTION. *The spectrum $\sigma(T)$ of T is nondense in the complex*

plane.

This means that the resolvent set $\rho(T)$ of T is dense in the plane. The chief purpose of this assumption is to prove the following lemma which asserts that the analytic function $(\xi - T)^{-1}x$ is single valued; if this fact is already known then Assumption 1.1 may easily, in most of what follows, be discarded. Since 1.1 is the underlying assumption for practically all of § 1, *it will not be explicitly stated in the lemmas to follow.* The other assumptions 1.7 and 1.14, and others in § 2, will however be indicated parenthetically when they are used.

1.2. LEMMA. *For each $x \in X$ the analytic function $T(\xi)x$ defined on $\rho(T)$ has a unique maximal single valued analytic extension.*

Let f, g be two vector-valued analytic functions defined on open sets $D(f)$, $D(g)$, respectively. We suppose that $D(f)D(g) \supset \rho(T)$ and that

$$f(\xi) = T(\xi)\, x = g(\xi) \quad \text{for} \quad \xi \in \rho(T).$$

Let $\xi_0 \in D(f)\, D(g)$. By 1.1, there is a sequence of points $\xi_n \in \rho(T)$ with $\xi_n \longrightarrow \xi_0$, and so $f(\xi_0) = g(\xi_0)$. Thus, if $\rho(x)$ is the union of all open sets containing $\rho(T)$ upon which $T(\xi)x$ has an analytic extension, we have uniquely defined upon $\rho(x)$ an analytic extension of $T(\xi)x$.

1.3. DEFINITIONS. By $x(\xi)$ we shall mean the unique maximal single valued analytic extension of $T(\xi)x$ whose existence is established in 1.2. The symbol $\rho(x)$ will be used for the domain of definition of $x(\xi)$, and the symbol $\sigma(x)$ will be used for the set of singularities of $x(\xi)$. Thus $\sigma(x)$ is the complement of $\rho(x)$, and $\rho(x) \supset \rho(T)$, $\sigma(x) \subset \sigma(T)$.

1.4. DEFINITION. By $[x]$ we shall mean the smallest closed linear manifold containing all of the vectors $T(\xi)x$, $\xi \in \rho(T)$.

1.5. LEMMA. *For every $x \in X$ we have:*

1.5.1. $x \in [x]$;

1.5.2. $f(T)\,[x] \subset [x]$, $f \in F(\sigma(T))$;

1.5.3. $x(\xi) \in [x]$, $\xi \in \rho(x)$;

1.5.4. $[y] \subset [x]$, $y \in [x]$.

Let C be a large circle such that

$$x = \frac{1}{2\pi i} \int_C T(\xi)x\,d\xi \in [x];$$

this proves 1.5.1. Let $y \in [x]$ and $f \in F(\sigma(T))$. Since y may be approximated by sums of the form $\sum \alpha_j T(\xi_j)x$, $f(T)y$ may be approximated by sums of the form

$$\sum \alpha_j T(\xi_j) f(T)x = \sum \alpha_j \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)T(\xi)x}{\xi_j - \xi} \, d\xi,$$

where Γ is chosen, in the domain of regularity of f, to include $\sigma(T)$ and exclude the points ξ_j. Thus $f(T)y \in [x]$, and 1.5.2 is proved. Next let $\xi_0 \in \rho(x)$ and, using (1.1), choose a sequence $\xi_n \in \rho(T)$ with $\xi_n \longrightarrow \xi_0$. Thus

$$T(\xi_n)x \longrightarrow x(\xi_0),$$

and since $[x]$ is closed we have $x(\xi_0) \in [x]$. Finally if $y \in [x]$ we have, by 1.5.2, $T(\xi)y \in [x]$, $\xi \in \rho(T)$, and thus $[y] \subset [x]$. This completes the proof of 1.5.

1.6. LEMMA. *For $x, y \in X$ we have*

$$\sigma(x + y) \subset \sigma(x) \cup \sigma(y),$$

and for $\xi \in \rho(x)\,\rho(y)$ we have

$$x(\xi) + y(\xi) = (x + y)(\xi).$$

On the open set $\rho(x)\,\rho(y)$, the function $x(\xi) + y(\xi)$ is an analytic extension of

$$T(\xi)x + T(\xi)y = T(\xi)(x + y), \quad \xi \in \rho(T).$$

Thus $\rho(x + y) \supset \rho(x)\,\rho(y)$, and for $\xi \in \rho(x)\,\rho(y)$ we have, by 1.2,

$$x(\xi) + y(\xi) = (x + y)(\xi).$$

The second assumption which is needed in most of §1 is:

1.7. ASSUMPTION. *If σ is a closed set of complex numbers, then the set $[\sigma]$ of all vectors x with $\sigma(x) \subset \sigma$ is also closed.*

1.8. LEMMA. (*Assumption* 1.7.) *If σ is a closed set of complex numbers, then $[\sigma]$ is a closed linear manifold, $T[\sigma] \subset [\sigma]$, and the spectrum of T when considered as an operator in $[\sigma]$ is contained in σ.*

That $[\sigma]$ is a closed linear manifold follows from 1.6 and 1.7. Since

$$Tx(\mu) = T(\mu)Tx,$$

for μ in $\rho(T)$ we have $\rho(x) \subset \rho(Tx)$ or $\sigma(Tx) \subset \sigma(x)$, and thus $T[\sigma] \subset [\sigma]$. Now let $x \in [\sigma]$, $\xi \in \sigma'$ (the complement of σ in the whole plane), $\xi_n \in \rho(T)$, and $\xi_n \longrightarrow \xi$. Since for $\mu \in \rho(T)$ we have

$$T(\mu)x = (\xi_n - T)T(\mu)T(\xi_n)x,$$

it follows that $\rho(x) \subset \rho(T(\xi_n)x)$ and thus $T(\xi_n)x \in [\sigma]$. Since

$$T(\xi_n)x = x(\xi_n) \longrightarrow x(\xi)$$

and $[\sigma]$ is closed, we have $x(\xi) \in [\sigma]$. Thus, since

$$(\xi - T)x(\xi) = x,$$

it follows that

$$(\xi - T)[\sigma] = [\sigma].$$

To see that $\xi - T$ is one-to-one on $[\sigma]$, suppose that

$$(\xi - T)y = 0, \ y \in [\sigma].$$

Then

$$y(\lambda) = \frac{y}{(\lambda - \xi)} \text{ and } \sigma(y) \subset (\xi) \cap \sigma = \phi,$$

the void set. This means that $y(\lambda)$ is analytic for all λ and thus that $y = 0$. Hence, if $\xi \in \sigma'$ then $\xi - T$ is a one-to-one map of $[\sigma]$ into all of itself.

1.9. LEMMA. (*Assumption* 1.7.) *For every pair* σ_1, σ_2 *of disjoint closed sets, there is a constant* $K(\sigma_1, \sigma_2)$ *such that*

$$|x(\xi)| \leq K(\sigma_1, \sigma_2)\,|x|, \quad \xi \in \sigma_1, \ x \in [\sigma_2].$$

By 1.8, σ_1 is contained in the resolvent set of T when considered as an operator in $[\sigma_2]$. Since $x(\xi)$ is the value of this resolvent at the point $\xi \in \sigma_1$ when operating on $x \in [\sigma_2]$, the present lemma follows from the preceeding one.

1.10. LEMMA. (*Assumption* 1.7.) *For every* $x \in X$ *we have* $T[x] \subset [x]$, *and when* T *is regarded as an operator in the space* $[x]$ *it has* $\sigma(x)$ *for its*

spectrum and $\rho(x)$ *for its resolvent set.*

It was proved in 1.5.2 that $T[x] \subset [x]$. Let ρ_1 be the resolvent set of T as an operator in $[x]$. Using 1.5.2 again, we readily show that $\rho(T) \subset \rho_1$; and since $T(\xi)x$ is analytic on ρ_1 (since $x \in [x]$, by 1.5.1), we have $\rho_1 \subset \rho(x)$. We shall now show that for every $y \in [x]$ we have $\rho(y) \supset \rho(x)$, which means that for every $y \in [x]$ the function $T(\xi)y$ defined for $\xi \in \rho(T)$ has an analytic extension to $\rho(x)$. Elements of the form

$$(*) \qquad y = \sum \alpha_j T(\xi_j)x, \quad \xi_j \in \rho(T)$$

are dense in $[x]$, and for such y we have, for $\mu \in \rho(T)$,

$$T(\mu)y = \sum \alpha_j T(\xi_j)\ T(\mu)x.$$

Thus $T(\mu)y$ has the analytic extension $\sum \alpha_j T(\xi_j)x(\mu)$, $\mu \in \rho(x)$, and so, for y of the form (*), we have $\rho(y) \supset \rho(x)$, $\sigma(y) \subset \sigma(x)$. Let $y \in [x]$, and let y_n be a sequence of vectors of the form (*) with $y_n \longrightarrow y$. Since $y_n - y_m$ has the form (*), we have $\sigma(y_n - y_m) \subset \sigma(x)$. Let N be a neighborhood whose closure $\overline{N} \subset \rho(x)$, so that $\overline{N}$ and $\sigma(x)$ are closed disjoint sets. By 1.9, then,

$$|y_n(\xi) - y_m(\xi)| = |(y_n - y_m)(\xi)| \leq K(\overline{N}, \sigma(x))\ |y_n - y_m| \longrightarrow 0$$

uniformly for $\xi \in N$. The function

$$f(\xi) = \lim_n y_n(\xi)$$

is analytic on N, and for every $\xi \in \rho(T)N$ we have

$$f(\xi) = \lim_n y_n(\xi) = \lim_n (\xi - T)^{-1} y_n = (\xi - T)^{-1} y.$$

Hence $f(\xi) = y(\xi)$, $\xi \in N$, and $\rho(y) \supset \rho(x)$. Finally we let $\xi_0 \in \rho(x)$ and show that $\xi_0 - T$ is a one-to-one map of $[x]$ into all of itself. Let $y \in [x]$; then since $\xi_0 \in \rho(x) \subset \rho(y)$ we have, by 1.5.3 and 1.5.4, $y(\xi_0) \in [y] \subset [x]$. Since

$$(\xi - T)y(\xi) = y$$

for $\xi \in \rho(T)$, this same equation must hold for $\xi \in \rho(y)$; in particular,

$$(\xi_0 - T)y(\xi_0) = y.$$

Thus $(\xi_0 - T)[x] = [x]$. To see that $\xi_0 - T$ is a one-to-one map on $[x]$, let $y \in [x]$ and $(\xi_0 - T)y = 0$. For large ξ, we have the expansion

$$T(\xi) = \sum_{n=0}^{\infty} \frac{(T - \xi_0)^n}{(\xi - \xi_0)^{n+1}};$$

hence $T(\xi)\, y = y/(\xi - \xi_0)$. Thus if $y \neq 0$ we have $\sigma(y)$ consisting of the single point $\xi_0 \in \rho(x) \subset \rho(y)$, a contradiction since $\rho(y)$ and $\sigma(y)$ are disjoint. Thus it has been proved that for every $\xi \in \rho(x)$ the operator $\xi - T$ is a one-to-one map of $[x]$ into all of itself, and hence $\rho(x) \subset \rho_1 \subset \rho(x)$.

1.11. LEMMA. (*Assumption* 1.7.) *If* $y \in [x]$ *then* $\sigma(y) \subset \sigma(x)$.

This was proved (in the form $\rho(y) \supset \rho(x)$) during the course of the proof of Lemma 1.10.

1.12. LEMMA. (*Assumption* 1.7.) *The set* $\sigma(x)$ *is void if and only if* $x = 0$.

If $x = 0$ it follows from Definition 1.3 that $\sigma(x)$ is void. Conversely, if $\sigma(x)$ is void then by (7) the spectrum of T as an operator in the space $[x]$ is void. This, according to Taylor's result [3], implies that $[x]$ consists of the zero vector alone. Hence $x = 0$.

1.13. LEMMA. (*Assumption* 1.7.) *Let* σ *be a set of complex numbers, and* σ' *its complement. If* $x + y = x_1 + y_1$, *where* $\sigma(x), \sigma(x_1) \subset \sigma$ *and* $\sigma(y), \sigma(y_1) \subset \sigma'$, *then* $x = x_1$, $y = y_1$.

The sets

$$\sigma_1 = \sigma(x) \cup \sigma(x_1), \quad \sigma_2 = \sigma(y) \cup \sigma(y_1)$$

are bounded, closed, and disjoint. Since, by 1.6, $\sigma(x + y) \subset \sigma_1 \cup \sigma_2$, there is an admissible contour C containing σ_1 and excluding σ_2 which lies in $\rho(x + y)$. Thus

$$\frac{1}{2\pi i} \int_C (x + y)(\xi)\, d\xi = \frac{1}{2\pi i} \int_C x(\xi)\, d\xi + \frac{1}{2\pi i} \int_C y(\xi)\, d\xi$$

Since $y(\xi)$ is regular in the closed domain bounded by C, the second integral on the right side of the above equality is zero. Since $\sigma(x)$ is contained within the domain bounded by C we see, from 1.10, that the first integral on the right of the above equality is equal to x. Hence

$$\frac{1}{2\pi i}\int_C (x+y)(\xi)\,d\xi = x,$$

and similarly

$$\frac{1}{2\pi i}\int_C (x_1+y_1)(\xi)\,d\xi = x_1.$$

Thus $x = x_1$, $y = y_1$.

In most of what follows we shall need besides Assumption 1.7 the following:

1.14. ASSUMPTION. *There is a constant K, depending only upon T, such that for every pair x, y of vectors with $\sigma(x)$, $\sigma(y)$ disjoint we have*

$$|x| \leq K|x+y|.$$

1.15. DEFINITION. By s_1 we shall mean the family of all sets σ with the property that vectors of the form $x+y$ with $\sigma(x) \subset \sigma$, $\sigma(y) \subset \sigma'$ are dense in X. Clearly, if $\sigma \in s_1$ then the complement $\sigma' \in s_1$.

1.16. LEMMA. (*Assumptions* 1.7, 1.14.) *For $\sigma \in s_1$ there is one and only one bounded projection E_σ on X with the properties $E_\sigma x = x$ if $\sigma(x) \subset \sigma$; $E_\sigma x = 0$ if $\sigma(x) \subset \sigma'$. This projection has the further properties that*

$$E_\sigma + E_{\sigma'} = I,\quad E_\sigma E_{\sigma'} = 0,\quad |E_\sigma| \leq K.$$

Vectors of the form $z = x+y$ with $\sigma(x) \subset \sigma$, $\sigma(y) \subset \sigma'$ are dense in X. In view of 1.13 it is permissible to define, on this dense set, $E_\sigma z = x$. From 1.14 it follows that $|E_\sigma z| \leq K|z|$. Now if

$$z_1 = x_1 + y_1 \text{ with } \sigma(x_1) \subset \sigma,\ \sigma(y_1) \subset \sigma',$$

then

$$z + z_1 = x + x_1 + y + y_1,$$

and, by 1.6, $\sigma(x+x_1) \subset \sigma$, $\sigma(y+y_1) \subset \sigma'$. Thus

$$E_\sigma(z+z_1) = E_\sigma z + E_\sigma z_1,$$

and E_σ is additive and continuous on a dense linear set. Thus $E_\sigma z$ is uniquely

defined for $z \in X$ by the requirements that $E_\sigma z$ is continuous in z. For elements z of the original dense set we have

$$E_\sigma^2 z = E_\sigma x = x = E_\sigma z, \text{ where } z = x + y,\ \sigma(x) \subset \sigma,\ \sigma(y) \subset \sigma'.$$

Thus $E_\sigma^2 = E_\sigma$. It is also clear that

$$E_\sigma E_{\sigma'} = 0 \text{ and } E_\sigma + E_{\sigma'} = I.$$

If A_σ is another bounded projection with the properties

$$A_\sigma x = x \text{ if } \sigma(x) \subset \sigma \text{ and } A_\sigma x = 0 \text{ if } \sigma(x) \subset \sigma',$$

then for $z = x + y$, where $\sigma(x) \subset \sigma$, $\sigma(y) \subset \sigma'$ we have $A_\sigma z = x = E_\sigma z$, and hence $A_\sigma z = E_\sigma z$ for every $z \in X$.

1.17. LEMMA. (*Assumptions* 1.7, 1.14.) *If* $\sigma \in s_1$ *and* $f \in F(\sigma(T))$, *then* $f(T)E_\sigma = E_\sigma f(T)$.

