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SOME SPECIAL EQUATIONS IN A FINITE FIELD

L. CARLITZ

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1. The equation (1.1). Let $f_i(u)$, $i = 1, \dots, r$ denote polynomials with coefficients in the finite field $GF(q)$, $q = p^n$. We consider the equation

$$(1.1) \quad f_1(\xi_1) + \dots + f_r(\xi_r) = \alpha \quad (\xi_i, \alpha \in GF(q));$$

let N denote the number of solutions of (1.1).

For $\beta \in GF(q)$, put

$$e(\beta) = e^{2\pi i t(\beta)/p}, \quad t(\beta) = \beta + \beta^p + \dots + \beta^{p^{n-1}}.$$

Then we may write

$$(1.2) \quad qN = \sum_{\beta} e(-\alpha\beta) \sum_{\xi_1, \dots, \xi_r} e(\beta f_1(\xi_1) + \dots + \beta f_r(\xi_r)),$$

where the summation extends over all numbers β, ξ_i of $GF(q)$. Now put

$$(1.3) \quad S(f) = \sum_{\xi} e(f(\xi)),$$

where f is any polynomial with coefficients in $GF(q)$. Then (1.2) becomes

$$qN = q^r + \sum_{\beta \neq 0} e(-\alpha\beta) \prod_{i=1}^r S(\beta f_i).$$

2. Estimate for N . If $\deg f \leq 2$, $S(f)$ can be evaluated explicitly. However, we are primarily interested in the case $\deg f > 2$. An estimate for $S(f)$ is given by the following:

THEOREM 1. *If $k = \deg f < p$, then*

$$(2.1) \quad S(f) = O(q^{1-1/k}) \quad (q \rightarrow \infty).$$

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Mordell [7] has proved (2.1) in the case $n = 1$, that is, $q = p$. However, examination of his proof shows that (2.1) holds for all $n \geq 1$ provided we have $\deg f < p$.

If we substitute from (2.1) in (1.4) we have at once:

THEOREM 2. *The number of solutions of (1.1), where $\deg f_i = k_i < p$, is given by*

$$(2.2) \quad N = q^{r-1} + O(q^{r-w}) \quad \left(w = \frac{1}{k_1} + \cdots + \frac{1}{k_r} \right).$$

This result is trivial unless $w > 1$, which will evidently be satisfied for r sufficiently large.

Hua and Vandiver [5] and Weil [9] have discussed the number of solutions of (1.1) in the special case $f_i(x) = x^{k_i}$; their results are considerably better than (2.2).

If $g_i(u)$, $i = 1, \dots, r$, denote a second set of polynomials with coefficients in $GF(q)$ and such that $\deg g_i < k_i$, then an estimate can be obtained for the weighted sum

$$S_g = \sum_{\xi_1, \dots, \xi_r} e(g_1(\xi_1) + \cdots + g_r(\xi_r)),$$

where the summation is extended over all ξ_i satisfying (1.1). Indeed, we have

$$\begin{aligned} qS_g &= \sum_{\beta} e(-\alpha\beta) \sum_{\xi_1, \dots, \xi_r} e \left\{ \sum_{i=1}^r (\beta f_i(\xi_i) + g_i(\xi_i)) \right\} \\ &= \sum_{\beta} e(-\alpha\beta) \prod_{i=1}^r S(\beta f_i + g_i), \end{aligned}$$

in the notation of (1.3); consequently if at least one $g_i(x)$ is of degree ≥ 1 , it follows that

$$S_g = O(q^{r-w}) \quad \left(w = \frac{1}{k_1} + \cdots + \frac{1}{k_r} \right).$$

If all $k_i = 2$ then an explicit formula can be obtained for S_g .

3. Some special case. Let

$$(3.1) \quad f(x) = \alpha_1 x^{e_1} + \dots + \alpha_k x^{e_k} \quad (\alpha_i \in GF(q));$$

Mordell has proved that

$$(3.2) \quad S(f) = O(q^{1-1/(2k)}) \quad (q \rightarrow \infty)$$

in the special case $q = p$. Negative values of e_i are permitted; however, in that case it is assumed that in the definition of $S(f)$, the summation is over $\xi \neq 0$. Clearly this does not affect the estimate (3.2). Here again we find that Mordell's proof applies to the general case. We state:

THEOREM 3. *If the integers e_i in (3.1) are incongruent (mod $q-1$), then (3.2) holds.*

(We remark that Min [6, p.139, Lemma 1] states that (2.1) is valid, without mentioning the restriction $k < p$. However, his proof does not seem adequate. For example, for $k = p$, the system

$$\sum_{i=1}^p x_i^j = \sum_{i=1}^p y_i^j \quad (j = 1, \dots, p)$$

does not imply that the y 's are a permutation of the x 's.)

