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THE NUMBER OF FARTHEST POINTS

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1. Introduction. Consider a set S in a metric space E. *For each point* $x \in E$, let $y(x)$ *denote a point of* S *which has maximum distance from* x, and let $Y(x)$ be the set of all $y(x)$ with that property. It is our purpose here to study sets S for which certain restrictions are placed on the number of points in $Y(x)$. In §2 we analyze those sets S in the Minkowski plane for which $Y(x)$ has exactly one element for each $x \in S$. In §3 we characterize those sets in the Euclidean plane E_2 for which $Y(x)$ has at least two elements for each $x \in S$.

In order to achieve these ends we first establish some introductory results which hold in rather general spaces.

DEFINITION 1. Let S be a set in a metric space. If S is contained in a sphere of radius r, then its *r-convex hull* is the intersection of all closed spheres of radius r which contain S.

A set S is *r-convex* if it coincides with its r-convex hull [2, p. 128].

LEMMA 1. *Let* S *be a set of diameter* d *in a linear metric space. Then for each* $x \in S$ *the set* $Y(x)$ *lies in the boundary of the* d-*convex hull of* S.

Proof. If $Y(x) \neq 0$, choose any point $y(x)$. Then S is contained in a sphere with center at x and with radius $d(x, y)$, where $d(x, y)$ denotes the distance from x to y. Since for $x \in S$ we have $d(x, y) \leq d$, there exists a point z on the ray $\overrightarrow{yx}$ such that the sphere with center z and with radius $d = d(z, y)$ contains S. The point y is thus clearly on the boundary of the d-convex hull.

NOTE. By virtue of Lemma 1, all results for compact S below will hold under the less restrictive assumption that S contain the intersection of its closure with the boundary of its d-convex hull.

COROLLARY 1. *Let* S *be a set in a linear metric space. Then for each* x *the set* $Y(x)$ *is contained in the boundary of the convex hull of* S.

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This is an immediate consequence of the fact that S is contained in the sphere with center x and radius $d(x, y(x))$, provided $Y(x) \neq 0$.

LEMMA 2. *Suppose S is a set in a linear metric space, and let T be a set such that $Y(x) \neq 0$ for each $x \in T$. Then $d(x, y(x))$ is a continuous function of x on T.*

Proof. Since $|d(x, z) - d(u, z)| \leqq d(x, u)$, and since

$$|\max_{z \in S} d(x, z) - \max_{z \in S} d(u, z)| = |d(x, y(x)) - d(u, y(u))|,$$

we have $|d(x, y(x)) - d(u, y(u))| < \epsilon$ if $d(x, u) < \epsilon$.

LEMMA 3. *Let S be a compact set in a linear metric space. If $x_i \longrightarrow x$, then all limit points of the sequence $\{y(x_i)\}$ lie in $Y(x)$.*

Proof. Let $y_i = y(x_i)$ be a sequence of points. Let y be a limit point of the sequence $\{y_i\}$. Then the continuity of $d(x, y(x))$ implies that $d(x, y) \geq d(x, q)$ for all $q \in S$. Hence we have $y \in Y(x)$.

LEMMA 4. *Let S be a compact set in a linear metric space, and suppose $y(x)$ is single-valued on a set T. Then $y(x)$ is a continuous mapping of T into S.*

Proof. Since $y(x)$ is single-valued, Lemma 3 implies that if $x_i \longrightarrow x$, then $y(x_i) \longrightarrow y(x)$.

2. Sets in M_2 on which $y(x)$ is single-valued. Let M_2 be a two-dimensional Minkowski space [2, p. 23]. We restrict our attention here to connected sets S in M_2. (See §4 for remarks about disconnected sets.)

THEOREM 1. *Let S be a continuum (compact connected set) in M_2. If $y(x)$ is single-valued on S, then the set sum*

$$\sum_{x \in S} Y(x)$$

is the entire boundary B of the convex hull of S; and this convex hull is d-convex, where d is the diameter of S.

Proof. According to Corollary 1, we have

$$\sum_{x \in S} Y(x) \subseteq B.$$

By Lemma 4, the mapping $y(x)$ yields a continuous mapping of S into B. Now the only connected sets in a simple closed curve are: (1) a point, (2) a simple arc, (3) the whole closed curve. For cases (1) and (2), let

$$A \equiv \sum_{x \in S} Y(x);$$

then the mapping $y(x)$ of A into itself must have a fixed point $x_0 = y(x_0)$, so that $\{x_0\} = Y(x_0) = S$, in which case the theorem is trivial. Thus $A = B$ in all three cases. Moreover, since by Lemma 1 the set $A = B$ lies in the boundary of of the d-convex hull of S, the boundary of the d-convex hull must coincide with B.