Let $z = x + y$, $\sigma(x) \subset \sigma$, $\sigma(y) \subset \sigma'$. Then

$$f(T)z = f(T)x + f(T)y.$$

By 1.5.1 and 1.5.2, $f(T)x \in [x]$; and by 1.11, $\sigma(f(T)x) \subset \sigma(x) \subset \sigma$. Similarly, $\sigma(f(T)y) \subset \sigma(y) \subset \sigma'$. So

$$E_\sigma f(T)z = f(T)x = f(T)E_\sigma z.$$

Since the vectors z are dense, the lemma is proved.

1.18. LEMMA. (*Assumptions* 1.7, 1.14.) *We have* $\sigma(E_\sigma x) \subset \sigma(x)$, $\sigma \in s_1$, $x \in X$.

We have, by 1.17,

$$T(\xi)E_\sigma x = E_\sigma T(\xi)x,\ \xi \in \rho(T),$$

and hence the analytic function $T(\xi)E_\sigma x$ has the analytic extension $E_\sigma x(\xi)$ for $\xi \in \rho(x)$. Thus $\rho(E_\sigma x) \supset \rho(x)$ and $\sigma(E_\sigma x) \subset \sigma(x)$.

1.19. DEFINITION. For $\sigma \in s_1$, define $X_\sigma = E_\sigma X$.

1.20. DEFINITION. If M is a closed linear manifold in X for which $TM \subset M$, we use the symbol $\sigma(M)$ for the spectrum of T when considered as an operator in M, and the symbol $\rho(M)$ for the resolvent set of T as an operator in M.

1.21. THEOREM. (*Assumptions* 1.7, 1.14.) *If* $\sigma \in s_1$, *then* $TX_\sigma \subset X_\sigma$ *and* $\sigma(X_\sigma) \subset \bar{\sigma}$, *where* $\bar{\sigma}$ *is the closure of* σ.

It follows from 1.17 that $TX_\sigma \subset X_\sigma$. Let $\xi \notin \bar{\sigma}$. We shall first show that $\xi - T$ is one-to-one on X_σ. If $x \in X_\sigma$, $(\xi - T)x = 0$, then $x(\lambda) = x/(\lambda - \xi)$ since for all large λ we have

$$T(\lambda) = \sum_{n=0}^{\infty} \frac{(T - \xi)^n}{(\lambda - \xi)^{n+1}}.$$

Since $x \in X_\sigma$, we have $x = E_\sigma x$; and since $x(\lambda)$ is everywhere regular except possibly at the point $\xi \in \sigma'$, we have $\sigma(x) \subset \sigma'$, from which it follows that $E_\sigma x = 0$. Thus $\xi - T$ is one-to-one on X_σ. We next show that $(\xi - T)X_\sigma = X_\sigma$. Let

$$x \in X_\sigma \text{ and } x_n + y_n \longrightarrow x, \ \sigma(x_n) \subset \sigma, \ \sigma(y_n) \subset \sigma'.$$

Then

$$x = E_\sigma x = \lim_n E_\sigma(x_n + y_n) = \lim_n E_\sigma x_n = \lim_n x_n.$$

Let $y_n = x_n(\xi)$, so that

$$y_n - y_m = x_n(\xi) - x_m(\xi) = (x_n - x_m)(\xi),$$

and hence, by 1.9,

$$|y_n - y_m| \leq K_1 |x_n - x_m| \longrightarrow 0.$$

Let $y = \lim_n y_n$, so that

$$x = \lim_n x_n = \lim (\xi - T)y_n = (\xi - T)y.$$

It remains to be shown that $y \in X_\sigma$. Since $\xi \in \rho(x_n)$, we see from 1.5.3 that

$$y_n = x_n(\xi) \in [x_n],$$

and thus 1.11 gives $\sigma(y_n) \subset \sigma(x_n) \subset \sigma$. Thus $y_n = E_\sigma y_n \in X_\sigma$ and $y \in X_\sigma$. We have shown that if $\xi \notin \bar{\sigma}$ then $\xi - T$ is a one-to-one map of X_σ into all of itself; that is, $\sigma(X_\sigma) \subset \bar{\sigma}$.

1.22. LEMMA. (*Assumptions* 1.7, 1.14.) *If* $\sigma \in s_1$, *then* $\sigma(E_\sigma x) \subset \bar{\sigma}\sigma(x)$

for every $x \in X$.

In view of 1.18 it will suffice to show that $\sigma(E_\sigma x) \subset \overline{\sigma}$. From 1.21 it follows that $\rho(E_\sigma x) \supset \overline{\sigma}'$, and thus $\sigma(E_\sigma x) \subset \overline{\sigma}$.

1.23. DEFINITION. The symbol s_2 will be used for the family of all sets σ having the following property. For every $x \in X$ and every $\epsilon > 0$ there are vectors x_1, x_1' with $\sigma(x_1) \subset \sigma(x)\sigma$, $\sigma(x_1') \subset \sigma(x)\sigma'$ and $|x_1 + x_1' - x| < \epsilon$.

1.24. LEMMA. *The family* s_2 *is a Boolean algebra and* $s_2 \subset s_1$.

That $s_2 \subset s_1$ is clear from the definition of these classes. Let σ_1, $\sigma_2 \in s_2$, $x \in X$ and $\epsilon > 0$. We then have

$$x = x_1 + x_1' + u_1, \quad \sigma(x_1) \subset \sigma(x)\sigma_1, \quad \sigma(x_1') \subset \sigma(x)\sigma_1', \quad |u_1| < \epsilon/2;$$

$$x_1' = x_2 + x_2' + u_2, \quad \sigma(x_2) \subset \sigma(x_1')\sigma_2, \quad \sigma(x_2') \subset \sigma(x_1')\sigma_2', \quad |u_2| < \epsilon/2;$$

$$x = x_1 + x_2 + x_2' + u_1 + u_2.$$

Using 1.6 we see that

$$\begin{aligned}\sigma(x_1 + x_2) \subset \sigma(x_1) \cup \sigma(x_2) &\subset (\sigma(x)\sigma_1) \cup (\sigma(x_1')\sigma_2) \\ &\subset (\sigma(x)\sigma_1) \cup (\sigma(x)\sigma_2) \\ &= \sigma(x)\,(\sigma_1 \cup \sigma_2),\end{aligned}$$

and

$$\sigma(x_2') \subset \sigma(x_1')\sigma_2' \subset \sigma(x)\sigma_1'\,\sigma_2' = \sigma(x)\,(\sigma_1 \cup \sigma_2)'.$$

Thus $\sigma_1 \cup \sigma_2 \in s_2$. It is clear from 1.23 that s_2 is closed under complementation and that the void set and the whole plane are in s_2. Thus s_2 is closed under crosscut; that is, $\sigma_1\,\sigma_2 \in s_2$ if σ_1, $\sigma_2 \in s_2$, and s_2 is a Boolean algebra.

1.25. THEOREM. (*Assumptions* 1.7, 1.14.) *On the Boolean algebra* s_2 *the projections* E_σ *have the following properties*:

$$E_{\sigma_1} \cup E_{\sigma_2} = E_{\sigma_1 \cup \sigma_2}, \quad E_{\sigma_1} E_{\sigma_2} = E_{\sigma_1\,\sigma_2}, \quad E_\sigma' = E_{\sigma'};$$

$$E_{\sigma(T)} = I, \quad E_\phi = 0, \text{ where } \phi \text{ is the void set.}$$

If the projections E_σ, $\sigma \in s_2$ are ordered in the usual fashion (that is, $E_{\sigma_1} \subset E_{\sigma_2}$ means $E_{\sigma_2} E_{\sigma_1} = E_{\sigma_1}$ or equivalently $X_{\sigma_1} \subset X_{\sigma_2}$) then, by definition, $E_{\sigma_1} \cup E_{\sigma_2}$ is the smallest projection containing E_{σ_1} and E_{σ_2}. It may be given by the formula

$$E_{\sigma_1} \cup E_{\sigma_2} = E_{\sigma_1} + E_{\sigma_2} - E_{\sigma_1} E_{\sigma_2}.$$

This formula is readily derived from the relation

$$E_{\sigma_1} E_{\sigma_2} = E_{\sigma_2} E_{\sigma_1},$$

which of course will be established as soon as we have shown that

$$E_{\sigma_1} E_{\sigma_2} = E_{\sigma_1 \sigma_2}.$$

Now let $x \in X$, $\epsilon > 0$, σ_1, $\sigma_2 \in s_2$. We have

$$x = x_1 + x_1' + u, \quad x_1' = x_2 + x_2' + v, \quad x_1 = y_2 + y_2' + w,$$

where $|u|$, $|v|$, $|w| < \epsilon$, and

$$\sigma(x_1) \subset \sigma(x)\sigma_1, \quad \sigma(x_1') \subset \sigma(x)\sigma_1',$$

$$\sigma(x_2) \subset \sigma(x_1')\sigma_2 \subset \sigma(x)\sigma_1' \sigma_2 \subset (\sigma_1 \sigma_2)',$$

$$\sigma(x_2') \subset \sigma(x_1')\sigma_2' \subset \sigma(x)\sigma_1' \sigma_2' \subset (\sigma_1 \cup \sigma_2)' \subset (\sigma_1 \sigma_2)',$$

$$\sigma(y_2) \subset \sigma(x_1)\sigma_2 \subset \sigma(x)\sigma_1 \sigma_2,$$

$$\sigma(y_2') \subset \sigma(x_1)\sigma_2' \subset \sigma(x)\sigma_1 \sigma_2' \subset (\sigma_1 \sigma_2)'\sigma_1.$$

Place $z = y_2 + y_2' + x_2 + x_2'$, $y = u + v + w$, so that $x = z + y$ and $|y| < 3\epsilon$. Remembering that $E_\sigma x = x$ for every x with $\sigma(x) \subset \sigma$ and $E_\sigma x = 0$ if $\sigma(x) \subset \sigma'$, we see from the above inclusion relations that

$$E_{\sigma_1} z = y_2 + y_2', \quad E_{\sigma_2} z = y_2 + x_2,$$

$$E_{\sigma_2} E_{\sigma_1} z = E_{\sigma_1} E_{\sigma_2} z = E_{\sigma_1 \sigma_2} z = y_2,$$

$$(E_{\sigma_1} + E_{\sigma_2} - E_{\sigma_1} E_{\sigma_2}) z = y_2 + y_2' + x_2 = E_{\sigma_1 \cup \sigma_2} z.$$

Hence

$$|(E_{\sigma_1}E_{\sigma_2} - E_{\sigma_1\sigma_2})x| = |(E_{\sigma_1}E_{\sigma_2} - E_{\sigma_1\sigma_2})y| \leq 3K(K+1)\epsilon.$$

Since ϵ is independent of x, we have

$$E_{\sigma_1}E_{\sigma_2} = E_{\sigma_1\sigma_2} = E_{\sigma_2}E_{\sigma_1}.$$

Also,

$$|((E_{\sigma_1} \cup E_{\sigma_2}) - E_{\sigma_1\cup\sigma_2})x| \leq 4K|y| < 12K\epsilon,$$

so that $E_{\sigma_1} \cup E_{\sigma_2} = E_{\sigma_1\cup\sigma_2}$. The remaining conclusions have been proved in 1.16.

1.26. DEFINITION. (Assumptions 1.7, 1.14.) The symbol s_3 will be used for those sets $\sigma \in s_1$ for which there exist closed sets μ_n, $\nu_n \in s_2$ with $\nu_n \subset \sigma$, $\mu_n \subset \sigma'$, $n = 1, 2, \cdots$ and

$$x = \lim_n (E_{\nu_n} + E_{\mu_n})x, \quad x \in X.$$

1.27. LEMMA. (*Assumptions* 1.7, 1.14.) *The family* s_3 *is a Boolean algebra and* $s_3 \subset s_2$.

If $\sigma \in s_3$ and μ_n, ν_n are as in 1.26, then by 1.22 we have

$$\sigma(E_{\nu_n}x) \subset \nu_n\,\sigma(x) \subset \sigma\,\sigma(x),\ \sigma(E_{\mu_n}x) \subset \mu_n\,\sigma(x) \subset \sigma'\sigma(x),$$

and so $\sigma \in s_2$; that is, $s_3 \subset s_2$. It is clear that s_3 is closed under complementation; hence, in order to show that s_3 is a Boolean algebra, it will suffice to show that it is closed under the operation of forming unions. Let $\sigma_1, \sigma_2 \in s_3$ and $\nu(i, n)$, $\mu(i, n)$, $(i = 1, 2;\ n = 1, 2, \cdots)$ be closed s_2 sets with $\nu(i, n) \subset \sigma_i$, $\mu(i, n) \subset \sigma_i'$ $(i = 1, 2)$ and

$$x = \lim_n \left(E_{\nu(i,n)} + E_{\mu(i,n)}\right)x \qquad (i = 1, 2).$$

Then

$$x = E_{\nu(1,n)}x + E_{\mu(1,n)}x + u_n, \quad \text{and } u_n \longrightarrow 0.$$

Thus

$$E_{\sigma_1'}x = E_{\mu(1,n)}x + E_{\sigma_1'}u_n \quad \text{and} \quad E_{\mu(1,n)}x \longrightarrow E_{\sigma_1'}x.$$

This last fact shows that the sequence v_n defined by the next equation has the property that $v_n \longrightarrow 0$:

$$E_{\mu(1,n)}x = E_{\nu(2,n)}E_{\mu(1,n)}x + E_{\mu(2,n)}E_{\mu(1,n)}x + v_n.$$

Upon substituting the above expression for $E_{\mu(1,n)}x$ into the formula defining u_n, we see by using 1.25 that

$$x = E_{\nu_n}x + E_{\mu_n}x + u_n + v_n,$$

where

$$\nu_n = \nu(1,n) \cup \nu(2,n)\,\mu(1,n) \text{ and } \mu_n = \mu(2,n)\,\mu(1,n).$$

Since ν_n, μ_n are closed s_2 sets (by 1.24) with $\nu_n \subset \sigma_1 \cup \sigma_2$, and $\mu_n \subset \sigma_1' \sigma_2' = (\sigma_1 \cup \sigma_2)'$, it follows that $\sigma_1 \cup \sigma_2$ is an s_3 set, and the lemma is established.

1.28. LEMMA. (*Assumptions* 1.7, 1.14.) *Let* $\sigma \in s_3$, $x \in X$, $\epsilon > 0$. *Then there are sets* μ, $\nu \in s_2$ *with* μ *open,* ν *closed,* $\mu \supset \sigma \supset \nu$, *and such that*

$$|E_\omega x| < \epsilon, \quad \omega \subset \mu - \nu, \quad \omega \in s_2;$$

$$|E_\sigma x - E_{\sigma_1}x| < \epsilon, \quad \mu \supset \sigma_1 \supset \nu, \quad \sigma_1 \in s_2.$$

Since σ is an s_3 set, there are closed sets ν, $\mu' \in s_2$ with

$$\nu \subset \sigma, \quad \mu' \subset \sigma', \quad x = E_\nu x + E_{\mu'}x + u, \text{ and } |u| < \epsilon.$$

Then

$$|E_\mu x - E_\nu x| = |E_\mu u| < K\epsilon.$$

Let $\omega \subset \mu - \nu$, $\omega \in s_2$. Then

$$|E_\omega x| = |E_\omega E_{\mu-\nu}x| = |E_\omega(E_\mu - E_\nu)x| < K^2\epsilon;$$

this proves the first conclusion. Now

$$E_\sigma - E_{\sigma_1} = E_{\sigma-\sigma\sigma_1} + E_{\sigma\sigma_1} - E_{\sigma_1-\sigma\sigma_1} - E_{\sigma\sigma_1} = E_{\sigma-\sigma\sigma_1} - E_{\sigma_1-\sigma\sigma_1},$$

and since $\sigma - \sigma\sigma_1 \subset \mu - \nu$, $\sigma_1 - \sigma\sigma_1 \subset \mu - \nu$, the second conclusion follows from the first conclusion.

1.29. LEMMA. (*Assumption* 1.7, 1.14.) *If* σ_m, $\sigma \in s_3$; $\sigma_m \subset \sigma_{m+1}$ $(m = 1, 2, \cdots)$; *and* $\sigma_m \longrightarrow \sigma$; *then* $E_{\sigma_m}x \longrightarrow E_\sigma x$, $x \in X$.