By means of Theorem 3 we obtain at once:

THEOREM 4. *Let $f_i(x)$, $i = 1, \dots, r$, be polynomials of the type (3.1), with k replaced by k_i , and let no two exponents in $f_i(x)$ be congruent (mod $q-1$). Then the number of solutions of*

$$(3.3) \quad f_1(\xi_1) + \dots + f_r(\xi_r) = \alpha \quad (\xi_i \neq 0)$$

is given by

$$(3.4) \quad q^{r-1} + O(q^{r-w}) \quad \left(w = \frac{1}{2k_1} + \dots + \frac{1}{2k_r} \right).$$

Once again we have $w > 1$ for r sufficiently large.

The most interesting case of (3.1) is perhaps $f(x) = \alpha x + \beta x^{-1}$. The corresponding sum $S(f)$ is the Kloosterman sum

$$(3.5) \quad K(\alpha, \beta) = \sum_{\xi \neq 0} e(\alpha \xi + \beta \xi^{-1}).$$

Theorem 4 now implies:

THEOREM 5. *The number of solutions $\xi_i \neq 0$ of*

$$(3.6) \quad \alpha_1 \xi_1 + \frac{\beta_1}{\xi_1} + \cdots + \alpha_r \xi_r + \frac{\beta_r}{\xi_r} = \alpha \quad (\alpha \alpha_i \beta_i \neq 0)$$

is given by

$$(3.7) \quad q^{r-1} + O(q^{3r/4}).$$

Indeed if we make use of André Weil's estimate [10] for (3.5)

$$|K(\alpha, \beta)| \leq 2q^{1/2},$$

then (3.7) can be replaced by

$$(3.7)' \quad q^{r-1} + O(q^{r/2}),$$

which is significant for $r \geq 3$.

4. Another special case. Let $p > 2$. Theorem 4 applies to $f(x) = x^2 + x^{-2}$, and indeed (3.7) furnishes an asymptotic formula for the number of solutions of

$$(4.1) \quad \alpha_1 \xi_1^2 + \frac{\beta_1}{\xi_1^2} + \cdots + \alpha_r \xi_r^2 + \frac{\beta_r}{\xi_r^2} = \alpha \quad (\alpha \alpha_i \beta_i \neq 0).$$

However it is of interest to note that certain exact results can be obtained. Let N_1 and N_2 denote the number of solutions of (3.6) and (4.1), respectively. On the one hand

$$(4.2) \quad qN_1 = q^r + \sum_{\beta \neq 0} e(-\alpha\beta) \prod_{i=1}^r K(\beta\alpha_i, \beta\beta_i);$$

on the other hand

$$(4.3) \quad qN_2 = q^r + \sum_{\beta \neq 0} e(-\alpha\beta) \prod_{i=1}^r K_2(\beta\alpha_i, \beta\beta_i),$$

where

$$(4.4) \quad K_2(\alpha, \beta) = \sum_{\xi \neq 0} e(\alpha \xi^2 + \beta \xi^{-2}).$$

Let $\psi(\xi) = +1$ or -1 according as ξ is a square or a non-square of $GF(q)$. Then (4.4) implies

$$K_2(\alpha, \beta) = \sum_{\xi \neq 0} (1 + \psi(\xi)) e(\alpha \xi + \beta \xi^{-1}) = K(\alpha, \beta) + L(\alpha, \beta),$$

where

$$(4.5) \quad L(\alpha, \beta) = \sum_{\xi \neq 0} \psi(\xi) e(\alpha \xi + \beta \xi^{-1}).$$

Now it is not difficult to evaluate $L(\alpha, \beta)$ explicitly (compare [8, p.102]). We have

$$L(\alpha, \beta) = \begin{cases} 0 & (\psi(\alpha\beta) = -1) \\ G(1)(e(2\gamma) + e(-2\gamma)) & (\alpha\beta = \gamma^2). \end{cases}$$