Since there is no continuous mapping without fixed points of a closed two-cell into itself, Lemma 2 and Theorem 1 imply that, for single-valued $y(x)$, the connected bounded set S must contain the entire boundary of its convex hull, but not all of the interior of that hull (unless S consists of a single point). It may suffice, in some cases, to delete one single point from the interior of a convex set; for instance, in the case of a circular disc in E_2, the deletion of the center makes $y(x)$ single-valued throughout.

In the remaining theorems and lemmas we restrict our attention to sets in E_2.

DEFINITION 2. By a *normal* to a convex curve C at a point $x \in C$ we mean a line perpendicular to a line of support to C at x.

NOTATION. We designate a line of support at x by $L(x)$, and the corresponding normal by $N(x)$. Further, for a point $y \in S$, we let $x(y)$ be a point in S such that $y = y(x)$, and let $X(y)$ be the set of all $x(y)$.

THEOREM 2. *Suppose S is the boundary of a compact convex set in E_2, and suppose $y(x)$ is single-valued on S. Then:*

(1) *The set $X(y)$ consists of all points of intersection of the normals to S at y with $S - y$. If S has a tangent at y, then $x(y)$ is single-valued and continuous at y.*

(2) *The mapping $x(y)$ is monotonic; that is, the order of $x(y_1)$, $x(y_2)$, $x(y_3)$ on S has the same sense as that of y_1, y_2, y_3.*

Proof. (1) If $x = x(y)$, then the circle with center x and radius $d(x, y)$ contains S. Hence the tangent to this circle at the point y is also a line of support to S, and the radius lies in a normal to S at y.

Now, let $y_i \longrightarrow y$, $y_i \in S$, and choose $x_i = x(y_i)$. Then, due to the continuity of the mapping $y(x)$, each limit point of $\{x_i\}$ is in $X(y)$. Thus if S has a tangent at a point y, then the mapping $x(y)$ is one-to-one and continuous at y.

To complete the proof of (1), suppose S has a corner at y. Then the farthest points of intersection from y of the normals at y with S fill out a closed subarc of S, which we denote by S_1; the end-points of S_1 we denote by u_l and u_r. There exists a sequence $y_i \in S$ with $y_i \longrightarrow y$ such that the normals to S at the y_i are unique and approach the left normal at y. Hence, by the above, $x(y_i)$ converges to u_l, and hence $u_l \in X(y)$. Similarly, $u_r \in X(y)$. The three lines determined by u_l, u_r, and y divide the plane into seven closed sets, and the arc S_1 is contained in that unbounded one which has $u_l u_r$ as part of its boundary. We denote that set by A. Since each of the two circles with centers u_l and u_r which pass through y contains S, it follows by the law of cosines that $y(u) = y$ for all $u \in A$. Hence $S_1 \subseteq X(y)$. According to Theorem 1, the curve S contains no straight line-segment, and thus any normal to S intersects S in exactly two points. Hence the common part $(S - S_1) \cdot X(y)$ is the null set, so that $S_1 = X(y)$.

(2) The above facts, together with the fact that each $u \in S$ is contained in some $X(y)$, imply that the transformation $x(y)$ maps connected sets into connected sets, even though the mapping need not be single-valued and therefore not necessarily continuous. The single-valuedness of $y(x)$ implies that if $y_1 \neq y_2$, then $X(y_1) \cdot X(y_2) = 0$. If the transformation $x(y)$ failed to be monotonic, it would have a fixed point $y = x(y)$; but this is impossible unless S is a single point. Hence condition (2) must hold.

COROLLARY 2. *Suppose C is the boundary of a compact convex set S. Let $\alpha\beta$ be a diameter of C, and let $N(\alpha, \beta)$ designate the common normal to C through α and β. Then $y(x)$ is single-valued on C if and only if for every pair of points u, $v \in C$ which lie on the same side of $N(\alpha, \beta)$, the normals $N(u)$ and $N(v)$ intersect at an interior point of S.*

Proof. First observe that, for any compact convex set S with $\alpha\beta$ as a diameter, if $x \cdot \alpha\beta = 0$, then x and $y(x)$ must lie on opposite sides of $N(\alpha, \beta)$.

To prove the necessity, observe that α and β are involutory points in the sense that

$$y(y(\alpha)) = \alpha \text{ and } y(y(\beta)) = \beta.$$

Hence the necessity follows from the monotonicity of $y(x)$ as described in

Theorem 2.

To prove the sufficiency, first choose $x \in C - (C \cdot \alpha\beta)$. Suppose $y(x)$ is not single-valued, and choose $u, v \in Y(x)$. As mentioned above, $y(x)$ and x lie on opposite sides of $N(\alpha, \beta)$. A circle with center x and radius $d(x, u)$ is tangent to C at both u and v, and the normals $N(u)$ and $N(v)$ intersect at x, which is not interior to S. Hence $y(x)$ is single-valued for $x \in C - (C \cdot \alpha\beta)$. By continuity it follows also that

$$y(\alpha) = \beta, \; y(\beta) = \alpha.$$

This completes the proof.