Let $x \in X$, $\epsilon > 0$ be arbitrary. Let $\epsilon_n > 0$ and $\sum_{n=1}^{\infty} \epsilon_n < \epsilon$. Using 1.28, pick open sets $\mu_n \in s_2$ with $\mu_n \supset \sigma_n$ and

$$|E_\omega x| < \epsilon_n, \quad \omega \subset \mu_n - \sigma_n, \quad \omega \in s_2.$$

We shall now show that

$$(*) \qquad |E_\omega x| < \sum_{j=1}^{n} \epsilon_j, \quad \omega \subset (\mu_1 + \cdots + \mu_n) - \sigma_n, \quad \omega \in s_2.$$

The statement (*) is true for $n = 1$. Assume that it is true for n and let

$$\omega \subset (\mu_1 + \cdots + \mu_{n+1}) - \sigma_{n+1}, \quad \omega \in s_2,$$

so that

$$\omega = \omega[(\mu_1 + \cdots + \mu_n) - \sigma_n] + \omega(\mu_{n+1} - \sigma_{n+1})$$

and $\omega = \omega_1 + \omega_2$, where

$$\omega_1 = \omega[(\mu_1 + \cdots + \mu_n) - \sigma_n](\mu_{n+1} - \sigma_{n+1})', \quad \omega_2 = \omega(\mu_{n+1} - \sigma_{n+1}).$$

By our induction assumption we have

$$|E_{\omega_1}x| < \sum_{j=1}^{n} \epsilon_j,$$

and since $\omega_2 \subset \mu_{n+1} - \sigma_{n+1}$ we have

$$|E_{\omega_2}x| < \epsilon_{n+1}.$$

Since ω_1 and ω_2 are disjoint, it follows that

$$|E_\omega x| \leq |E_{\omega_1}x| + |E_{\omega_2}x| < \sum_{j=1}^{n+1} \epsilon_j;$$

this proves (*). Now let $\xi_n = \mu_1 + \cdots + \mu_n$, so that ξ_n is open, increasing with n, $\xi_n \supset \sigma_n$, and $|E_\omega x| < \epsilon$ for every $\omega \in s_2$ with $\omega \subset \xi_n - \sigma_n$. Using 1.28, let

μ, $\nu \in s_2$, μ open, ν closed, $\mu \supset \sigma \supset \nu$, and $|E_\sigma x - E_{\sigma_1} x| < \epsilon$ for every $\sigma_1 \in s_2$ with $\mu \supset \sigma_1 \supset \nu$. We have

$$\mu \supset \bigcup_{m=1}^{\infty} \xi_m \mu \supset \mu \bigcup_{m=1}^{\infty} \sigma_m = \mu\sigma \supset \sigma \supset \nu.$$

Since ν is closed, there is an integer k_0 such that

$$\mu \supset \bigcup_{m=1}^{k} \xi_m \mu = \xi_k \mu \supset \nu, \qquad (k \geq k_0).$$

Thus

$$|E_\sigma x - E_{\xi_k \mu} x| < \epsilon, \qquad (k \geq k_0),$$

and since $\xi_k \mu - \sigma_k \subset \xi_k - \sigma_k$ we have, from (*),

$$|E_{\xi_k \mu} x - E_{\sigma_k} x| = |E_{\xi_k \mu - \sigma_k} x| < \epsilon, \qquad (k = 1, 2, \cdots).$$

Hence

$$|E_\sigma x - E_{\sigma_k} x| < \epsilon, \qquad (k \geq k_0);$$

this proves the lemma.

1.30. THEOREM. (*Assumptions* 1.7, 1.14.) *For each* $x \in X$ *the set function* $E_\sigma x$ *is countably additive on the Boolean algebra* s_3.

The conclusion of the theorem means that if σ, $\sigma_n \in s_3$, $\sigma_n \sigma_m$ is void for $n \neq m$, $\sigma = \bigcup_1^\infty \sigma_n$, then

$$\sum_{n=1}^{\infty} E_{\sigma_n} x = E_\sigma x.$$

The lemma is an immediate consequence of 1.25, 1.27, and 1.29.

1.31. DEFINITION. By a *Borel algebra* of sets we shall mean a Boolean algebra of sets which is closed under the operation of taking denumerable unions.

1.32. DEFINITION. The smallest Borel algebra of sets containing the Boolean algebra s_3 will be called the family of sets *measurable* T and will be denoted by $m(T)$.

In part of what follows we shall assume:

1.33. ASSUMPTION. *The space X is weakly complete.*

1.34. THEOREM. (*Assumptions* 1.7, 1.14, 1.33.) *The function E_e defined on s_3 to the set of bounded projections on X has a unique extension to $m(T)$ with the properties (the first of which ensures uniqueness):*

(i) $E_e x$ *is countably additive on* $m(T)$, $x \in X$;

(ii) $|E_e| \leq K$, $e \in m(T)$;

(iii) $E_{e_1} E_{e_2} = E_{e_1 e_2}$, $e_1, e_2 \in m(T)$;

(iv) $E_{e_1 \cup e_2} = E_{e_1} \cup E_{e_2}$, $e_1, e_2 \in m(T)$;

(v) $E_e' = E_{e'}$, $E_{\sigma(T)} = I$, $E_\phi = 0$, $e \in m(T)$, ϕ *void*;

(vi) $f(T)E_e = E_e f(T)$, $e \in m(T)$, $f \in F(\sigma(T))$.

For point sets e_n, e we mean by $e_n \longrightarrow e$ or $\lim_n e_n = e$ that

$$e = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} e_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} e_n,$$

and we recall that if

$$a_n \longrightarrow a, \quad b_n \longrightarrow b$$

then

$$a_n b_n \longrightarrow ab, \quad a_n \cup b_n \longrightarrow a \cup b, \quad a_n' \longrightarrow a'.$$

We define a transfinite sequence of Boolean algebras $\beta_0, \beta_1, \cdots$ as follows: $\beta_0 = s_3$ and β_α consists of all e such that there exists a sequence

$$e_n \in \bigcup_{\gamma < \alpha} \beta_\gamma$$

with $e_n \longrightarrow e$. Thus $m(T) = \bigcup_{\gamma < \omega} \beta_\gamma$, where ω is the first ordinal whose cardinal is that of a nonenumerable class. For each $x \in X$ and $x^* \in X^*$ there is, according to a well-known theorem of Hahn, a uniquely defined countably

additive numerical set function $m(e, x, x^*)$ on $m(T)$ such that

$$m(e, x, x^*) = x^*E_e x, \quad e \in s_3 = \beta_0, \quad x \in X, \quad x^* \in X^*.$$

We first show that for every $x \in X$ and $e \in m(T)$ there is a unique vector $x_e \in X$ such that

$$x^*x_e = m(e, x, x^*), \quad x^* \in X^*.$$

This is true for $e \in \beta_0$. Assume that it is true for $e \in \bigcup_{\gamma<\alpha} \beta_\gamma$ and let $e \in \beta_\alpha$, $e_n \in \bigcup_{\gamma<\alpha} \beta_\gamma$, $e_n \longrightarrow e$. Then

$$x^*x_{e_n} = m(e_n, x, x^*) \longrightarrow m(e, x, x^*), \quad x^* \in X^*.$$

Since X is weakly complete, there is a vector x_e with

$$x^*x_e = m(e, x, x^*), \quad x^* \in X^*.$$

This last equation shows that x_e is independent of the sequence $e_n \longrightarrow e$ and also is uniquely defined. Next consider the statements:

$$|m(e, x, x^*)| \leq K|x| \, |x^*|;$$

$$m(e, x_1 + x_2, x^*) = m(e, x_1, x^*) + m(e, x_2, x^*);$$

$$m(e, \alpha x, x^*) = \alpha m(e, x, x^*), \quad \alpha \text{ scalar.}$$

These relations hold for $e \in \beta_0$, and since $m(e, x, x^*)$ is continuous on $m(T)$ in the topology $e_n \longrightarrow e$ it is seen by induction that they hold for any $e \in m(T)$. They show that for fixed $e \in m(T)$ the vector x_e is linear and continuous in x; that is, for $e \in m(T)$ there is a bounded linear operator E_e on X with $E_e x = x_e$. Hence we have

$$x^*E_e x = m(e, x, x^*), \quad |E_e| \leq K, \quad e \in m(T), \quad x \in X, \quad x^* \in X^*.$$

The uniqueness of E_e follows from the uniqueness of $m(e, x, x^*)$ asserted by Hahn's theorem. That $E_e x$ is countably additive on $m(E)$ in the strong topology of operators and not merely in the weak topology follows from a theorem of Orlicz concerning weakly complete spaces. Banach has restated the theorem of Orlicz in a form to hold on any Banach space and it reads as follows [2]:

ORLICZ-BANACH THEOREM. *If all the partial sums of* $\sum x_n$ *converge weakly to an element, then the series* $\sum x_n$ *is unconditionally convergent.*

The countable additivity of $E_e x$ is a corollary. For let

$$e_n e_m = \phi,\ n \neq m,\ e_n \in m(T),\ e = \bigcup_{n=1}^{\infty} e_n .$$

For every set π of integers, let

$$e_\pi = \bigcup_{n \in \pi} e_n .$$

Then we have the weak series convergence:

$$\sum_{n \in \pi} E_{e_n} x = E_{e_\pi} x .$$

Thus, according to the Orlicz-Banach theorem, $\sum E_{e_n} x$ converges unconditionally in the strong vector topology. The sum is, of course, $E_e x$ since

$$x^* E_e x = \sum x^* E_{e_n} x,\ x^* \in X^* .$$

Thus we have proved statements (i) and (ii). Statement (iii) holds for e_1, $e_2 \in \beta_0$. We suppose that

$$E_a E_b = E_{ab},\ a, b \in \bigcup_{\gamma < \alpha} \beta_\gamma ,$$

and let

$$a, b_n \in \bigcup_{\gamma < \alpha} \beta_\gamma \text{ with } b_n \longrightarrow b \in \beta_\alpha .$$

Then

$$x^* E_a E_b x = m(b, x, x^* E_a) = \lim_n (b_n, x, x^* E_a) = \lim_n x^* E_a E_{b_n} x$$

$$= \lim_n x^* E_{a b_n} x = \lim_n m(a b_n, x, x^*) = m(ab, x, x^*) = x^* E_{ab} x .$$

Thus

$$E_a E_b = E_{ab} \text{ for } a \in \bigcup_{\gamma < \alpha} \beta_\gamma ,\ b \in \beta_\alpha .$$

Next choose

$$a_n \in \bigcup_{\gamma < a} \beta_\gamma \text{ with } a_n \longrightarrow a \in \beta_a.$$

Then

$$x^*E_aE_bx = m(a, E_bx, x^*) = \lim_n m(a_n, E_bx, x^*) = \lim_n x^*E_{a_n}E_bx$$

$$= \lim_n x^*E_{a_n b}x = \lim_n m(a_n b, x, x^*) = m(ab, x, x^*) = x^*E_{ab}x\ .$$

This proves (iii). Statements (v) and (vi) are readily proved by induction, and (iv) follows from (iii) and (v).

1.35. DEFINITION. If for each $e \in B$, the Borel sets in the complex plane, there is a bounded linear operator E_e on X, then the function E_e on B to the ring of operators on X is called a *resolution of the identity* in case $E_{e'} = I - E_e$, $E_{e_1}E_{e_2} = E_{e_1 e_2}$ for $e, e_1, e_2 \in B$, and x^*E_ex is countably additive on B for every $x \in X$, $x^* \in X^*$.

1.36. LEMMA. *A resolution of the identity has the further properties*

(i) E_ex *is countably additive on* B, $x \in X$;

(ii) $\sup_{e \in B} |E_e| < \infty$;

(iii) $E_{e_1}E_{e_2} = E_{e_1e_2}$, $E_{e_1 \cup e_2} = E_{e_1} \cup E_{e_2}$, $e_1, e_2 \in B$;

(iv) $E_e' = E_{e'}$, $E_\phi = 0$, $E_p = I$, $e \in B$, ϕ *void*, p = *the whole plane.*

Statement (i) follows from the Orlicz-Banach theorem, and (ii) from the principle of uniform boundedness. $E_\phi = 0$ since E_e is additive in e; hence $I = E_\phi' = E_p$. The second part of (iii) follows from the first part and (iv).

1.37. DEFINITION. A resolution of the identity E_e is called a *resolution of the identity for the linear operator* T in case

$$TE_e = E_eT \quad \text{and} \quad \sigma(E_eX) \subset \overline{e}, \quad e \in B.$$

1.38. LEMMA. *Let* X *be weakly complete, and let* T *be a bounded linear operator in* X *whose spectrum is nondense. Then* T *has a resolution of the identity if and only if it satisfies the conditions* 1.7, 1.14 *and*:

1.39. *For every complex number* λ *and every* $\epsilon > 0$ *there is an* s_3 *set of diameter* $< \epsilon$ *and containing* λ *as an interior point.*

Furthermore, when T *has a resolution of the identity* E_e *it is unique and has*

the following properties:

(i) *if* σ *is closed, then* $E_\sigma x = x$ *if and only if* $\sigma(x) \subset \sigma$;

(ii) $\sigma(E_\sigma x) \subset \bar{\sigma}\,\sigma(x)$, $\sigma \in B$, $x \in X$.

To prove the sufficiency of the conditions it is, in view of 1.34, sufficient to show that $\sigma(X_e) \subset \bar{e}$, $e \in B$, where $X_e = E_e X$. Let ξ be a complex number not in $\bar{e}\,\sigma(T)$, and with each $\lambda \in \bar{e}\,\sigma(T)$ associate an s_3 set σ_λ whose diameter is less than 1/2 the distance from ξ to $\bar{e}\,\sigma(T)$ and such that λ is in the interior of σ_λ. A finite number $\sigma_1, \cdots, \sigma_n$ of these sets σ_λ covers $\bar{e}\,\sigma(T)$, and since s_3 is a Boolean algebra the set $\sigma = \bigcup_{i=1}^n \sigma_i \in s_3$. Since $s_3 \subset s_1$, we see from 1.21 that $\sigma(X_\sigma) \subset \bar{\sigma}$. But since

$$E_e = E_e E_{\sigma(T)} = E_{e\sigma(T)} \subset E_{\bar{e}\sigma(T)} \subset E_\sigma,$$

we have $X_e \subset X_\sigma$. Since $\xi \notin \bar{\sigma}$ and $\sigma(X_\sigma) \subset \bar{\sigma}$, the operator $\xi - T$ is one-to-one on X_σ to all of X_σ, and hence likewise on the invariant subspace X_e. Thus

$$\xi \in \rho(X_e),\ (\bar{e}\,\sigma(T))' \subset \rho(X_e),\ \bar{e} \supset \bar{e}\,\sigma(T) \supset \sigma(X_e).$$

It will now be shown that if T has a resolution of the identity E_e, then it is unique. Let A_e also be a resolution of the identity for T. Let σ, σ_1 be disjoint closed sets of complex numbers. Since $\sigma(E_{\sigma_1} X) \subset \sigma_1$, the function $T(\xi) E_{\sigma_1} x$ analytic on $\rho(T)$ has an analytic extension to σ_1'. Hence also the function

$$T(\xi) A_\sigma E_{\sigma_1} x = A_\sigma T(\xi) E_{\sigma_1} x$$

analytic on $\rho(T)$ has an analytic extension to σ_1'. Since $\sigma(A_\sigma X) \subset \sigma$, the function $T(\xi) A_\sigma E_{\sigma_1} x$ has an analytic extension to σ'. Thus

$$\rho(A_\sigma E_{\sigma_1} x) \supset \sigma_1' \cup \sigma' = (\sigma_1 \sigma)' = \text{the whole plane};$$

that is, $\sigma(A_\sigma E_{\sigma_1} x)$ is void. By 1.12 we have $A_\sigma E_{\sigma_1} = 0$. Likewise $E_{\sigma_1} A_\sigma = 0$. Now there are closed sets $\sigma_n \subset \sigma_{n+1} \longrightarrow \sigma_1'$, and hence $A_{\sigma_n} x \longrightarrow A_{\sigma_1'} x = A_{\sigma_1'} x$. Then

$$0 = A_{\sigma_n} E_{\sigma_1} x = E_{\sigma_1} A_{\sigma_n} x \longrightarrow E_{\sigma_1} A_{\sigma_1'} x = A_{\sigma_1'} E_{\sigma_1} x.$$

So

$$0 = E_{\sigma_1}(I - A_{\sigma_1}) = (I - A_{\sigma_1}) E_{\sigma_1},\ E_{\sigma_1} = E_{\sigma_1} A_{\sigma_1} = A_{\sigma_1} E_{\sigma_1}.$$