As for the Gauss sum $G(1)$, we note [1, §3]

$$(4.7) \quad G(\alpha) = \sum_{\xi} e(\alpha \xi^2) = \psi(\alpha) G(1) \quad (\alpha \neq 0; G^2(1) = q\psi(-1)).$$

Then by (4.3), (4.4), and (4.5),

$$(4.8) \quad qN_2 = q^r + \sum_{\beta \neq 0} e(-\alpha\beta) \prod_{i=1}^r (K(\beta\alpha_i, \beta\beta_i) + L(\beta\alpha_i, \beta\beta_i)).$$

Comparison of (4.2) and (4.8) leads at once to:

THEOREM 6. *If $\psi(\alpha_i \beta_i) = -1$, $i = 1, \dots, r$, then the number of solutions of (4.1) is equal to the number of solutions of (3.6).*

5. Quadratic forms. In the remainder of the paper we shall be concerned with a quadratic form

$$(5.1) \quad Q(u_1, \dots, u_r) = \sum_{i,j=1}^r \alpha_{ij} u_i u_j \quad (\alpha_{ij} \in GF(q), \delta = |\alpha_{ij}| \neq 0).$$

We recall that the number of solutions $N_Q(\alpha)$ of

$$(5.2) \quad Q(\xi_1, \dots, \xi_r) = \alpha$$

is given [4, pp. 47-48] by

$$(5.3) \quad \begin{cases} q^{2s-1} + (q^s - q^{s-1}) \psi((-1)^s \delta) & (\alpha = 0) \\ q^{2s-1} - q^{s-1} \psi((-1)^s \delta) & (\alpha \neq 0) \end{cases}$$

for $r = 2s$;

$$(5.4) \quad q^{2s} + q^s \psi((-1)^s \alpha \delta)$$

for $r = 2s + 1$, where in (5.4) it is understood that $\psi(0) = 0$.

Now let $f(u_1, \dots, u_t)$ denote an arbitrary polynomial with coefficients in $GF(q)$, and let $N_f(\alpha)$ denote the number of solutions ζ_1 of

$$(5.5) \quad f(\zeta_1, \dots, \zeta_t) = \alpha.$$

Clearly the number of solutions ξ_i, ζ_j of

$$(5.6) \quad Q(\xi_1, \dots, \xi_t) = f(\zeta_1, \dots, \zeta_t)$$

is given by

$$(5.7) \quad N = \sum_{\alpha} N_Q(\alpha) N_f(\alpha).$$

We shall now show that the right member of (5.7) can be evaluated in certain cases.

In the first place let $f = u^k$. Then (5.7) becomes

$$N = \sum_{\xi} N_Q(\xi^k) = N_Q(0) + \sum_{\xi \neq 0} N_Q(\xi^k).$$

Now apply (5.3) and we get, for $r = 2s$,

$$N = (q^{2s-1} + (q^s - q^{s-1}) \psi((-1)^s \delta)) + (q-1) (q^{2s-1} - q^{s-1} \psi((-1)^s \delta)),$$

which is simply

$$(5.8) \quad N = q^{2s} \quad (r = 2s).$$

Similarly, application of (5.4) in the case $r = 2s + 1$ yields

$$(5.9) \quad N = \begin{cases} q^{2s+1} & (k \text{ odd}) \\ q^{2s+1} + q^s(q-1)\psi((-1)^s\delta) & (k \text{ even}). \end{cases}$$

This proves:

THEOREM 7. *The number of solutions of*

$$Q(\xi_1, \dots, \xi_r) = \zeta^k \quad (k \geq 1)$$

is furnished by (5.8) and (5.9).

A slight generalization of Theorem 7 is contained in:

THEOREM 8. *The number of solutions of*

$$Q(\xi_1, \dots, \xi_r) = \zeta_1^{k_1} \dots \zeta_t^{k_t} \quad (k_i \geq 1)$$

is given by

$$N = q^{t+2s-1} + \{(q^s - q^{s-1})q^t - q^s(q-1)^t\}\psi((-1)^s\delta)$$

for $r = 2s$;

$$N = \begin{cases} q^{t+2s} & ((k_1, \dots, k_t) \text{ odd}) \\ q^{t+2s} + q^s(q-1)^t\psi((-1)^s\delta) & ((k_1, \dots, k_t) \text{ even}) \end{cases}$$

for $r = 2s + 1$.