In the following we shall extend the generalized notions of curvature described by Bonnesen and Fenchel [2, pp. 143-144]. Choose a point $x \in C$, where C is a closed convex curve together with a line of support $L(x)$. The circle tangent to $L(x)$ at x and passing through a point $p \in C - x$ must have its center $z(p)$ on the normal $N(x)$ to $L(x)$ at x. Establish an order on $N(x)$ in terms of the distance from x, and let

$$\left.\begin{aligned} E_s(x, \delta(x)) &\equiv \sup_p z(p) \\ E_l(x, \delta(x)) &\equiv \inf_p z(p) \end{aligned}\right\} \quad p \in \delta(x) - x,$$

where $\delta(x)$ is an arc of C containing x. We define four types of centers of curvature as follows:

$$E_s(x) \equiv E_s(x, C), \quad E_l(x) = E_l(x, C).$$

$$E_o(x) \equiv \lim_{\delta(x) \to x} E_s(x, \delta(x)), \quad E_i(x) \equiv \lim_{\delta(x) \to x} E_l(x, \delta(x)).$$

Clearly $E_l(x) \leq E_i(x) \leq E_o(x) \leq E_s(x)$ relative to $N(x)$.

DEFINITION 3. The sets

$$\sum E_s(x), \; \sum E_o(x), \; \sum E_i(x), \text{ and } \sum E_l(x)$$

(x ranges over C) are respectively called the *superior evolute*, the *outer evolute*, the *inner evolute*, and the *inferior evolute* of S, and are denoted by E_s, E_o, E_i, E_l.

THEOREM 3. *Suppose C is the boundary of the compact convex set $S \subset E_2$. If $y(x)$ is single-valued on C, then the superior evolute, and hence all four* *evolutes, of C must be contained in S.*

Proof. Since $y(x)$ is single-valued for each point $x_1 \in C$, the proof of Theorem 2 implies that for any normal $N(x_1)$, the set $N(x_1) \cdot (C - x_1)$ consists of a single point, denoted by x'. Choose $p \in C - x_1$. Since

$$d(x', p) < d(x', x_1) = d(x', y(x')),$$

it is clear that the perpendicular bisector B of the segment $x_1 p$ intersects the segment $x_1 x'$. Hence

$$B \cdot x_1 x' = z(p) \in S.$$

THEOREM 4. *Suppose the inner evolute of the boundary C of the compact convex set S is contained in $S - C$. Then $y(x)$ is single-valued on C.*

Proof. Suppose there exists an $x \in C$ such that $y(x)$ is not single-valued. Choose $u, v \in Y(x)$. The circle with center x and radius $d(x, u)$ contains S and is tangent to C at u and v. Hence the arc uv of $C - x$ contains a point w of minimal distance from x. The circle with center x and radius $d(x, w)$ is tangent to C at w, while a neighboring arc of w on C lies outside or on that circle. Hence C has a unique normal at w and $E_i(w) \geq x$, so that $E_i(w)$ is on or outside C.

Theorems 3 and 4 do not determine the single-valuedness of $y(x)$ on S if E_i, E_o, and E_s lie in S and contain points of C. This situation can be described as follows:

THEOREM 5. *Let S be a compact convex set with boundary C such that E_s (and hence each of the evolutes) of C lies in S. Then $y(x)$ fails to be single-valued on C if and only if there exists a point $x \in C$ which lies on E_i, E_o, and E_s, and which is the center of a circular arc contained in C.* [1]

Proof. To prove sufficiency, suppose there exists a point $x \in C$ which is the center of a circular arc $C_1 \subset C$, and suppose $y(x)$ is single-valued on C. Then according to Theorem 2 the single-valuedness of $y(x)$ implies $x \in X(y)$ for each $y \in C_1$. Hence $C_1 \subseteq Y(x)$, a contradiction.

To prove necessity, assume $y(x)$ is not single-valued on C. Choose u, $v \in Y(x)$, and let w be a nearest point to x of the arc C_1 of $C - x$ joining u and v. In the proof of Theorem 4 we saw that $E_i(w) \geq x$; but since the evolutes are

[1] By "center of a circular arc" we mean the center of the circle to which the arc belongs.

in S, we have

$$E_i(w) = E_o(w) = E_s(w) = x.$$

(Since E_s is bounded, C can contain no straight line segments.) Hence the circle with center x and radius $d(x, w)$ contains S. Thus $d(x, w) \geq d(x, u)$. From the definition of w it now follows that $d(x, z) = d(x, u)$ for each $z \in C_1$. Hence C_1 Hence C_1 is circular arc in C with center at x.

