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NORMAL k-TUPLES

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1. Introduction. This paper is an extension to k dimensions of most of the known theorems on normal numbers, along with several new results. Certain results are obtained showing some sufficient conditions under which the sum of normal numbers is normal.

DEFINITION 1. A *k-tuple* β is the k-tuple $(\alpha_1, \alpha_2, \cdots, \alpha_k)$, each α_i being a real number.

DEFINITION 2. The nth *k-digit* of a k-tuple to base r is

$$b^n = (a_1^n, a_2^n, \cdots, a_k^n),$$

where a_s^n is the nth digit of the fractional part of α_s to base (or scale) r.

DEFINITION 3. A k-tuple β is said to be *simply normal to the base* r if the number n_c of occurrences of the k-digit c in the first n k-digits of the fractional part of β has the property

$$\lim_{n \to \infty} \frac{n_c}{n} = \frac{1}{r^k}$$

for each of the r^k possible values of c.

DEFINITION 4. A k-tuple β is said to be *normal to the scale* r if β, $r\beta$, $r^2\beta$, $\cdots$ are each simply normal to all the scales $r, r^2, \cdots$, where

$$r^s\beta = (r^s\alpha_1, r^s\alpha_2, \cdots, r^s\alpha_k).$$

NOTE. If $\beta_k = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is normal, then any m-tuple, $m \leq k$, having any distinct m of the α_i as components is normal.

2. The correspondent and its use. We make the following definition.

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DEFINITION 5. The *correspondent number to the scale* r to a k-tuple β is the number $\alpha = .a_1 a_2 \cdots$ to the scale r such that

$$a_{(n-1)k+1}, \cdots, a_{nk}$$

is the nth k-digit of β to the scale r.

THEOREM 1. *A k-tuple is normal to the scale r if and only if its correspondent number to the scale r is normal to the scale r.*

Proof. Suppose β to be a normal k-tuple. Its correspondent α is simply normal to the scales r^k, r^{2k}, $\cdots$. Also $r^k\alpha$, $r^{2k}\alpha$, $\cdots$ are simply normal to the scales r^k, r^{2k}, $\cdots$. Thus α is simply normal to the scale r^k, and by a result[1] of Wall [4] is normal to the scale r.

Let α, a normal number, be the correspondent of β. Then α, $r^k\alpha$, $r^{2k}\alpha$, $\cdots$ are simply normal to the scales r^k, r^{2k}, $\cdots$, and β is normal to the scale r.

COROLLARY 1. *Given integers $0 \leq c_1 < c_2 < \cdots < c_j < m$, and a normal k-tuple β, delete all k-digits except those in positions congruent to c_1 or c_2 or $\cdots$ or c_j* (mod m). *The resulting k-tuple is normal.*

COROLLARY 2. *If β is a normal k-tuple to scale r^s, then β is normal to scale r.*

THEOREM 2. *A necessary and sufficient condition that a k-tuple β be be normal to scale r is that every sequence of v k-digits occur with a frequency of $1/r^{kv}$ for all v.*

Proof. By application of Theorem 1 and the Niven-Zuckerman paper [3], this result follows immediately.

THEOREM 3. *A necessary and sufficient condition that a k-tuple β be normal to scale r is that β be simply normal to the scales r, r^2, $\cdots$.*

Proof. This theorem follows immediately from Theorem 1 and a theorem of S. S. Pillai [4].

THEOREM 4. *Let P be any permutation of the digits* $0, 1, \cdots, r-1$, *and let $P\alpha$ be the number obtained from α by performing this permutation on all the digits of α.*

If $\beta = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is normal to the scale r, then so are:

[1] This result can be obtained easily by a counting process.

(1) $(\alpha_1, \cdots, \alpha_{i-1}, P\alpha_i, \alpha_{i+1}, \cdots, \alpha_k)$,

(2) *the k-tuple obtained from β by the application of P to the k-digits b_m, b_{2m}, b_{3m}, $\cdots$, giving β'.*

Proof. Since applying P to α_i replaces each sequence of digits of α_i by a prescribed other sequence of digits, (1) follows immediately.

To prove (2), one proceeds as follows. Form γ, the correspondent number to β. Then γ is normal to scale r^k. Now, considering γ to be written to scale r^k, form the m-dimensional correspondent δ of γ. This is normal to scale r, but the mth component of δ is formed from the imth k-digits of β $(i = 1, 2, \cdots)$. Performing the alterations (2) on β will merely replace each digit of the mth component of δ by another uniquely. Call the resulting number δ'. Thus a one-to-one correspondence will exist between sequences of m-digits of δ and those of δ'.