In the next place let f denote a polynomial such that $f(\zeta_2, \dots, \zeta_t)$ never vanishes. Then since for $r = 2s$, $\alpha \neq 0$, $N_Q(\alpha)$ is independent of α , we see that the number of solutions of (5.6) is given by

$$(5.10) \quad N = q^t \{q^{2s-1} - q^{s-2}\psi((-1)^s\delta)\}$$

for $r = 2s$. On the other hand, for $r = 2s + 1$ we get

$$(5.11) \quad N = q^{t+3s} + q^s\psi((-1)^s\delta) \sum_{\zeta_1, \dots, \zeta_t} \psi(f(\zeta_1, \dots, \zeta_t)).$$

We state:

THEOREM 9. *Let f be a polynomial such that $f(\zeta_1, \dots, \zeta_t)$ never vanishes. Then the number of solutions of (5.6) is furnished by (5.10) and (5.11).*

Note that the right member of (5.10) is independent of the polynomial f . It follows from (5.11) that the number of solutions of

$$Q(\xi_1, \dots, \xi_{2s+1}) = f^{2m+1}(\zeta_1, \dots, \zeta_t)$$

is the same for all values of m . Other special cases that lead to simple explicit results are contained in the following two theorems:

THEOREM 10. *Let f be a polynomial such that $f(\zeta_1, \dots, \zeta_t)$ never vanishes. Then the number of solutions of*

$$Q(\xi_1, \dots, \xi_{2s+1}) = f^2(\zeta_1, \dots, \zeta_t)$$

is given by

$$q^{t+2s} + q^{t+s} \psi((-1)^s \delta).$$

THEOREM 11. *Let f be a polynomial such that $f(\zeta_1, \dots, \zeta_t)$ never vanishes. Then the number of solutions ξ_i, η, ζ_j of*

$$Q(\xi_1, \dots, \xi_r) = \eta^k f(\zeta_1, \dots, \zeta_t) \quad (k \geq 1)$$

is q^{t+2s} for $r = 2s$, while for $r = 2s + 1$ the number of solutions is given by

$$(5.12) \quad \begin{cases} q^{t+2s} & (k \text{ odd}) \\ q^{t+2s} + q^s(q-1) \psi((-1)^s \delta) \sum_{\zeta_1, \dots, \zeta_t} \psi(f(\zeta_1, \dots, \zeta_t)) & (k \text{ even}). \end{cases}$$

In particular if f is the square of a polynomial then the second of (5.12) reduces to

$$q^{t+2s} + q^{t+s}(q-1) \psi((-1)^s \delta).$$

It is clear how Theorem 11 can be generalized to give the number of solutions of

$$Q(\xi_1, \dots, \xi_r) = \eta_1^{k_1} \dots \eta_w^{k_w} f(\zeta_1, \dots, \zeta_t).$$

A word may be added about a generalization of a different kind. Let Q_i denote quadratic forms in r_i indeterminates and of discriminant $\delta_i \neq 0$. Then we can treat such equations as

$$(5.13) \quad Q_1(\xi_1, \dots, \xi_{r_1}) Q_2(\eta_1, \dots, \eta_{r_2}) = \alpha.$$

For example, the number of solutions of (5.13) for $\alpha \neq 0$ is evidently

$$(5.14) \quad \sum_{\beta \neq 0} N_{Q_1}(\beta) N_{Q_2}(\alpha/\beta),$$

which can be evaluated by means of (5.3) and (5.4). In particular if r_1 and r_2 are both even, then (5.14) becomes

$$(q-1) (q^{2s_1-1} - q^{s_1-1} \psi((-1)^{s_1} \delta_1)) (q^{2s_2-1} - q^{s_2-1} \psi((-1)^{s_2} \delta_2)),$$

where $r_i = 2s_i$. In similar fashion we can determine the number of solutions of, say,

$$(5.15) \quad Q_1 Q_2 + \dots + Q_{2w-1} Q_{2w} = \alpha,$$

where no two Q 's have any unknowns in common.