Similarly,

$$A_{\sigma_1} = E_{\sigma_1} A_{\sigma_1} = E_{\sigma_1}.$$

Since $A_\sigma x$ and $E_\sigma x$ are both countably additive on B and coincide for closed sets σ, they must coincide for all $\sigma \in B$. We shall now show that if T has a resolution of the identity E_e, and σ is closed, then $E_\sigma x = x$ if and only if $\sigma(x) \subset \sigma$. Let σ be closed and $E_\sigma x = x$. Since $\sigma(X_\sigma) \subset \sigma$, the function

$$T(\xi) E_\sigma x = T(\xi) x$$

analytic on $\rho(T)$ has an analytic extension to σ'. Thus $\rho(x) \supset \sigma'$, $\sigma(x) \subset \sigma$. Conversely, let $\sigma(x) \subset \sigma$, where σ is closed. Let σ_n be closed, $\sigma_n \subset \sigma_{n+1} \longrightarrow \sigma'$, so that

$$x = E_\sigma x + E_{\sigma'} x = E_\sigma x + E_{\sigma'} x = E_\sigma x + \lim_n E_{\sigma_n} x.$$

Since $\sigma_n \sigma(T)$ and σ are disjoint, closed, and $\sigma_n \sigma(T)$ is bounded, there is an admissible contour C_n surrounding $\sigma_n \sigma(T)$ and excluding σ. Also, since

$$\sigma(X_{\sigma_n}) \subset \sigma_n \sigma(T),$$

we have

$$E_{\sigma_n} x = \frac{1}{2\pi i} \int_{C_n} (E_{\sigma_n} x)(\xi)\, d\xi = \frac{1}{2\pi i} \int_{C_n} E_{\sigma_n} x(\xi)\, d\xi.$$

However, since $\sigma(x) \subset \sigma$, the function $x(\xi)$ is analytic on and within C_n. Thus

$$E_{\sigma_n} x = 0 \quad \text{and} \quad x = E_\sigma x.$$

In this proof we have used the equality

$$(E_{\sigma_n} x)(\xi) = E_{\sigma_n} x(\xi),$$

which is clear from 1.2 since both functions are analytic extensions of

$$T(\xi) E_{\sigma_n} x = E_{\sigma_n} T(\xi) x$$

on $\rho(T)$. We shall next show that $\sigma(E_\sigma x) \subset \overline{\sigma} \sigma(x)$ for every $\sigma \in B$ and $x \in X$. Since $\sigma(X_\sigma) \subset \overline{\sigma}$, it is clear that $\sigma(E_\sigma x) \subset \overline{\sigma}$. Since

$$(E_\sigma x)(\xi) = E_\sigma x(\xi) \text{ for } \xi \in \rho(T),$$

we have $\rho(E_\sigma x) \supset \rho(x)$, $\sigma(E_\sigma x) \subset \sigma(x)$. Thus $\sigma(E_\sigma x) \subset \overline{\sigma}\sigma(x)$. The necessity of 1.7 follows immediately from (i). Next we shall show the necessity of 1.14. As pointed out in 1.36 (ii) it follows from the principle of uniform boundedness that

$$\sup_{e \in B} |E_e| \equiv K < \infty.$$

Let $\sigma = \sigma(x)$, $\sigma_1 = \sigma(y)$ be disjoint. Then, by (i), $E_\sigma x = x$; and, by (ii) and 1.12, $E_\sigma y = 0$. Thus

$$|x| = |E_\sigma(x + y)| \leq K|x + y|.$$

Finally we prove the necessity of 1.39. Let σ be a circle and its interior. Let σ_n be closed and $\sigma_n \subset \sigma_{n+1} \longrightarrow \sigma'$. Then $E_\sigma x + E_{\sigma_n} x \longrightarrow x$. Since

$$\sigma(E_\sigma x) \subset \sigma\sigma(x) \quad \text{and} \quad \sigma(E_{\sigma_n} x) \subset \sigma_n \sigma(x) \subset \sigma'\sigma(x),$$

we see that σ is an s_3 set.

II. Operators whose spectra lie in a rectifiable Jordan curve

In order to apply the final lemma of §1 we find it necessary to restrict further the nature of the spectrum $\sigma(T)$. Later we shall be interested in specific cases where the spectrum lies either in a straight line segment or in a circle, and these two cases may be treated simultaneously by restricting the spectrum in the manner described in the next paragraph. When this is done and a rate of growth is imposed upon the resolvent (Assumption 2.1), it is possible to give conditions, of a nature much more applicable than those of the preceding lemma, which will ensure the existence of a resolution of the identity. This may be accomplished in a variety of ways, and some of the sets of conditions given are necessary as well as sufficient.

Throughout §2 it is assumed that the spectrum $\sigma(T)$ is contained in a closed rectifiable Jordan curve Γ_0. In order to be able to manipulate in a fairly simple fashion the analytical operations involved, we suppose further that Γ_0 is embedded in a one parameter family $\Gamma_\delta (-\delta_0 \leq \delta \leq \delta_0,\ 0 < \delta_0 \leq 1/2)$ of closed rectifiable Jordan curves, with Γ_{δ_1} interior to Γ_{δ_2} for $-\delta_0 \leq \delta_1 < \delta_2 \leq \delta_0$. The curve Γ_δ is defined by a function

$$\xi = \xi(\lambda, \delta),\ -1 \leq \lambda \leq 1, \text{ with } \xi(-1, \delta) = \xi(1, \delta).$$

We suppose that the parameter δ has been chosen in such a way that $|\delta|$ is the distance measured along the arc $\xi(\lambda, \delta)$ from the point $\xi(\lambda, 0)$ to the point $\xi(\lambda, \delta)$, and that the arcs $\xi(\lambda, \delta)$, $-\delta_0 \leq \delta \leq \delta_0$, for different values of λ, do not intersect. Finally we suppose that for each $\delta \in [-\delta_0, \delta_0]$ the point $\xi(\lambda, \delta)$ traces, as λ increases from -1 to $+1$, the curve Γ_δ in a counterclockwise direction.

2.1. ASSUMPTION. *The spectrum $\sigma(T)$ of T is contained in the rectifiable Jordan curve Γ_0 described above, and the rate of growth of the resolvent $T(\xi)$ for $\xi = \xi(\lambda, \delta)$ near the spectrum is restricted by the condition*

$$\limsup_{\delta \to 0} \; |\delta^{\nu(\lambda)} T(\xi)| < \infty, \quad -1 \leq \lambda \leq 1,$$

where $\nu(\lambda)$ is a nonnegative function defined for $-1 \leq \lambda \leq 1$.

Since the function $\nu(\lambda)$ may be increased without destroying the above property, and since $\delta_0 \leq 1/2$, every operator T satisfying 2.1 has an index function $\nu(\lambda)$ according to the following definition.

2.2. DEFINITION. Any nonnegative integer-valued function $\nu(\lambda)$ satisfying the condition

$$|\delta^{\nu(\lambda)} \, T(\xi)| \leq 1, \quad 0 < |\delta| < \delta_0, \quad \lambda \in [-1, 1],$$

will be called an *index function* for T.

It might be pointed out that if $\nu(\lambda)$ is defined only on the set $\Lambda \subset [-1, 1]$ consisting of all those λ for which $\xi(\lambda, 0) \in \sigma(T)$, and the above inequality is valid for $\lambda \in \Lambda$, then T has an index function. It is not assumed that $\nu(\lambda)$ is bounded, and it is erroneous to conclude that T has a bounded index function providing $\nu(\lambda)$ is bounded on Λ. Elementary operators exist for which every index function is unbounded and at the same time every index function is bounded on Λ.

2.3. LEMMA. (*Assumption* 2.1.) *There is an index function $\nu(\lambda)$ for T with the property that every interval of positive length contains an interval of positive length upon which $\nu(\lambda)$ is constant.*

Let Δ be a closed subinterval of $[-1, 1]$. Let Δ_n be the set of all $\lambda \in \Delta$ such that

$$|\delta^n \, T(\xi)| \leq 1, \quad 0 < |\delta| \leq \delta_0.$$

Since for fixed $\delta \neq 0$ the point $\xi = \xi(\lambda, \delta)$, and thus $T(\xi)$, is continuous in λ, it follows that Δ_n is closed. By 2.1 we see that $\Delta = \cup\Delta_n$, and thus the desired conclusion follows from the category theorem of Baire.

2.4. LEMMA. (*Assumption* 2.1.) *Let* $\nu(\lambda)$ *be an index function for* T: $0 < \delta \leq \delta_0$; $\lambda_1, \lambda_2 \in [-1, 1]$; $\xi_1 = \xi(\lambda_1, 0)$, $\xi_2 = \xi(\lambda_2, 0)$ *distinct points of* Γ_0; *and* $\nu_1 = \nu(\lambda_1)$, $\nu_2 = \nu(\lambda_2)$. *Let* $C(\lambda_1, \lambda_2)$ *be the rectifiable contour (oriented counterclockwise) composed of the following four arcs. There are two cases: If* $\lambda_1 < \lambda_2$ *the arcs are*

$$\xi(\lambda_1, \mu),\ -\delta \leq \mu \leq \delta;\quad \xi(\lambda_2, \mu),\ -\delta \leq \mu \leq \delta;$$

$$\xi(\lambda, \delta),\ \lambda_1 \leq \lambda \leq \lambda_2;\quad \textit{and}\quad \xi(\lambda, -\delta),\ \lambda_1 \leq \lambda \leq \lambda_2;$$

whereas if $\lambda_2 < \lambda_1$ *we use the arcs*

$$\xi(\lambda_1, \mu),\ -\delta \leq \mu \leq \delta;\quad \xi(\lambda_2, \mu),\ -\delta \leq \mu \leq \delta;$$

$$\xi(\lambda, \delta),\quad \lambda \notin (\lambda_2, \lambda_1);\quad \textit{and}\quad \xi(\lambda, \delta),\quad \lambda \notin (\lambda_2, \lambda_1).$$

Let $P(\xi)$ *be a polynomial in* ξ. *Then*

$$I(\lambda, \lambda_2) = \frac{1}{2\pi i} \int_{C(\lambda_1, \lambda_2)} P(\xi)\,(\xi - \xi_1)^{\nu_1}\,(\xi - \xi_2)^{\nu_2}\,T(\xi)\,d\xi$$

exists as a Riemann integral, is independent of δ, *and has the properties*

$$\lim_{0<|\lambda_2-\lambda_1|\to 0} I(\lambda_1, \lambda_2) = 0,\quad \sigma(I(\lambda_1, \lambda_2)x) \subset [\xi_1, \xi_2],$$

where $[\xi_1, \xi_2]$ *is the closed subarc of* Γ_0 *consisting of all points of* Γ_0 *which are inside or on the contour* $C(\lambda_1, \lambda_2)$.

The integrand is defined and continuous at every point of the contour $C(\lambda_1, \lambda_2)$ except at the points ξ_1 and ξ_2. Since $\nu(\lambda)$ is an index function for T, the integrand is bounded on the curve $C(\lambda_1, \lambda_2)$. Hence $I(\lambda_1, \lambda_2)$ exists. It is clearly independent of δ since the integrand has its only singularities on the curve Γ_0. Now let $\lambda_1 < \lambda_2$ so that $C(\lambda_1, \lambda_2)$ consists of the arcs AB, BC, CD, DA, where A, B, C, D are given by the complex numbers

$$\xi(\lambda_2, -\delta),\ \xi(\lambda_1, -\delta),\ \xi(\lambda_1, \delta),\ \xi(\lambda_2, \delta).$$

Let

$$K = \text{l.u.b.}\,|P(\xi)|,\ \xi \in \Gamma_{\delta_0};\quad K_\delta = \text{l.u.b.}\,|T(\xi)|,\ \xi \in \Gamma_\delta.$$

Let $0 < \epsilon < \delta_0$, $\delta = \epsilon/2$, and let γ_ϵ be such that for $0 < \lambda_2 - \lambda_1 < \gamma_\epsilon$ we have the lengths of the arcs AB, CD, and $\xi_1\xi_2$ (on Γ_0) all less than δ and also less than $\epsilon/K(\delta)$. Then the integrand $f(\xi)$ defining $I(\lambda_1, \lambda_2)$ for $\xi = \xi(\lambda, \mu)$ satisfies

$$|f(\xi)| \le K|\xi - \xi_2|^{\nu_2}\,|\mu|^{\nu_1}\ |T(\xi)| \le K\,|\xi - \xi_2|^{\nu_2},\ \xi \in BC.$$

Since, for $\xi \in BC$, we have

$$|\xi - \xi_2| \le |\xi_2 - \xi_1| + |\xi_1 - \xi| \le \delta + \mu \le 2\delta < 1,$$

it follows that for $\xi \in BC$, and likewise for $\xi \in DA$, we have the bound

$$|f(\xi)| \le K.$$

For ξ on AB or CD, we have

$$|f(\xi)| \le KK_\delta\,|\xi - \xi_2|^{\nu_2}\,|\xi - \xi_1|^{\nu_1} \le KK_\delta.$$

Thus if $0 < \lambda_2 - \lambda_1 < \gamma_\epsilon$, then

$$|I(\lambda_1, \lambda_2)| \le \frac{1}{2\pi}\,[4\delta K + 2KK_\delta\,\epsilon/K_\delta] = 2K\epsilon/\pi.$$

Now let $x \in X$, $\eta \in \rho(T)$, and η outside of $C(\lambda_1, \lambda_2)$. Then

$$\begin{aligned} &T(\eta)\,I(\lambda_1, \lambda_2)x \\ &= \frac{1}{2\pi i}\int_{C(\lambda_1,\lambda_2)} P(\xi)\,(\xi - \xi_1)^{\nu_1}\,(\xi - \xi_2)^{\nu_2}\,(\eta - \xi)^{-1}\,T(\xi)x\,d\xi \\ &\quad + \frac{1}{2\pi i}\,T(\eta)x\int_{C(\lambda_1,\lambda_2)} P(\xi)\,(\xi - \xi_1)^{\nu_1}\,(\xi - \xi_2)^{\nu_2}\,(\eta - \xi)^{-1}\,d\xi \\ &= \frac{1}{2\pi i}\int_{C(\lambda_1,\lambda_2)} P(\xi)\,(\xi - \xi_1)^{\nu_1}\,(\xi - \xi_2)^{\nu_2}\,(\eta - \xi)^{-1}\,T(\xi)x\,d\xi, \end{aligned}$$

and the last integral gives an analytic extension of $T(\eta)\,I(\lambda_1, \lambda_2)x$ for all η outside $C(\lambda_1, \lambda_2)$.