Given $\epsilon > 0$, there exists an N such that every fixed sequence of m-digits of length s, say c_s, of δ satisfies the condition

$$\left|\frac{n_{c_s}}{n} - \frac{1}{r^{ms}}\right| < \epsilon \text{ for } n > N,$$

where n_{c_s} is the number of occurrences of the sequence c_s among the first n m-digits of δ. Under the correspondence, since a fixed sequence in δ corresponds to a fixed sequence (not necessarily the same) in δ', we have

$$\left|\frac{n'_{c_s}}{n} - \frac{1}{r^{ms}}\right| < \epsilon \qquad \text{for } n > N.$$

Thus δ' is normal to scale r. Thus by Theorem 1, β' is normal to scale r.

3. An application of uniform distribution theory. We now need some more definitions.

DEFINITION 6. The symbol $\{f(x)\}$ represents the nonnegative fractional part of the real function of a real variable $f(x)$.

DEFINITION 7. [2, p. 90]. Let n be a given integer, and $\mathfrak{Z}$ an infinite sequence of intervals $Q \cdots$ $(a \leq x < b)$ (a and b integers). Let the number $N = N(Q)$ be the number of lattice points $[x]$ of Q, where $N(Q)$ increases without bound the Q run through $\mathfrak{Z}$.

To each Q let there correspond a system of n real functions $f_i(x)$, which are defined for each lattice point $[x]$ of Q. This function system

$$F(x, n) = [f_i(x)]$$

is said to be *uniformly distributed* (mod 1), or u.d. (mod 1), in the intervals Q of $\mathfrak{J}$ if for each fixed set of numbers $\gamma_1, \gamma_2, \cdots, \gamma_n$, with $0 \leq \gamma_i \leq 1$, the number

$$N' = N'(Q) = N'(Q; \gamma_1, \gamma_2, \cdots, \gamma_n)$$

of lattice points $[x]$ of Q for which

$$0 \leq \{f_i\} \leq \gamma_i \qquad (i = 1, 2, \cdots, n)$$

satisfies the condition

$$\lim \frac{N'(Q)}{N(Q)} = \prod_{i=1}^{n} \gamma_i$$

as Q runs through the infinite sequence $\mathfrak{J}$.

LEMMA A [2, p. 90]. *If the system* $[f_i(x)]$ *is defined at each lattice point* $[x]$ *of the intervals* Q, *and* α_i, β_i *are real numbers, where*

$$\alpha_i \leq \beta_i \leq \alpha_i + 1 \qquad (i = 1, 2, \cdots, n),$$

and

$$N'(Q) = N'(Q; \alpha_1, \beta_1; \alpha_2, \beta_2; \cdots; \alpha_n, \beta_n)$$

is the number of lattice points of Q *satisfying*

$$\alpha_i \leq \{f_i(x)\} < \beta_i \qquad (i = 1, 2, \cdots, n),$$

and $[f_i(x)]$ *is* u.d. (mod 1), *then*

$$\lim \frac{N'(Q)}{N(Q)} = \prod_{i=1}^{n} (\beta_i - \alpha_i).$$

LEMMA B [2, p. 92, Th. 7]. *The system* $[f_i(x)]$ *is* u.d. (mod 1) *if and only if, for each fixed set of integers* $(h_1, h_2, \cdots, h_k) \neq (0, 0, \cdots, 0)$,

$$\lim \frac{1}{N(Q)} \sum_{(x) \subset Q} e^{2\pi i(h_1 f_1(x) + \cdots + h_k f_k(x))} = 0.$$

LEMMA C [2, p. 94]. *The real function $f(x)$ of a real variable x is* u.d. (mod 1) *if for each fixed $q = 1, 2, \cdots$, the function $f(x+q) - f(x)$ is* u.d. (mod 1).

THEOREM 5.[2] *The k-tuple $\beta = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is normal to scale r if and only if the function system defined by*

$$f_i(x) = \alpha_i r^x \qquad (i = 1, 2, \cdots, k)$$

is u.d. (mod 1).

Proof. Assume that

$$[f_1(x), f_2(x), \cdots, f_k(x)] = F(x; k)$$

is u.d. (mod 1). Consider the sequence of k-digits

$$c = c_1 c_2 \cdots c_s = [a_1 a_2 \cdots a_s, b_1 b_2 \cdots b_s, \cdots, d_1 d_2 \cdots d_s],$$

the a_i from α_1, b_i from $\alpha_2, \cdots$, d_i from α_k. We shall count the occurrences of c in β. Let

$$\epsilon_1 = \sum_{i=1}^{s} \frac{a_i}{r^i}, \quad \epsilon_2 = \sum_{i=1}^{s} \frac{b_i}{r^i}, \cdots, \quad \epsilon_k = \sum_{i=1}^{s} \frac{d_i}{r^i}; \text{ and } \eta = \frac{1}{r^s}.$$

The frequency of occurrence of the sequence of digits c in β is the frequency with which

$$\epsilon_i \le \{f_i(x)\} < \epsilon_i + \eta$$

for all $i = 1, 2, \cdots, k$. From the conclusion of Lemma A this frequency is

$$\prod_{i=1}^{k} (\epsilon_i + \eta - \epsilon_i) = \prod_{i=1}^{k} \eta = \frac{1}{r^{sk}}.$$

By Theorem 2, β is normal to scale r.