6. Bounds ($t = 1$). Returning to (5.11) and (5.12), we remark that since an exact formula for such sums as

$$(6.1) \quad \sum_{\zeta_1, \dots, \zeta_t} \psi(f(\zeta_1, \dots, \zeta_t))$$

is usually not available, it is natural to look for a bound. We shall consider only the case $t = 1$. Then for the sum

$$T(f) = \sum_{\zeta} f(\zeta),$$

it follows from a theorem of Weil [10] that

$$(6.2) \quad T(f) = O(q^{1/2});$$

by more elementary methods one can prove the weaker estimate [3]

$$T(f) = O(q^{1-\Theta_k}) \quad (k = \deg f),$$

where $\Theta_3 = 1/4$, $\Theta_k = 3/2(k+4)$ for $k \geq 4$.

Thus applying (6.2) or (6.3) we obtain asymptotic results for (5.11) and (5.12) with $t = 1$.

7. Extension of results of §5. The results of §5 can be extended by making use of known results on the number of solutions of

$$(7.1) \quad Q(U_1, \dots, U_r) = \alpha$$

in polynomials $U_i \in GF[q, x]$ of degree $< m$; Q has its usual meaning. For simplicity we limit our attention to the case $r = 2s$. Cohen [2, p.556, Cor. 3] has proved that the number of solutions of (7.1) with $r = 2s$ is

$$(7.2) \quad \begin{cases} (q^s - \lambda) q^{(s-1)(2m-1)} & (\alpha \neq 0) \\ \lambda^m q^{ms} + (q^s - \lambda) q^{(s-1)(2m-1)} \sum_{z=0}^{m-1} \lambda^z q^{-z(s-2)} & (\alpha = 0), \end{cases}$$

where $\lambda = \psi((-1)^s \delta)$. Then we have:

THEOREM 12. *The number of solutions of*

$$(7.3) \quad Q(U_1, \dots, U_{2s}) = \zeta_1^{k_1} \dots \zeta_t^{k_t} \quad (k_1 \geq 1)$$

in polynomials U_i of degree $< m$ is

$$(q^t - (q-1)^t) \left\{ \lambda^m q^{ms} + (q^s - \lambda) q^{(s-1)(2m-1)} \sum_{z=0}^{m-1} \lambda^z q^{-z(s-2)} \right\} \\ + (q-1)^t (q^s - \lambda) q^{(s-1)(2m-1)},$$

where $\lambda = \psi((-1)^s \delta)$.

The proof is like that of Theorem 7.

THEOREM 13. *Let f be a polynomial such that $f(\zeta_1, \dots, \zeta_t)$ never vanishes. Then the number of solutions of*

$$Q(U_1, \dots, U_{2s}) = f(\zeta_1, \dots, \zeta_t)$$

in polynomials U_i of degree $< m$ and $\zeta_j \in GF(q)$ is

$$q^{t+(s-1)(2k-1)} (q^s - \lambda).$$

THEOREM 14. *Let f be a polynomial such that $f(\zeta_1, \dots, \zeta_t)$ never vanishes. Then the number of solutions of*

$$Q(U_1, \dots, U_{2s}) = \eta_1^{k_1} \dots \eta_t^{k_t} f(\zeta_1, \dots, \zeta_w) \quad (k_i \geq 1),$$

with $\deg U_i < m$, is q^w times the number of solutions of (7.3).

The proof of these theorems is immediate.

Finally we mention problems like (5.13) and (5.15) in which the unknowns are polynomials. Thus for example the number of solutions of

$$Q_1(U_1, \dots, U_{2s_1}) U_2(V_1, \dots, V_{2s_2}) = f(\zeta_1, \dots, \zeta_t),$$

with $\deg U_i < m_i$, $\deg U_2 < m_2$, where f never vanishes, is equal to

$$q^{t+(s_1-1)(2m_1-1)+(s_2-1)(2m_2-1)} (q^{s_1} - \lambda_1) (q^{s_2} - \lambda_2) (q - 1),$$

where $\lambda_i = \psi((-1)^{s_i} \delta_i)$, and δ_i is the discriminant of Q_i .

It may also be mentioned that in a problem like (7.3) we may restrict some of the U_i to be primary of degree m ; the final formula is similar to that obtained in Theorem 12. The same remark applies to the other theorems of this section.

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