2.5. LEMMA. (*Assumption* 2.1.) *The operator* T *satisfies the conditions* 1.1 *and* 1.7.

Condition 1.1 clearly is satisfied since a rectifiable Jordan curve is nondense in the plane. To prove 1.7, let σ be a closed subset of the spectrum, $x_n \in [\sigma]$, $x_n \longrightarrow x$. We make an indirect proof by supposing that there is a point $\xi \in \sigma(x)\sigma'$. According to the Heine-Borel theorem there are closed disjoint subarcs $\Delta_1, \cdots, \Delta_p$ of Γ_0 with $\sigma \subset \Delta = \Delta_1 \cup \cdots \cup \Delta_p$ and $\xi \in \Delta'$. Let

$$-1 \leq \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_p < \mu_p < 1$$

be such that the arc Δ_j, $(j = 1, \cdots, p)$ is defined by $\xi(\lambda, 0)$, $\lambda_j \leq \lambda \leq \mu_j$. Let $C_j = C(\lambda_j, \mu_j)$ as defined in 2.4. Since $x_n(\xi)$ has its singularities in the set Δ, we see that

$$\int_C \prod_{j=1}^{p} (\xi - \xi_j)^{\nu(\lambda_j)} (\xi - \zeta_j)^{\nu(\mu_j)} x_n(\xi) d\xi = 0,$$

where

$$\xi_j = \xi(\lambda_j, 0), \quad \zeta_j = \xi(\mu_j, 0),$$

and C is any contour of the form $C = C(\lambda_1, \lambda_2)$ providing $[\lambda_1, \lambda_2]$ is disjoint with Δ. Since, by 2.4,

$$I\left(\lambda_j - \frac{1}{n}, \lambda_j\right) \longrightarrow 0, \quad I\left(\mu_j, \mu_j + \frac{1}{n}\right) \longrightarrow 0,$$

we have,

$$\prod_{j=1}^{p} (T - \xi_j)^{\nu(\lambda_j)} (T - \zeta_j)^{\nu(\mu_j)} x_n$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{p} \int_{C_j} \prod_{k=1}^{p} (\xi - \xi_k)^{\nu(\lambda_k)} (\xi - \zeta_k)^{\nu(\mu_k)} x_n(\xi) d\xi.$$

Since the convergence, as $n \longrightarrow \infty$, of the integrands on the right side of the above equality is bounded, we may in this equation replace x_n by x and conclude from 2.4 and 1.6 that

$$\sigma\left(\prod_{j=1}^{p} (T - \xi_j)^{\nu(\lambda_j)} (T - \zeta_j)^{\nu(\mu_j)} x\right) \subset \Delta.$$

The desired contradiction will be obtained as soon as we show that the above inclusion implies that $\sigma(x) \subset \Delta$. But this implication follows immediately by induction from the statement

$$\sigma(z) \subset \sigma((\xi - T)z) \cup (\xi),$$

which is verified as follows. Since

$$T(\mu)z = (\xi - \mu)^{-1} \{T(\mu)(\xi - T)z - z\}, \quad \mu \neq \xi,$$

any point μ other than $\mu = \xi$ to which $T(\mu)(\xi - T)z$ has an analytic extension must be in $\rho(z)$; that is,

$$\rho((\xi - T)z) \subset \rho(z) \cup (\xi).$$

Thus

$$\sigma((\xi - T)z) \supset \sigma(z) \cap (\xi)'.$$

2.6. LEMMA. (*Assumption* 2.1.) *Let* $\nu(\lambda)$ *be an index function for* T. *For every complex number* ξ *and every nonnegative integer* n, *define*

$$\mathfrak{M}_\xi^n = \mathop{\mathcal{E}}_x [(T - \xi)^n x = 0], \quad \mathfrak{N}_\xi^n = (T - \xi)^n X.$$

Then for $\xi = \xi(\lambda, 0)$ *we have*

$$\mathfrak{M}_\xi^{\nu(\lambda)} = \mathfrak{M}_\xi^n, \quad \overline{\mathfrak{N}}_\xi^{\nu(\lambda)} = \overline{\mathfrak{N}}_\xi^n, \qquad n \geq \nu(\lambda).$$

Let $\xi_1 = \xi(\lambda, \delta)$; then

$$(\xi - T)\, T(\xi_1) = (\xi - \xi_1)\, T(\xi_1) + I.$$

Now assume, for the purpose of induction, that (the preceeding identity is the case when $j = 1$)

$$(\xi - T)^j\, T(\xi_1) = (\xi - \xi_1)^j\, T(\xi_1) + (\xi - \xi_1)^{j-1} + (\xi - \xi_1)^{j-2} (\xi - T)$$

$$+ \cdots + (\xi - \xi_1)(\xi - T)^{j-2} + (\xi - T)^{j-1}.$$

Multiplying this by $\xi - T$, we have

$$(\xi - T)^{j+1} T(\xi_1) = (\xi - \xi_1)^j [(\xi - \xi_1) T(\xi_1) + I] + (\xi - \xi_1)^{j-1} (\xi - T) + \cdots + (\xi - \xi_1)(\xi - T)^{j-1} + (\xi - T)^j,$$

and hence

$$(*) \quad (\xi - T)^{j+1} T(\xi_1) = (\xi - \xi_1)^{j+1} T(\xi_1) + (\xi - \xi_1)^j + (\xi - \xi_1)^{j-1} (\xi - T) + \cdots + (\xi - \xi_1)(\xi - T)^{j-1} + (\xi - T)^j.$$

Now in (*) put $j = \nu(\lambda)$, and operate on a vector $x \in \mathfrak{M}_\xi^{\nu(\lambda)+1}$. We get

$$0 = (\xi - \xi_1)^{\nu(\lambda)+1} T(\xi_1)x + (\xi - \xi_1)^{\nu(\lambda)} x + \cdots + (\xi - \xi_1)(\xi - T)^{\nu(\lambda)-1} x + (\xi - T)^{\nu(\lambda)} x.$$

If we let $\delta \longrightarrow 0$, then $\xi_1 \longrightarrow \xi$; and since δ measures the arc $\xi\xi_1$, we have

$$|(\xi - \xi_1)^{\nu(\lambda)+1} T(\xi_1)x| = \left| \delta^{\nu(\lambda)+1} T(\xi_1) \left(\frac{\xi - \xi_1}{\delta} \right)^{\nu(\lambda)+1} \right| \leq \delta \longrightarrow 0.$$

This shows that $x \in \mathfrak{M}_\xi^{\nu(\lambda)}$. Thus

$$\mathfrak{M}_\xi^{\nu(\lambda)+1} \subset \mathfrak{M}_\xi^{\nu(\lambda)},$$

and hence

$$\mathfrak{M}_\xi^{\nu(\lambda)} = \mathfrak{M}_\xi^n, \quad n \geq \nu(\lambda).$$

Using (*) again except now with x arbitrary, we may write

$$(\xi - T)^{\nu(\lambda)} x = (\xi - T)^{\nu(\lambda)+1} T(\xi_1)x + O(\delta),$$

where $O(\delta)$ is a vector which approaches zero with δ. Thus

$$\mathfrak{N}_\xi^{\nu(\lambda)} \subset \overline{\mathfrak{N}}_\xi^{\nu(\lambda)+1} \subset \overline{\mathfrak{N}}_\xi^{\nu(\lambda)},$$

and so

$$\overline{\mathfrak{N}}_\xi^{\nu(\lambda)} = \overline{\mathfrak{N}}_\xi^{n}, \quad n \geq \nu(\lambda).$$

2.7. DEFINITION. (Assumption 2.1.) Let $\nu(\lambda)$ be an index function for T. As shown in 2.6, the manifolds $\mathfrak{M}_\xi^{\nu(\lambda)}$ and $\overline{\mathfrak{N}}_\xi^{\nu(\lambda)}$ are independent of the index function $\nu(\lambda)$. They will henceforth be designated by the symbols $\mathfrak{M}_\xi$, $\mathfrak{N}_\xi$, respectively.

Of the three principal conditions of Lemma 1.38, namely 1.7, 1.14, and 1.39, the condition 1.7 has already been shown (by 2.5) to be a consequence of 2.1. Neither one of the other two conditions is a consequence of 2.1 alone, and we concentrate our attention now on restating 1.39 in a more applicable form. The following assumption 2.8, which may be used to replace 1.39, turns out to be necessary as well as sufficient. It has the disadvantage of not always being easily applicable. *It should be noted, however, that in the case of operators with only continuous spectra it is trivially satisfied. Or more generally, if every subarc of* Γ_0 *contains points either in the resolvent set or in the continuous spectrum then* 2.8 *is automatically satisfied.*

2.8. ASSUMPTION. (*See Assumption* 2.1.) *For every* ξ *in a dense set on* Γ_0,

$$\mathfrak{M}_\xi + \mathfrak{N}_\xi \text{ is dense in } X.$$

2.9. LEMMA. (*Assumptions* 2.1, 1.14.) *The set of points* ξ *on the curve* Γ_0 *for which* $\mathfrak{M}_\xi\ \mathfrak{N}_\xi \neq 0$ *is nondense on* Γ_0. *Moreover,* $\mathfrak{M}_\xi\ \mathfrak{N}_\xi = 0$ *for every* ξ, *interior to a subarc of* Γ_0 *upon which some index function is constant. For such* ξ, *the set* $\mathfrak{M}_\xi \oplus \mathfrak{N}_\xi$ *is closed. Thus if* 2.8 *is satisfied then*

$$\mathfrak{M}_\xi \oplus \mathfrak{N}_\xi = X,$$

for every ξ *in a set dense on* Γ_0.

In view of 2.3, the first statement is a consequence of the second. Accordingly, let $\nu(\lambda)$ be an index function which is constant on the interval $[\lambda_1, \lambda_2]$, and let $\xi = \xi(\lambda, 0)$ where $\lambda_1 < \lambda < \lambda_2$. Let

$$\nu = \nu(\lambda), \quad \xi_n = \xi(\lambda_n, 0), \quad \zeta_n = \xi(\mu_n, 0),$$

where

$$\lambda_1 < \lambda_n < \lambda < \mu_n < \lambda_2 \quad \text{and} \quad \lambda_n \longrightarrow \lambda, \quad \mu_n \longrightarrow \lambda.$$

Take $P(\xi) \equiv 1$ in 2.4, so that

$$I(\lambda_n, \mu_n) + I(\mu_n, \lambda_n) = (T - \xi_n)^\nu (T - \zeta_n)^\nu \longrightarrow (T - \xi)^{2\nu}.$$

Thus, it is seen from 2.4 that the vector $(T - \xi)^{2\nu}x$ is the limit of vectors $x_n = I(\mu_n, \lambda_n)x$ and $\xi \in \rho(x_n)$. Thus 2.6 and 2.7 show that every vector $x \in \mathfrak{N}_\xi$ is the limit of a sequence $\{x_n\}$ with $\xi \in \rho(x_n)$. Now if $x \in \mathfrak{M}_\xi \mathfrak{N}_\xi$ we have $(T - \xi)^\nu x = 0$, and hence $\sigma(x) \subset (\xi)$. If $\xi \in \rho(x_n)$ and $x_n \longrightarrow x$, then $\xi \in \rho(-x_n)$; and, by 1.14,

$$|x| \leq K\,|x - x_n| \longrightarrow 0, \quad x = 0, \text{ and } \mathfrak{M}_\xi \mathfrak{N}_\xi = 0.$$

Finally we show that $\mathfrak{M}_\xi \oplus \mathfrak{N}_\xi$ is closed. Let

$$x_n + y_n \longrightarrow z, \text{ where } x_n \in \mathfrak{M}_\xi, \ y_n \in \mathfrak{N}_\xi.$$

Since $y_n - y_m \in \mathfrak{N}_\xi$, there are vectors u_{nm} with

$$|y_n - y_m - u_{nm}| < (nm)^{-1}, \quad \xi \in \rho(u_{nm}).$$

By 1.14, then,

$$|x_n - x_m| \leq K\,|x_n - x_m - u_{nm}| \leq K\{|x_n - x_m + y_n - y_m| + (nm)^{-1}\} \longrightarrow 0.$$

2.10. LEMMA. (*Assumptions* 2.1, 1.14, 2.8.) *For every pair* ξ, ξ_1 *of distinct points in a set dense on* Γ_0 *we have*

$$\mathfrak{M}_\xi \oplus \mathfrak{M}_{\xi_1} \oplus \mathfrak{N}_\xi \mathfrak{N}_{\xi_1} = X;$$

and $\mathfrak{N}_\xi \mathfrak{N}_{\xi_1}$ *is the closure of the manifold* $(T - \xi)^\nu (T - \xi_1)^{\nu_1} X$, *where* $\nu = \nu(\lambda)$, $\nu_1 = \nu(\lambda_1)$, $\xi = \xi(\lambda, 0)$, $\xi_1 = \xi(\lambda_1, 0)$, *and* $\nu(\lambda)$ *is an index function for* T.

For every ξ in a set Γ dense on Γ_0 we have, by 2.9, projections A_ξ and A'_ξ with

$$A_\xi + A'_\xi = I, \quad A_\xi X = \mathfrak{M}_\xi, \quad A'_\xi X = \mathfrak{N}_\xi.$$

Since for $\mu \in \rho(T)$, we have $T(\mu)\mathfrak{N}_\xi \subset \mathfrak{N}_\xi$ and $T(\mu)\mathfrak{M}_\xi \subset \mathfrak{M}_\xi$, it follows that $T(\mu)A_\xi x = A_\xi T(\mu)x$, and thus $\rho(x) \subset \rho(A_\xi x)$. Since

$$(T - \xi_1)^{\nu_1} A_{\xi_1} = 0,$$

we have $\sigma(A_{\xi_1} x) \subset (\xi_1)$, and hence $\xi \in \rho(A_{\xi_1} x) \subset \rho(A_\xi A_{\xi_1} x)$. Since $\sigma(A_\xi y) \subset (\xi)$, we have $y \in X$; then $A_\xi A_{\xi_1} = 0$ by 1.12 and 2.5. Similarly $A_{\xi_1} A_\xi = 0$. Thus

$$I = (A_\xi + A'_\xi)(A_{\xi_1} + A'_{\xi_1}) = A_\xi + A'_\xi + A'_\xi A'_{\xi_1},$$

and this proves the first statement of the lemma. Now let $x \in \mathfrak{N}_\xi^\nu \mathfrak{N}_{\xi_1}^{\nu_1}$, so that

$$x = (T - \xi)^\nu y, \quad y = A_{\xi_1} y + A'_{\xi_1} y, \quad x = A'_{\xi_1} x.$$

By 2.7 there are vectors v_n with $(T - \xi_1)^{\nu_1} v_n \longrightarrow A'_{\xi_1} y$ and

$$x = \lim_n (T - \xi)^\nu [A_{\xi_1} y + (T - \xi_1)^{\nu_1} v_n],$$

$$x = A'_{\xi_1} x = \lim_n (T - \xi)^\nu (T - \xi_1)^{\nu_1} v_n.$$

Thus $\mathfrak{N}_\xi^\nu \mathfrak{N}_{\xi_1}^{\nu_1}$ as well as its closure $\mathfrak{N}_\xi \mathfrak{N}_{\xi_1}$ is contained in the closure

$$(T - \xi)^\nu (T - \xi_1)^{\nu_1} X.$$

Obviously $(T - \xi)^\nu (T - \xi_1)^{\nu_1} X \subset \mathfrak{N}_\xi \mathfrak{N}_{\xi_1}$, and so the proof is complete.

2.11. THEOREM. (*Assumptions* 1.14, 2.1, 2.8.) *Every Borel set is measurable T; and if X is weakly complete then T has a resolution of the identity.*

Let $-1 \leq \alpha < \beta < \gamma < \delta < 1$, and choose $\lambda_1, \lambda_2, \mu_1, \mu_2$ so that

$$\alpha < \lambda_1 < \lambda_2 < \beta, \quad \gamma < \mu_1 < \mu_2 < \delta,$$

and such that there is an index function $\nu(\lambda)$ which is constant on the intervals $[\lambda_1, \lambda_2]$, $[\mu_1, \mu_2]$. This is possible in view of 2.3. Since an index function may be increased without destroying the property of being an index function, we shall suppose that $\nu(\lambda)$ has the constant value ν on both of the intervals $[\lambda_1, \lambda_2]$, $[\mu_1, \mu_2]$. Let

$$\xi_1 = \xi(\lambda_1, 0),\ \xi_2 = \xi(\lambda_2, 0),\ \zeta_1 = \xi(\mu_1, 0),\ \zeta_2 = \xi(\mu_2, 0),$$

and

$$f(\xi) = (\xi - \xi_1)^{\nu}\,(\xi - \xi_2)^{\nu}\,(\xi - \zeta_1)^{\nu}\,(\xi - \zeta_2)^{\nu}.$$

Then for appropriate choices of the polynomial $P(\xi)$ in 2.4 we have

$$f(T) = I(\lambda_1, \lambda_2) + I(\mu_1, \mu_2) + I(\mu_2, \lambda_1) + I(\lambda_2, \mu_1).$$

By 2.10 there are points λ, μ with $\lambda_1 < \lambda < \lambda_2$, $\mu_1 < \mu < \mu_2$, and such that for $\xi = \xi(\lambda, 0)$, $\zeta = \xi(\mu, 0)$ we have

$$(*) \qquad \mathfrak{M}_{\xi} \oplus \mathfrak{M}_{\zeta} \oplus \mathfrak{N}_{\xi}\,\mathfrak{N}_{\zeta} = X.$$

Now if we let $\lambda_1 \longrightarrow \lambda$, $\lambda_2 \longrightarrow \lambda$, $\mu_1 \longrightarrow \mu$, $\mu_2 \longrightarrow \mu$ then by 2.4 we have

$$I(\lambda_1, \lambda_2) \longrightarrow 0,\ I(\mu_1, \mu_2) \longrightarrow 0,$$

and

$$(T - \xi)^{2\nu}\,(T - \zeta)^{2\nu}x = \lim f(T)x = \lim \{I(\mu_2, \lambda_1)x + I(\lambda_2, \mu_1)x\}.$$

Also, by 2.4, we have $\sigma(I(\mu_2, \lambda_1)x) \subset [\zeta_2, \xi_1]$ and $\sigma(I(\lambda_2, \mu_1)x) \subset [\xi_2, \zeta_1]$, where, for two points $\xi' = \xi(\lambda', 0)$, $\xi'' = \xi(\lambda'', 0)$ on Γ_0, the symbol $[\xi', \xi'']$ means the closed subarc of Γ_0 defined by $\xi(\lambda, 0)$, $\lambda' \leq \lambda \leq \lambda''$, if $\lambda' < \lambda''$ and closed subarc $\xi(\lambda, 0)$, $\lambda \notin (\lambda'', \lambda')$, if $\lambda'' < \lambda'$.