Now assume that $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ is normal to scale r. We must show that the frequency of x's such that

$$\{f_i(x)\} < \eta_i \qquad \text{for } 0 \le \eta_i \le 1$$

[2]This theorem in one dimension was proved by D. D. Wall [5].

is $\eta_1 \eta_2 \cdots \eta_k$.

Case 1. Suppose first that η_1 and η_2 and $\cdots$ and η_k are terminating decimals in the scale r. Extend the shorter ones among these terminating decimals to the length of the longest, say m, by adding 0's. There are $\eta_i r^m$ sequences of m digits which, regarded as decimals, are less than η_i; thus there are $\eta_1 \eta_2 \cdots \eta_k r^{km}$ that must be counted. However, each of these sequences of k-digits of length m occurs in β with frequency $1/r^{km}$, and thus the frequency is $\eta_1 \eta_2 \cdots \eta_k$, as was desired.

Case 2. Suppose now that η_i is nonterminating for some i. Pick a sequence of terminating decimals $\eta_i^j \longrightarrow \eta_i$ for every $i = 1, 2, \cdots, k$. We know by Case 1 that the frequency for each j is

$$\eta_1^j \eta_2^j \cdots \eta_k^j \longrightarrow \eta_1 \eta_2 \cdots \eta_k \qquad \text{as } j \longrightarrow \infty .$$

COROLLARY. *The k-tuple $\beta = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is normal if and only if $\sum_{i=1}^k h_i \alpha_i$ is a normal number for all*

$$(h_1, h_2, \cdots, h_k) \neq (0, 0, \cdots, 0).$$

Proof. The result follows from Lemma B and Theorem 5.

DEFINITION 8. If $\beta = (\alpha_1, \alpha_2, \cdots, \alpha_k)$, then $(m_i)\beta$ is defined to be the k-tuple $(m_1 \alpha_1, m_2 \alpha_2, \cdots, m_k \alpha_k)$.

THEOREM 6. *If $\beta = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is a normal k-tuple, and A is a nonsingular $k \times k$ matrix with rational elements, then the transformed k-tuple $\gamma = \beta A$ is normal.*

Proof. By Theorem 5 and Lemma B, when we take lattice points of the form $(m_1 h_1, m_2 h_2, \cdots, m_k h_k)$, β is transformed to a new normal k-tuple if each component α_i is multiplied by a nonzero integer m_i. To show that multiplication of a component by a nonzero rational preserves normality, we first note that any such rational can be expressed in one of the forms b/r^s, $b/r^s(r^t - 1)$, where b is an integer. (For $1/b$ has a scale r expansion containing a period of length t, while s depends on the point at which periodicity begins.)

Now multiplication by an integer b preserves normality, as shown above. Division by r^s is normality-preserving from the definition. We consider division by $r^t - 1$. Let δ be normal to scale r. Then it is normal to scale r^t. Taking δ as written to scale r^t, we conclude that $(r^{tq} - 1)\delta/(r^t - 1)$, for any positive integer q, is also normal to scale r^t, since the operation involved is multiplication by an integer. Then by the corollary to Theorem 5,

$$\frac{r^{tq}-1}{r^t-1}\sum_{i=1}^{k} h_i\,\alpha_i\,r^{tx}$$

is u. d. (mod 1) for all

$$(h_1, h_2, \cdots, h_k) \neq (0, 0, \cdots, 0).$$

By Lemma C,

$$\sum_{i=1}^{k} \frac{h_i\,\alpha_i\,r^{tx}}{r^t-1}$$

is u. d. (mod 1). Thus by the corollary to Theorem 5, $\delta/(r^t-1)$ is normal to scale r^t, and hence, by Corollary 2 to Theorem 1, to scale r.

We have shown that multiplication of α_i by a rational preserves normality. By choosing $h_i = h_i' + h_j'$ in Lemma B, we can show that replacement of α_j by $\alpha_i + \alpha_j$ preserves normality. Interchange of two α's does not affect normality. Thus the elementary operations of which multiplication by the matrix A is composed preserve normality.

THEOREM 7. *Almost all k-tuples are normal.*

Proof. This theorem follows immediately from the foregoing Theorem 5, and Theorems 8 and 15 of [2; pp. 92 and 94].

THEOREM 8. *The set of numbers simply normal to no scale is noncountable.*

Proof. It is not difficult to show that the set of numbers simply normal to a given scale forms a set of the first category [1, p. 134]; the sum of a countable number of sets of the first category is also of the first category [1, p. 137], and thus the complementary set has the cardinal number of the continuum [1, p. 136], and is thus noncountable.

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