Since $2\nu(\lambda)$ is also an index function, it is seen from 2.10 that $\mathfrak{N}_{\xi}\,\mathfrak{N}_{\zeta}$ is the closure of $(T - \xi)^{2\nu}\,(T - \zeta)^{2\nu}X$, and hence every vector in $\mathfrak{N}_{\xi}\,\mathfrak{N}_{\zeta}$ is the limit of a sequence of vectors of the form $x + y$ with $\sigma(x) \subset [\xi, \zeta]$, $\sigma(y) \subset [\xi, \zeta]'$. Since $\sigma(x) \subset (\xi)$ if $x \in \mathfrak{M}_{\xi}$, we have, from 1.6 and $(*)$ above, the fact that every vector in X is the limit of a sequence of vectors of the form $x + y$ with $\sigma(x) \subset [\xi, \zeta]$, $\sigma(y) \subset [\xi, \zeta]'$. This shows that $[\xi, \zeta]$ is an s_1 set for T (see Definition 1.15). The above argument shows also that $[\zeta, \xi]$ is an s_1 set for T. We shall next show that $\sigma = [\xi, \zeta]$ is an s_2 set for T. If the intervals (α, β) and (α, δ) that we started with above are replaced by the intervals $(\lambda - 1/n, \lambda)$ and $(\mu, \mu + 1/n)$, we see that there are points λ_n, μ_n, with $\lambda - 1/n < \lambda_n < \lambda$, $\mu < \mu_n < \mu + 1/n$, such that $\sigma_n \equiv [\zeta_n, \xi_n]$, where

$$\zeta = \xi(\mu_n, 0),\ \xi_n = \xi(\lambda_n, 0),$$

is an s_1 set. Now let

$$y = (T - \xi)^{2\nu} (T - \zeta)^{2\nu} x, \; y_n = (T - \xi)^{\nu} (T - \xi_n)^{\nu} (T - \zeta)^{\nu} (T - \zeta_n)^{\nu} x,$$

so that $y_n \longrightarrow y$, and for appropriate choices of $P(\xi)$ in 2.4 we have

$$y_n = I(\lambda_n, \lambda)x + I(\lambda, \mu)x + I(\mu, \mu_n)x + I(\mu_n, \lambda_n)x.$$

Thus by 2.4 we may write

$$y = I(\lambda, \mu)x + I(\mu_n, \lambda_n)x + z_n, \text{ where } z_n \longrightarrow 0.$$

Now from 2.4, 2.5, and 1.16 we see that

$$E_\sigma I(\lambda, \mu) = I(\lambda, \mu), \; E_\sigma I(\mu_n, \lambda_n) = 0,$$

$$E_{\sigma_n} I(\lambda, \mu) = 0, \; E_{\sigma_n} I(\mu_n, \lambda_n) = I(\mu_n, \lambda_n);$$

and since $|E_{\sigma_n}| \leq K$ we may write $y = E_\sigma y + E_{\sigma_n} y + v_n$ where $v_n \longrightarrow 0$. Since y is an arbitrary point in the manifold $(T - \xi)^{2\nu} (T - \zeta)^{2\nu} X$ whose closure is $\mathfrak{N}_\xi \mathfrak{N}_\zeta$, and since

$$|E_\sigma + E_{\sigma_n}| \leq 2K,$$

we have

$$y = E_\sigma y + \lim_n E_{\sigma_n} y$$

for every $y \in \mathfrak{N}_\xi \mathfrak{N}_\zeta$. For $x \in \mathfrak{M}_\xi \oplus \mathfrak{M}_\zeta$ we have $\sigma(x) \subset \sigma$; and so, by 1.16, we have $E_\sigma x = x$, $E_{\sigma_n} x = 0$. Hence, it follows from (*) that

$$(**) \qquad x = E_\sigma x + \lim_n E_{\sigma_n} x, \; x \in X.$$

Now 2.5 and 1.22 show that $\sigma(E_\sigma x) \subset \sigma\sigma(x)$, $\sigma(E_{\sigma_n} x) \subset \sigma_n \sigma(x) \subset \sigma' \sigma(x)$, so that σ is an s_2 set. The same argument shows that $[\zeta_n, \xi_n] = \sigma_n$ is an s_2 set and hence (**) shows that σ is an s_3 set for T. Thus we have proved that if $-1 \leq \alpha < \beta < \gamma < \delta < 1$ there are points λ, μ with $\alpha < \lambda < \beta$, $\gamma < \mu < \delta$, such that $\sigma = [\xi(\lambda, 0), \xi(\mu, 0)]$ is an s_3 set. This clearly implies the statement 1.39; hence every Borel set is measurable T. Theorem 2.11 then follows from 1.38.

III. The operational calculus

3.1. DEFINITION OF $\int_\sigma f(\lambda)\,dE_\lambda$. In what follows we shall be concerned with an integral, $\int f(\lambda)\,dE_\lambda$, where E_e is the resolution of the identity for an operator T. The functions f to be integrated are either scalar- or operator-valued; they will always be continuous, so that the Riemann integral will suffice. Although the applications to be made are to operators satisfying the preceding restrictions, it seems desirable to word the definitions and elementary properties of $\int f(\lambda)\,dE_\lambda$ in terms of an arbitrary operator T on an arbitrary space X subject to the single restriction that T has a resolution of the identity. Since $\sigma(T)$ is bounded, it may for any $\delta > 0$ be partitioned into disjoint, nonvoid Borel sets $\Delta_1, \cdots, \Delta_n$ whose diameters are less than δ. The norm $|\pi|$ of such a partitioning $\pi = (\Delta_1, \cdots, \Delta_n)$ is $|\pi| = \max_i \operatorname{diam} \Delta_i$. If for a scalar- or operator-valued function f defined on $\sigma(T)$ we have the sums $\sum_\pi f(\lambda_i)E_{\Delta_i}$ converging, as $|\pi| \longrightarrow 0$, to a limit independent of the choice of $\lambda_i \in \Delta_i$, the function f is said to be integrable. Of course the convergence of $\sum_\pi f(\lambda_i)E_{\Delta_i}$ as $|\pi| \longrightarrow 0$ may be in the weak, strong, or uniform topology of operators; but *for the functions we shall integrate, it is always in the uniform topology of operators*, so we need not concern ourselves here with the other cases. The integral is defined by

$$\int f(\lambda)\,dE_\lambda \equiv \lim_{|\pi| \to 0} \sum_\pi f(\lambda_i)E_{\Delta_i};$$

and for any Borel set σ in the plane we define $\int_\sigma f(\lambda)\,dE_\lambda$ to be $E_\sigma \int f(\lambda)\,dE_\lambda$.

3.2. LEMMA. *If for each e in a Borel algebra β there is a bounded linear operator A_e in the space X such that $x^*A_e x$ is countably additive on β for each $x \in X$, $x^* \in X^*$, then there is a constant $v(A)$ such that*

$$\sum_i |x^*A_{e_i}x| \leq v(A)\,|x|\,|x^*|, \quad e_i e_j \text{ void for } i \neq j, \quad e_i \in \beta.$$

Let $\pi = \{e_i\}$ be a finite or enumerable sequence of disjoint elements in β. For each $x \in X$, $x^* \in X^*$, define $U_\pi(x, x^*)$ as the point in the complex Banach space $\mathcal{l}_1$ (the space of absolutely convergent sequences) given by the sequence $\{x^*A_{e_i}x\}$ (if the sequence $\{e_i\}$ is finite we extend it to an infinite sequence by taking e_n to be the void set for all large n). For fixed x, π, the function $U_\pi(x, x^*)$ is additive, homogeneous, and closed; hence $U_\pi(x, x^*)$ is continuous in x^*. Similarly, $U_\pi(x, x^*)$ is continuous in x for fixed x^*, π. Thus for each π, $U_\pi(x, x^*)$ is simultaneously continuous in x, x^*. Since the numerical function

$x^*A_e x$ is countably additive on β, we have

$$(*) \qquad \sup_{\pi} |U_\pi(x, x^*)| < \infty, \quad x \in X, \ x^* \in X^*.$$

Let Z_n be the set of points (x, x^*) in the Cartesian product space $Z = X \times X^*$, where $|U_\pi(x, x^*)| \leq n$ for every π. Since $U_\pi(x, x^*)$ is continuous in (x, x^*), Z_n is closed. From $(*)$ we have $Z = \cup Z_n$, and so the Baire catagory theorem gives an integer n_0, a point $(x_0, x_0^*) \in Z$, and an $r_0 > 0$, such that

$$|U_\pi(x, x^*)| \leq n_0, \ \pi, \ |x - x_0| < r_0, \ |x^* - x_0^*| < r_0.$$

Now if $y \in X$, $y^* \in X^*$, and $|y|$, $|y^*| < r_0$, we have

$$U_\pi(y, y^*) = U_\pi(x_0 - y, x_0^* - y^*) = U_\pi(x_0 - y, x_0^*)$$

$$= U_\pi(x_0, x_0^* - y^*) + U_\pi(x_0, x_0^*),$$

and $|U_\pi(y, y'') \leq 4n_0$. Thus if $v(A) = 5n_0/r_0^2$, we have

$$|U_\pi(y, y^*)| \leq v(A)\,|y|\,|y^*|.$$

3.3. LEMMA. *Every continuous scalar function f on $\sigma(T)$ is integrable, and*

$$\left|\int f(\lambda)\,dE_\lambda\right| \leq \max_{\lambda \in \sigma(T)} |f(\lambda)|\, v(E),$$

$$\left|\int_\sigma f(\lambda)\,dE_\lambda\right| \leq \sup_{\lambda \in \sigma} |f(\lambda)|\, v(E),$$

where $v(E)$ is the constant of 3.2. *Also if α is an arbitrary parameter and $f(\alpha, \lambda)$ is continuous for $\lambda \in \sigma(T)$ uniformly with respect to α, then the sums $\sum_\pi f(\alpha, \lambda_i)E_{\Delta_i}$ for a partition $\pi = (\Delta_i, \cdots, \Delta_n)$, $\lambda_i \in \Delta_i$, converge, as $|\pi| \longrightarrow 0$, uniformly with respect to α.*

For two partitions $\pi = (\Delta_i, \cdots, \Delta_n)$, $\pi' = (\Delta_1', \cdots, \Delta_m')$ of $\sigma(T)$, and for $\lambda_i \in \Delta_i$, $\lambda_j' \in \Delta_j'$ $(i = 1, \cdots, n, \ j = 1, \cdots, m)$, we have

$$\sum_{i=1}^{n} f(\alpha, \lambda_i)E_{\Delta_i} - \sum_{j=1}^{m} f(\alpha, \lambda_j')E_{\Delta_j'} = \sum_{i=1}^{n}\sum_{j=1}^{m} \{f(\alpha, \lambda_i) - f(\alpha, \lambda_j')\}E_{\Delta_i\Delta_j'}.$$

If for $\epsilon > 0$, $\delta(\epsilon) > 0$ is chosen so that

$$|f(\alpha, \lambda) - f(\alpha, \lambda')| < \epsilon \text{ for } |\lambda - \lambda'| < \delta(\epsilon),$$

we have, by 3.2,

$$\left| \sum_{i=1}^{n} f(\alpha, \lambda_i) E_{\Delta_i} - \sum_{j=1}^{m} f(\alpha, \lambda_j') E_{\Delta_j} \right| < \epsilon \, v(E), \ \alpha, \ |\pi|, \ |\pi'| < \delta(\epsilon)$$

$$\left| \sum_{i=1}^{n} f(\alpha, \lambda_i) E_{\Delta_i} \right| \leq \max_{\lambda \in \sigma(T)} |f(\alpha, \lambda)| \, v(E),$$

and

$$\left| E_\sigma \sum_{i=1}^{n} f(\alpha, \lambda_i) E_{\Delta_i} \right| \leq \sup_{\lambda \in \sigma} |f(\alpha, \lambda)| \, v(E);$$

this proves the lemma.

While in our final results the only operator-valued functions we shall have to integrate are of the form $(T - \lambda I)^n f(\lambda)$, where f is a scalar function, it will, during the course of the proofs to follow, be necessary to integrate functions in a more extensive class. Accordingly, we consider functions of the following type. Let D_1 be an open set containing the closure $\overline{D}$ of an admissible open set $D \supset \sigma(T)$. Let C be the boundary of D. Let $f(\alpha, \lambda)$ be a scalar function defined for $\alpha \in D_1$, $\lambda \in \sigma(T)$, continuous similtaneously in both variables over their respective ranges, and analytic for $\alpha \in D_1$ uniformly with respect to $\lambda \in \sigma(T)$; that is,

$$\frac{f(\alpha + \eta, \lambda) - f(\alpha, \lambda)}{\eta} \longrightarrow \frac{\partial f(\alpha, \lambda)}{\partial \alpha}$$

uniformly with respect to $\lambda \in \sigma(T)$. Because the continuity in λ is uniform with respect to α on C, the operator-valued function

$$f(T, \lambda) = \frac{1}{2\pi i} \int_C f(\alpha, \lambda) \, T(\alpha) \, d\alpha$$

depends continuously on λ. It is this type of continuous operator-valued function defined by a scalar function $f(\alpha, \lambda)$ whose singularities in α stay uniformly

away from $\sigma(T)$ as λ varies over $\sigma(T)$ that we shall be integrating. For the sake of brevity we shall call functions $f(\alpha, \lambda)$ of the above type *T-uniform.*

3.4. LEMMA. *Let $f(\alpha, \lambda)$ be T-uniform. Then $f(T, \lambda)$ is integrable, and for every Borel set σ we have*

$$\int_\sigma f(T, \lambda)\, dE_\lambda = \frac{1}{2\pi i} \int_C \left(\int_\sigma f(\alpha, \lambda)\, dE_\lambda \right) T(\alpha)\, d\alpha.$$

Let $\pi = (\Delta_i, \cdots, \Delta_n)$, $\lambda_i \in \Delta_i$, be a partition of $\sigma(T)$. Then

$$\sum_j f(T, \lambda_j) E_{\Delta_j} = \sum_j \left(\frac{1}{2\pi i} \int_C f(\alpha, \lambda_j)\, T(\alpha)\, d\alpha \right) E_{\Delta_j}$$

$$= \frac{1}{2\pi i} \int_C \left(\sum_j f(\alpha, \lambda_j) E_{\Delta_j} \right) T(\alpha)\, d\alpha .$$

The desired results follow from 3.3.

3.5. LEMMA. *Let $f(\alpha, \lambda)$, $g(\alpha, \lambda)$ be T-uniform. Then $f(\alpha, \lambda) g(\alpha, \lambda)$ is T-uniform, and for a partition $\pi = (\Delta_i, \cdots, \Delta_n)$ of $\sigma(T)$ and points $\lambda_j, \lambda_j' \in \overline{\Delta}_j$ we have*

$$\lim_{|\pi| \to 0} \sum_\pi f(T, \lambda_j)\, g(T, \lambda_j') E_{\Delta_j} = \int f(T, \lambda)\, g(T, \lambda)\, dE_\lambda .$$

It is clear that there is a common domain of uniform analyticity. Let C be its boundary. For $\epsilon > 0$, fix $\delta > 0$ such for every pair $\lambda, \lambda' \in \sigma(T)$ with $|\lambda - \lambda'| < \delta$ we have

$$|f(\alpha, \lambda)[g(\alpha, \lambda') - g(\alpha, \lambda)]| \leq \epsilon, \quad \alpha \in C.$$

From 3.2, if $|\pi| < \delta$ then

$$\sum_\pi f(\alpha, \lambda_i)[g(\alpha, \lambda_j') - g(\alpha, \lambda_i)] E_{\Delta_j} | \leq \epsilon\, v(E), \quad \alpha \in C.$$

Now

$$f(T, \lambda_j)\, g(T, \lambda_j') = f(T, \lambda_j)\, g(T, \lambda_j) + f(T, \lambda_j)[g(T, \lambda_j'] - g(T, \lambda_j)]$$

and

$$|\sum f(T, \lambda_j)[\, g(T, \lambda_j') - g(T, \lambda_j)]\, E_{\Delta_j}|$$

$$= \left| \frac{1}{2\pi i} \int_C T(\alpha)\, d\alpha \ (\sum f(\alpha, \lambda_j)\, [\, g(\alpha, \lambda_j') - g(\alpha, \lambda_j)]\, E_{\Delta_j}) \right|$$

$$\leq (\epsilon / 2\pi) \left(\max_{\alpha \in C} |\, T(\alpha)| \right) (\text{length } C)\ v(E).$$

Thus in order to prove the lemma, it suffices to show that

$$\lim_{\pi} \sum f(T, \lambda_j)\ g(T, \lambda_j) E_{\Delta_j} = \int f(T, \lambda)\ g(T, \lambda)\, dE_\lambda\ .$$

(Note that if $\lambda_j \in \Delta_j$ and not merely in $\overline{\Delta}_j$, there is nothing left to prove.) But this is clear since the function $f(T, \lambda)\, g(T, \lambda)$ is continuous in λ.

3.6. LEMMA. *If f is integrable (scaler- or operator-valued) and U is a bounded linear operator in X which commutes with E_e, $e \in B$, then Uf is integrable and*

$$U \int_\sigma f(\lambda)\, dE_\lambda = \int_\sigma Uf(\lambda)\, dE_\lambda,\ \sigma \in B.$$

The proof is clear from the definitions.

3.7. THEOREM. *Let X be arbitrary and T a bounded linear operator in X with the resolution of the identity E_e. For any closed set of points $\rho \in \rho(T)$, we have*

$$(\xi - T)^{-1} = \sum_{n=0}^{\infty} \int \frac{(T - \lambda)^n}{(\xi - \lambda)^{n+1}}\, dE_\lambda\ ,$$

where the sum converges in the uniform topology of operators and uniformly with respect to $\xi \in \rho$.

In view of the elementary identity

$$(\xi - T) \sum_{n=0}^{p} \frac{(T - \lambda)^n}{(\xi - \lambda)^{n+1}} = I - \frac{(T - \lambda)^{p+1}}{(\xi - \lambda)^{p+1}}$$

and 3.6, we have

$$(\xi - T) \sum_{n=0}^{p} \frac{(T-\lambda)^n}{(\xi-\lambda)^{n+1}}\, dE_\lambda = I - \int \frac{(T-\lambda)^{p+1}}{(\xi-\lambda)^{p+1}}\, dE_\lambda .$$

Now let δ = the distance from ρ to $\sigma(T)$. This is positive since $\sigma(T)$ is bounded and closed. Break $\sigma(T)$ into disjoint measurable parts $\Delta_1, \cdots, \Delta_n$ of which the diameters satisfy

$$\text{diameter } \Delta_j \leq \delta/4 \qquad (j = 1, \cdots, n).$$

Let C_j be a circle of diameter $\delta/2$ containing $\bar{\Delta}_j$ in its interior, so that

$$|(\alpha - \lambda)(\xi - \lambda)^{-1}| \leq 1/2, \quad \xi \in \rho, \ \lambda \in \Delta_j, \quad \alpha \in C_j .$$

Let $T_\alpha(\Delta_j)$ be the resolvent of T when considered as an operator in $X_{\bar{\Delta}_j}$. Since $\sigma(X_{\bar{\Delta}_j}) \subset \bar{\Delta}_j$ and

$$\left(\int_{\Delta_j} \frac{(T-\lambda)^p}{(\xi-\lambda)^p}\, dE_\lambda \right) X \subset X_{\bar{\Delta}_j} ,$$

we have, from 3.3,

$$\left| \int_{\Delta_j} \frac{(T-\lambda)^p}{(\xi-\lambda)^p}\, dE_\lambda \right|$$

$$= \left| \int_{\Delta_j} \left(\frac{1}{2\pi i} \int_{C_j} \frac{(\alpha-\lambda)^p}{(\xi-\lambda)^p}\, T_\alpha(\Delta_j)\, d\alpha \right) dE_\lambda \right|$$

$$= \left| \frac{1}{2\pi i} \int_{C_j} T_\alpha(\Delta_j)\, d\alpha \int_{\Delta_j} \frac{(\alpha-\lambda)^p}{(\xi-\lambda)^p}\, dE_\lambda \right|$$

$$\leq \frac{1}{2\pi} \max_{\alpha \in C_j} |T_\alpha(\Delta_j)|\ v(E)/2^p, \quad \xi \in \rho .$$

Since $\int = \int_{\Delta_1} + \cdots + \int_{\Delta_n}$, we have

$$(\xi - T) \sum_{n=0}^{p} \int \frac{(T-\lambda)^n}{(\xi-\lambda)^{n+1}}\, dE_\lambda \longrightarrow I$$

uniformly for $\xi \in \rho$. This proves the theorem.

3.8. DEFINITION. An operator is called a *spectral operator* if it has a resolution of the identity.

3.9. THEOREM. *If T is a bounded spectral operator, the resolution of the the identity E_e is unique, and for every $f \in F(T)$ we have*

$$f(T) = \sum_{n=0}^{\infty} \int \frac{f^{(n)}(\lambda)}{n!} (T - \lambda)^n \, dE_\lambda ,$$

where the integral exists as a Riemann integral in the uniform topology of operators and the sum converges in the uniform topology of operators.

We shall first show uniqueness. Let A_e, E_e be resolutions of the identity for T. Let σ_1, σ_2 be disjoint and each consist of a circle and its interior. Let $T^1(\xi)$ be the resolvent of T when considered as an operator on $E_{\sigma_1} X$. Let $T^2(\xi)$ be the resolvent of T when considered as an operator in $A_{\sigma_2} X$. Then for $\xi \notin \sigma_1$, $T^1(\xi) E_{\sigma_1} A_{\sigma_2}$ is a bounded linear operator in X and analytic for $\xi \in \sigma_1'$. Likewise for $\xi \in \sigma_2'$, $E_{\sigma_1} T^2(\xi) A_{\sigma_2}$ is a bounded linear operator in X and analytic for $\xi \in \sigma_2'$. Let $\xi \in (\sigma_1 \cup \sigma_2)'$. Since $\xi - T$ commutes with E_{σ_1}, we have

$$(\xi - T) E_{\sigma_1} T^2(\xi) A_{\sigma_2} = E_{\sigma_1} A_{\sigma_2} ,$$

and operating on the left with $T^1(\xi)$ we have

$$E_{\sigma_1} T^2(\xi) A_{\sigma_2} = T'(\xi) E_{\sigma_1} A_{\sigma_2} .$$

If $f(\xi)$ is defined to be

$$f(\xi) = E_{\sigma_1} T^2(\xi) A_{\sigma_2} = T^1(\xi) E_{\sigma_1} A_{\sigma_2}, \quad \xi \in (\sigma_1 \cup \sigma_2)'$$

$$= E_{\sigma_1} T^2(\xi) A_{\sigma_2}, \quad \xi \in \sigma_1$$

$$= T^1(\xi) E_{\sigma_1} A_{\sigma_2}, \quad \xi \in \sigma_2 ,$$

then f is an entire function. Since for large ξ, we have

$$f(\xi) = T(\xi) E_{\sigma_1} A_{\sigma_2} \longrightarrow 0 \quad \text{as} \quad |\xi| \longrightarrow \infty .$$

it follows that $f(\xi) = 0$ and $E_{\sigma_1} A_{\sigma_2} = 0$. By symmetry, $A_{\sigma_2} E_{\sigma_1} = 0$. Let σ_n, $(n = 3, 4, \cdots)$ be the set of those points of $\sigma(T)$ whose distance from σ_1 is $\geq 1/n$. Let $\delta_1, \cdots, \delta_{p(n)}$ each consist of a circle and its interior and be such that the set $\delta^n = \cup \delta_i$ is disjoint with σ_1 and covers σ_n. Then, since $E_{\sigma_1} A_{\delta_i} = 0$, we have $A_{\delta_i} \subset E'_{\sigma_1} X$ and

$$A_{\sigma_n} X \subset A_{\delta^n} X = (\bigcup_i A_{\delta_i}) X = \bigcup_i (A_{\delta_i} X) \subset E'_{\sigma_1} X .$$

Hence

$$E'_{\sigma_1} A_{\sigma_n} = A_{\sigma_n}, \; E_{\sigma_1} A_{\sigma_n} = 0 .$$

But $A_{\sigma_n} x \longrightarrow A_{\sigma_1'} x$, $x \in X$, and so

$$E_{\sigma_1} A'_{\sigma_1} = 0, \; E_{\sigma_1} A_{\sigma_1} = E_{\sigma_1} .$$

Also, since $A_{\delta_i} E_{\sigma_1} = 0$, we have $E_{\sigma_1} X \subset A_{\delta_i'} X$ and

$$E_{\sigma_1} X \subset \bigcap_i (A_{\delta_i'} X) = (\bigcap_i A_{\delta_i'}) X = A'_{\delta^n} X .$$

Thus

$$A'_{\delta^n} E_{\sigma_1} = E_{\sigma_1}, \; A_{\delta^n} E_{\sigma_1} = 0, \; A'_{\sigma_1} E_{\sigma_1} = 0, \; A_{\sigma_1} E_{\sigma_1} = E_{\sigma_1} ,$$

and therefore

$$E_{\sigma_1} = A_{\sigma_1} E_{\sigma_1} = E_{\sigma_1} A_{\sigma_1} = E_{\sigma_1} .$$

By symmetry,

$$A_{\sigma_1} = A_{\sigma_1} E_{\sigma_1} = E_{\sigma_1} A_{\sigma_1} = E_{\sigma_1} .$$

From this it readily follows that $A_\sigma = E_\sigma$ for any Borel set σ. Now let $f \in F(T)$. Let C be an admissible contour upon which f is analytic and such that

$$f(T) = \frac{1}{2\pi i} \int_C f(\xi) T(\xi) d\xi .$$

Now, by 3.7, we have

$$T(\xi) = \sum_{n=0}^{\infty} \int (T - \lambda)^n \, (\xi - \lambda)^{-n-1} \, dE_\lambda \, ,$$

and the series converges in the uniform topology of operators and uniformly for $\xi \in C$. So

$$f(T) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C f(\xi) \, d\xi \left(\int \frac{(T - \lambda)^n}{(\xi - \lambda)^{n+1}} \, dE_\lambda \right).$$

Since

$$\int \frac{(T - \lambda)^n}{(\xi - \lambda)^{n+1}} \, dE_\lambda = \sum_{n=0}^{n} \binom{n}{r} T^r \int \frac{\lambda^{(n-r)}}{(\xi - \lambda)^{n+1}} \, dE_\lambda \, ,$$

it follows from 3.3, that the Riemann sums approximating

$$\int (T - \lambda)^n \, (\xi - \lambda)^{-n-1} \, dE_\lambda$$

converge uniformly with respect to $\xi \in C$.

Hence

$$f(T) = \sum_{n=0}^{\infty} \int \frac{f^{(n)}(\lambda)}{n!} \, (T - \lambda)^n \, dE_\lambda \, .$$

From this point on we shall again restrict our attention to the case of an operator T whose spectrum lies in a rectifiable Jordan curve Γ_0 and whose resolvent satisfies the growth condition 2.1. It will be convenient to state the condition 1.14 in terms of residues as defined in the following:

3.10. DEFINITION. Let $\sigma(x)$, $\sigma(x^*x)$ be the sets of singularities of the functions $x(\xi) = (\xi - T)^{-1}x$, $x^*x(\xi)$, respectively. Let σ be open and closed in $\sigma(x)$, and

$$x_\sigma = \frac{1}{2\pi i} \int_C x(\xi) \, d\xi \, ,$$

where C is a rectifiable Jordan curve containing σ in its interior and having $\sigma(x)\sigma'$ in its exterior. Then the vector x_σ is called the σ – *residue* of $x(\xi)$.

Similarly if σ is open and closed in $\sigma(x^*x)$ then the scalar

$$(x^*x)_\sigma = \frac{1}{2\pi i} \int_C x^*x(\xi)\, d\xi$$

is called the σ – residue of $x^*x(\xi)$.

With this terminology, condition 1.14 asserts the existence of a constant K such that $|x_\sigma| \leq K\,|x|$.

3.11. THEOREM. *Let X be weakly complete, and let T be a bounded linear operator in X whose spectrum lies in the rectifiable Jordan curve Γ_0 and whose resolvent is restricted in its growth by Assumption* 2.1. *Then T is a spectral operator and satisfies the equation*

$$f(T) = \sum_{n=0}^{\infty} \int_{\sigma(T)} \frac{f^{(n)}(\xi)}{n!}\, (T-\xi)^n\, dE_\xi\,, \quad f \in F(T),$$

providing:

(i) (*The density condition.*) *For every ξ in a set dense on Γ_0 $\mathfrak{M}_\xi + \mathfrak{N}_\xi$ is dense in X.*

(ii) (*The boundedness condition.*) *There is a constant K such that all residues x_σ satisfy the inequality*

$$|x_\sigma| \leq K\,|x|, \quad x \in X.$$

This theorem is an immediate corollary of 2.11 and 3.8.

Conditions will now be given which are of a nature more applicable than (i) and (ii) of 3.11 and which are sufficient (and in some cases necessary) to imply (i) and (ii). We shall begin with a brief analysis of some conditions which are sufficient to imply the density condition (i).

3.12. THEOREM. *The operator T of Theorem* 3.11 *satisfies the density condition* (i) *of that theorem in case any one of the following is true*:

(i) *Every subarc (of positive length) of Γ_0 contains points either in the continuous spectrum or in the resolvent set.*

(ii) *No subarc (of positive length) of Γ_0 consists entirely of points in the*

point spectrum of the adjoint T^* of T.

(iii) *The space X is reflexive and $\nu(\lambda) = 1$ is an index function for T.*

(iv) *The space X is reflexive and the adjoint T^* satisfies the boundedness condition* (ii) *of* 3.11.

(v) *The operator T is completely continuous.*

The first statement is obvious since if ξ is in either the resolvent set or the continuous spectrum we have $\mathfrak{N}_\xi = X$. The second statement is equally clear since it is seen from the Hahn-Banach theorem that ξ is in the point spectrum of the adjoint if and only if $\mathfrak{N}_\xi \neq X$. Next, if (iii) holds, let

$$\xi = (\lambda, \delta), \quad \xi_0 = \xi(\lambda, 0),$$

so that

$$|(\xi - \xi_0)\, T(\xi)| \leq |\delta T(\xi)| \leq 1.$$

Now

$$(\xi - \xi_0)\, T(\xi)\, (\xi_0 - T) = \xi - \xi_0 - (\xi - \xi_0)^2\, T(\xi),$$

and hence

$$(*) \qquad \lim_{\delta \to 0} (\xi - \xi_0)\, T(\xi)\, (\xi_0 - T) = 0.$$

Now let x be an arbitrary vector in X. Since X is reflexive, the set

$$(\xi - \xi_0)\, T(\xi)x, \quad 0 < \delta \leq \delta_0,$$

is weakly compact, and there is a vector $y \in X$ and a sequence $\delta_n \longrightarrow 0$ such that for $\xi_n = \xi(\lambda, \delta_n)$ we have

$$(\xi_n - \xi_0)\, T(\xi_n)\, x \longrightarrow y \quad \text{weakly}.$$

The equation (*) shows that $y \in \mathfrak{M}_{\xi_0}$. To see that $x - y \in \mathfrak{N}_{\xi_0}$, let $x^* \mathfrak{N}_{\xi_0} = 0$. Then

$$x^*(\xi - \xi_0)\, T(\xi) = x^*,$$

and so $x^*(x-y)=0$. Hence $\mathfrak{M}_{\xi_0}+\mathfrak{N}_{\xi_0}=X$. To prove the fourth statement we note first that, since $\sigma(T)=\sigma(T^*)$ and $|T(\xi)|=|T^*(\xi)|$, the adjoint T^* satisfies 2.1, and any index function $\nu(\lambda)$ for T is also an index function for T^*. By 2.9, then, $\mathfrak{M}_\xi(T^*)\,\mathfrak{N}_\xi(T^*)=0$ for ξ interior to any interval upon which an index function is constant. Such ξ are by 2.3 dense on Γ_0. Let ξ be such a point on Γ_0, and let

$$x^*\mathfrak{M}_\xi = 0 = x^*\mathfrak{N}_\xi .$$

It will suffice to prove that $x^*=0$. Since $x^*\mathfrak{N}_\xi=0$, we have $x^*\in\mathfrak{M}_\xi(T^*)$. To see that $x^*\in\mathfrak{N}_\xi(T^*)$ (which will prove $x^*=0$), it will suffice, since X is reflexive, to show that $x^*x_0=0$ for every x_0 with $\mathfrak{N}_\xi(T^*)x_0=0$, that is, for every x_0 with

$$y^*(\xi-T)^\nu x_0 = 0,\quad y^*\in Y^*.$$

But such an x_0 is in $\mathfrak{M}_\xi$, and so $x^*x_0=0$. The final statement (v) follows from the fact that the spectrum of a completely continuous operator is at most denumerable.

N. B. As the above proof shows, the condition that X be reflexive (in (iv)) may be replaced by the statement that, for ξ in a dense set on Γ_0, the manifold $\mathfrak{N}_\xi(T^*)$ is regularly closed. Also in (iii) the condition of reflexivity may be replaced by the assumption that the set of vectors $(\xi-\xi_0)T(\xi)x$, $0<\delta<\delta_0$, is weakly compact.

3.13. THEOREM. *Let X be a reflexive space and T a bounded linear operator in X whose spectrum lies in the rectifiable Jordan curve Γ_0 and whose resolvent satisfies the growth condition* 2.1. *Then T is a spectral operator and satisfies the equation*

$$f(T)=\sum_{n=0}^{\infty}\int_{\sigma(T)}\frac{f^{(n)}(\xi)}{n!}(T-\xi)^n\,dE_\xi,\quad f\in F(T),$$

if and only if there is a constant K such that all the residues $(x^*x)_\sigma$ satisfy the inequality.

$$|(x^*,x)_\sigma|\le K|x|\,|x^*|.$$

The residue condition is clearly necessary since

$$(x^*x)_\sigma = x^*E_\sigma x.$$

To see that it is sufficient we note first that it implies the condition (ii) of Theorem 3.11. Since X is reflexive, the residue $(x^*, x)_\sigma$ is equal to the residue $(x^{**}, x^*)_\sigma$ calculated for the adjoint T^* (here $x^{**}x^* = x^*x$, $x^* \in X^*$) and so the residue condition of 3.13 implies that the adjoint T^* satisfies (ii) of 3.11. The present theorem then follows from 3.11 and 3.12 (iv).

We now turn our attention to stating the requirement (of 3.11 or 3.13) that the residues be bounded, in a form which, in some instances, is more readily applied.

3.14. THEOREM. *Let $\xi = \xi(\lambda, \delta)$ have continuous first partial derivatives, and let $\xi^- = \xi(\lambda, -\delta)$. Then the residues $(x^*, x)_\sigma$ and x_σ will have a bound of the form $K|x|\,|x^*|$ in case*

$$\text{l.u.b.}_{0<\delta<\delta_0} \int_{-1}^{+1} |x^*\left\{T(\xi)\frac{\partial\xi}{\partial\lambda} - T(\xi^-)\frac{\partial\xi^-}{\partial\lambda}\right\}x|\,d\lambda \leq M|x|\,|x^*|.$$

Let $0 \leq \lambda_j < \lambda_j' \leq 1$, let $C_j = C(\lambda_j, \lambda_j')$ be as in 2.4, and let C be the set C_j, $(j = 1, \cdots n)$. Suppose that C lies in the domain of analyticity of $x^*x(\xi)$. Then

$$\int_C x^*x(\xi)\,d\xi = \sum_{j=1}^{n} \int_{\lambda_j}^{\lambda_j'} x^*\left\{x(\xi)\frac{\partial\xi}{\partial\lambda} - x(\xi^-)\frac{\partial\xi^-}{\partial\lambda}\right\}d\lambda + I(\delta),$$

where $I(\delta)$ is a sum of integrals $\int x^*x(\xi)\,d\xi$ taken along the arcs $\xi(\lambda_j, \mu)$, $\xi(\lambda_j', \mu)$, $-\delta \leq \mu \leq \delta$. Thus $\lim_{\delta\to 0} I(\delta) = 0$, and

$$\limsup_{\delta\to 0} \left|\int_C x^*x(\xi)\,d\xi\right|$$

$$\leq \text{l.u.b.}_{0<\delta<\delta_0} \int_{-1}^{+1} \left|x^*\left\{T(\xi)\frac{\partial\xi}{\partial\lambda} - T(\xi^-)\frac{\partial\xi}{\partial\lambda}\right\}x\right|d\lambda \leq M\,|x|\,|x^*|.$$

The condition of 3.13 is far from necessary, and is not satisfied by the resolvent $T(\xi)$ if its rate of growth for ξ near $\sigma(T)$ is not that of the inverse of the distance from ξ to $\sigma(T)$. To avoid this objection a similar condition, as is evident from the above proof, may be stated.

3.15. THEOREM. *Let* $T(\xi) = U(\xi) + V(\xi)$, *where* $x^*V(\xi)x$ *is the derivative of a single valued analytic function at each point* ξ *where* $x^*T(\xi)x$ *is analytic. Then the residues* $(x^*, x^*)_\sigma$ *and* x_σ *will have a bound of the forms* $K|x|\,|x^*|$ *provided that* $U(\xi)$ *satisfies the condition of* 3.14.

Operators of finite type

As is to be expected from the analogy with the elementary divisor theory for a finite matrix, certain spectral operators should satisfy the formula

$$f(T) = \sum_{n=0}^{m-1} \int_{\sigma(T)} \frac{f^{(n)}(\xi)}{n!} (T - \xi)^n \, dE_\xi , \quad f \in F(T).$$

One might expect this to be true if the spectrum $\sigma(T)$ is nowhere dense and if the resolvent $T(\xi)$ has for ξ near $\sigma(T)$ the same rate of growth as

$$[\operatorname{dis}(\xi, \sigma(T))]^{-m}.$$

We have been able to prove this only in the case where $\sigma(T)$ is restricted to lie in a sufficiently smooth Jordan curve.

We shall assume throughout the following discussion that the function $\xi(\lambda, \mu)$ *defining the net described in* 2.1 *has continuous second partial derivatives.* The purpose of this assumption is to assure that the length of the contour $C(\lambda_1, \lambda_2)$ of 2.4 is at most $K\delta$, provided that $\lambda_1 < \lambda_2$ and $\delta = \lambda_2 - \lambda_1$. Also the diameter of $C(\lambda_1, \lambda_2)$ is at most $K\delta$ for $\delta = \lambda_2 - \lambda_1$.

3.16. LEMMA. (*Assumption* 2.1.) *Let* $d(\xi)$ *be the distance from* ξ *to the spectrum* $\sigma(T)$. *If* $|d^m(\xi)\ T(\xi)|$ *is bounded for* ξ *near* $\sigma(T)$, *then*

$$\int_{\sigma(T)} f(T, \xi)\ (T - \xi)^{2m}\ dE_\xi = 0$$

for every T*-uniform* $f(\alpha, \xi)$.

We may and shall assume that $\nu(\lambda) = m$ is an index function for T, so that

$$|\delta^m\ T_{(\xi)}| \leq 1, \quad 0 < |\delta| < \delta_0, \quad \lambda \in [-1, 1].$$

Let $\lambda_1 < \lambda_2$, $\lambda_2 - \lambda_1 < \delta_0$. Let $C(\lambda_1, \lambda_2)$ be the contour defined in 2.4 with $\delta = \lambda_2 - \lambda_1$. Let Δ be the closed subarc of Γ_0 defined by $\xi(\lambda, 0)$, $\lambda_1 \leq \lambda \leq \lambda_2$.

Let $I(\lambda_1, \lambda_2)$ be the integral defined in 2.4 with $\nu_1 = \nu_2 = m$ and $P(\xi) = 1$. Let $\lambda_n < \lambda_1$, $\lambda_2 < \mu_n$, $\lambda_n \longrightarrow \lambda_1$, $\mu_n \longrightarrow \lambda_2$. By 2.4, we have $I(\lambda_n, \lambda_1) \longrightarrow 0$, $I(\lambda_2, \mu_n) \longrightarrow 0$. Also, by 2.4, 1.39 (ii), 2.5, and 1.12, we have $E_\Delta I(\mu_n, \lambda_n) = 0$. If $\xi_j = \xi(\lambda_j, 0)$ $(j = 1, 2)$, then

$$(T - \xi_1)^m (T - \xi_2)^m = I(\lambda_n, \lambda_1) + I(\lambda_1, \lambda_2) + I(\lambda_2, \mu_n) + I(\mu_n, \lambda_n);$$

and so we have

$$E_\Delta (T - \xi_1)^m (T - \xi_2)^m = E_\Delta I(\lambda_1, \lambda_2).$$

But by 2.4, we have $\sigma(I(\lambda_1, \lambda_2)x) \subset \Delta$; hence by 1.39 (i) it is seen that $I(\lambda_1, \lambda_2) = E_\Delta I(\lambda_1, \lambda_2)$. Thus

$$(*) \qquad E_\Delta (T - \xi_1)^m (T - \xi_2)^m = I(\lambda_1, \lambda_2).$$

Now, since $\delta = \lambda_2 - \lambda_1$, there are constants K_1, K_2 such that

$$\max_{\lambda_1 \leq \lambda \leq \lambda_2} |\xi(\lambda, \pm\delta) - \xi(\lambda_1, 0)| \leq K_1 \delta,$$

and

$$\text{length } C(\lambda_1, \lambda_2) \leq K_2 \delta.$$

It follows from the definition of $I(\lambda_1, \lambda_2)$ therefore that

$$(\dagger) \qquad |I(\lambda_1, \lambda_2)| \leq K_3 \delta^{m+1}.$$

Let the interval $[-1, 1]$ be partitioned into n intervals $[\lambda_{j-1}, \lambda_j]$ each of length $2/n$, and let Δ_j be the corresponding subarcs of Γ_0 with end points ξ_{j-1}, ξ_j. Statements $(*)$ and $(\dagger)$ then give

$$\left| \sum_{j=1}^{n} f(T, \xi_j) (T - \xi_j)^m (T - \xi_{j-1})^m E_{\Delta_j} \right| \leq K_4 n^{-m}.$$

Hence, by 3.5, we have

$$\int_{\sigma(T)} F(T, \xi) (T - \xi)^{2m} \, dE_\xi = 0.$$

3.17. LEMMA. *Under the hypothesis of* 3.16, *we have*

$$(T - \xi)^m E_{(\xi)} = 0,$$

where (ξ) *is the set consisting of the single point* ξ.

From 3.16, we get

$$(T - \xi)^{2m} E_{(\xi)} = E_{(\xi)} \int_{\sigma(T)} (T - \mu)^{2m} \, dE_\mu = 0.$$

Thus

$$T(\alpha) E_{(\xi)} = \sum_{j=0}^{2m-1} \frac{(T - \xi)^j E}{(\alpha - \xi)^{j+1}} (\xi),$$

has a pole of order $\leq 2m$ at $\alpha = \xi$. Since $| d^m(\alpha) T(\alpha) |$ is bounded for α near ξ, the pole must be of order $\leq m$; that is, $(T - \xi)^m E_{(\xi)} = 0$.

3.18. LEMMA. *Under the hypothesis of* 3.16, *we have*

$$\int_{\sigma(T)} f(T, \xi) \, (T - \xi)^j \, dE_\xi = 0, \quad j \geq m$$

for every T-uniform function $f(\alpha, \xi)$.

For $\xi = \xi(\lambda, 0) \in \Gamma_0$ and $0 < |\delta| < \delta_0$, let $\xi_\delta = \xi(\lambda, \delta)$. Then

$$(\xi - T) \, T(\xi_\delta) = (\xi - \xi_\delta) \, T(\xi_\delta) + I.$$

Now assume for the purposes of induction (the above equality is the case $j = 1$,) that

$$(\xi - T)^j \, T(\xi_\delta) = (\xi - \xi_\delta)^j \, T(\xi_\delta) + (\xi - \xi_\delta)^{j-1} + (\xi - \xi_\delta)^{j-2} (\xi - T)$$

$$+ \cdots + (\xi - \xi_\delta) (\xi - T)^{j-2} + (\xi - T)^{j-1}.$$

Multiplying by $(\xi - T)$, we have

$$(\xi - T)^{j+1}\ T(\xi_\delta) = (\xi - T)^j\ (\xi - T)\ T(\xi_\delta) = (\xi - T)^j\ [(\xi - \xi_\delta)\ T(\xi_\delta) + I]$$

$$= (\xi - \xi_\delta)^{j+1}\ T(\xi_\delta) + (\xi - \xi_\delta)^j + (\xi - \xi_\delta)^{j-1}\ (\xi - T)$$

$$+ \cdots + (\xi - \xi_\delta)\ (\xi - T)^{j-1} + (\xi - T)^j .$$

Hence

$$f(T, \xi)\ [(\xi - T)^{j+1}\ T(\xi_\delta) - (\xi - T)^j]$$

$$= f(T, \xi)\ [(\xi - \xi_\delta)^{j+1}\ T(\xi_\delta) + (\xi - \xi_\delta)^j + \cdots + (\xi - \xi_\delta)\ (\xi - T)^{j-1}].$$

Thus we may state:

(*) If for some $j = 1, 2, \cdots$ we have

$$\int_{\sigma(T)} f(T, \xi)\ (\xi - T)^{j+1}\ T(\xi_\delta)\, dE_\xi = 0,\quad 0 < |\delta| < \delta_0 ,$$

then

$$\lim_{\delta \to 0} \int_{\sigma(T)} f(T, \xi)\ (\xi - \xi_\delta)^{j+1}\ T(\xi_\delta)\, dE_\xi = - \int_{\sigma(T)} f(T, \xi)\ (\xi - T)^j\, dE_\xi .$$

Now let $0 < |\delta_i| < \delta_0$, $(i = 1, 2, \cdots, m)$. By 3.16, then,

$$\int_{\sigma(T)} f(T, \xi)\ (T - \xi)^{2m}\ T(\xi_{\delta_1})\ T(\xi_{\delta_2}) \cdots T(\xi_{\delta_m})\, dE_\xi = 0.$$

To this equation we may apply (*) with $\delta = \delta_1$, and with $f(T, \xi)$ replaced by $f(T, \xi)\ T(\xi_{\delta_2}) \cdots T(\xi_{\delta_m})$. Thus,

$$\lim_{\delta_1 \to 0} \int_{\sigma(T)} f(T, \xi)\ (\xi - \xi_{\delta_1})^{2m}\ T(\xi_{\delta_1}) \cdots T(\xi_{\delta_m})\, dE_\xi$$

$$= - \int_{\sigma(T)} f(T, \xi)\ (\xi - T)^{2m-1}\ T(\xi_{\delta_2}) \cdots T(\xi_{\delta_m})\, dE_\xi .$$

Since

$$|(\xi - \xi_{\delta_1})^{2m}\ T(\xi_{\delta_1})| \leq K\, |\delta_1|^m ,$$

the integrand on the right side of the preceeding equation approaches zero with δ_1 and uniformly with respect to $\xi \in \sigma(T)$. Thus

$$\int_{\sigma(T)} f(T, \xi)\ (\xi - T)^{2m-1}\ T(\xi_{\delta_2}) \cdots T(\xi_{\delta_m})\ dE_\xi = 0.$$

A repetition of this process clearly yields the desired result:

$$\int_{\sigma(T)} f(T, \xi)\ (\xi - T)^j\ dE_\xi = 0, \quad j \geq m.$$

3.19. DEFINITION. Let m be a positive integer. A spectral operator T is said to be *of type m* in case

$$f(T) = \sum_{n=0}^{m-1} \int_{\sigma(T)} \frac{f^{(n)}(\xi)}{n!}\ (T - \xi)^n\ dE_\xi$$

for every f single valued and analytic on $\sigma(T)$, that is, for $f \in F(T)$.

Let us recall that for the case in hand (that is, $\nu(\lambda) = m$ is an index function), the manifolds $\mathfrak{M}_\xi$, $\mathfrak{N}_\xi$ are respectively the zeros and the closure of the range $(T - \xi)^m$. Then if $d(\xi)$ is the distance from ξ to the spectrum $\sigma(T)$ we may state:

3.20. THEOREM. *If X is weakly complete, T will be a spectral operator and of type m providing*

(i) *$d^m(\xi)\ T(\xi)$ is bounded for ξ near $\sigma(T)$,*

(ii) *for ξ in a set dense in Γ_0 the manifold $\mathfrak{M}_\xi + \mathfrak{N}_\xi$ is dense in X.*

(iii) *all residues x_σ have a bound of the form $K|x|$.*

This theorem follows immediately from 3.11 and 3.18.

N. B. 1. As before, the condition (ii) is automatically satisfied if T enjoys any one of the properties listed in 3.12. Also (iii) is satisfied if the resolvent $T(\xi)$ satisfies the mean rate of growth condition of 3.14 or 3.15.

N. B. 2. *In case X is not weakly complete* it is still true that E_Δ is defined for every closed subarc of Γ_0 (see proof of 2.11), and E_Δ is completely additive, in the strong topology of operators, on the Boolean algebra determined by such arcs. Thus the integral

$$\int_{\sigma(T)} \frac{f^{(n)}(\xi)}{n!} (T-\xi)^n \, dE_\xi$$

may be defined and the operational calculus developed even though E_e may not be defined as an operator in X for every Borel set e.

An immediate corollary is (see 3.13):

3.21. THEOREM. *If X is reflexive, then T will be a spectral operator of type m if and only if*

(i) *$d^m(\xi)\, T(\xi)$ is bounded for ξ near $\sigma(T)$,*

(ii) *all residues $(x^*, x)_\sigma$ have a bound of the form $K|x|\,|x^*|$.*

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