As seen earlier, if S is a simply connected set containing at least two points, then $y(x)$ is not single-valued on S. The situation is described more fully in the following theorem.

THEOREM 6. *Let S be a compact convex set in E_2 with boundary C. Then $y(z)$ is single-valued if $z \neq E_s(x)$ for all $x \in C$; and $y(z)$ is not single-valued if $z = E_s(x)$, $z \neq E_o(x)$ for some $x \in C$.*

Proof. Assume $y(z)$ is not single-valued; then there exist distinct points $u \in Y(z)$, $v \in Y(z)$, and the circle with center z and radius $d(z, u)$ contains S and is tangent to C at u and v. Hence $E_s(u) = E_s(v) = z$.

Now suppose there exists an $x \in C$ such that $z = E_s(x)$, $z \neq E_o(x)$. Then, since C is compact, there exists a point $u \neq x$, $u \in C$, such that

$$d(z, u) = d(z, x) = d(z, y(z)).$$

Hence $u \in Y(z)$, $x \in Y(z)$. Thus Theorem 6 is proved.

A few remarks about the four evolutes may be desirable at this point. The inferior and superior centers of curvature, $E_l(x)$ and $E_s(x)$, are determined by properties in the large. In fact, E_l contains the set of centers of those circles which are in S and which are tangent to C at not less than two points. Similarly E_s contains the sets of centers of those circles which contain C and which are tangent to C at not less than two points.

Since a convex curve C has curvature almost everywhere, we have $E_i(x) = E_o(x)$ for almost all $x \in C$. Let us define

$$E \equiv \sum E_i(x)\, E_o(x),$$

(x ranges over C), where, as usual, $E_i(x)\, E_o(x)$ denotes a closed segment. The number of normals to C through a point $x \in E_2$, as a function of x, is the same in each component of the complement of E. In the case where S is a compact convex set for which E is bounded, there are exactly two normals to C through each point x in the unbounded component of the complement of E (the lines joining y to the nearest and farthest points on C). However, from each point $y \notin E$ on $E_l(E_s)$ there are at least four normals to C. [According to Theorem 6, there are at least two normals to the two or more points of tangency u, v of the inscribed (circumscribed) circle with center at y. In addition, there are lines joining y to nearest (farthest) points on each of the two arcs of C joining u and v.] Thus E_l and E_s do not intersect the unbounded component of $\bar{E}$. These statements imply the following:

THEOREM 7. *Let C be the boundary of a compact convex set $S \subset E_2$. Then $E_s \subset S$ if and only if $E_o \subset S$. Also $E_s \subset S - C$ if and only if $E_o \subset S - C$.*

AN EXAMPLE. Consider the family of ellipses $C(e)$,

$$b^2 x_1{}^2 + a^2 x_2{}^2 = a^2 b^2, \ a \geqq b.$$

If the eccentricity e satisfies the condition $e \leq \sqrt{2}/2$, then $y(x)$ is single-valued on $C(e)$. If $e > \sqrt{2}/2$, then $y(x)$ is not single-valued at $x = (0, \pm b)$. In each case the inner and outer evolutes coincide; they form the familiar astroid with cusps at

$$\xi = (a_1, 0),\ \eta = (-a_1, 0),\ \tau = (0, b_1) \text{ and } \rho = (0, -b_1),$$

where $a_1 < a$, and $b_1 < b$ for $e < \sqrt{2}/2$ while $b_1 > b$ for $e > \sqrt{2}/2$. The superior evolute E_s is the closed line-segment $\rho\tau$, and E_l is the closed line-segment $\xi\eta$. If $e \neq 0$, then $y(x)$ is single-valued on the complement of the open segment $\rho\tau - \rho - \tau$.

3. Sets on which $Y(x)$ contains at least two points.

THEOREM 8. *Let $S \subset E_2$ be a compact set of diameter d, and let D denote the set of end-points of diameters of S. If $Y(x)$ has at least two elements for each $x \in D$, then $Y(x)$ consists of exactly two points for $x \in D$, and D contains a finite number of points. The d-convex hull of S coincides with the d-convex hull of D.* [Since the latter is a Reuleaux polygon (see below), D must contain an odd number of points.]

Proof. Let $\Sigma \equiv \{C(x)\}$ be the family of circular boundaries $C(x)$ with centers $x \in D$ and with radii d. Let $x \in D$; then

$$Y(x) = C(x) \cdot D.$$

Since

$$\operatorname{diam} Y(x) \leq \operatorname{diam} S = d,$$

there exists a *smallest* arc A of $C(x)$ which contains $Y(x)$, and which has a length not exceeding $\pi d/6$. Let x_1 and x_2 be the end-points of A. If a circle $C(x') \in \Sigma$ were to intersect $A - x_1 - x_2$, then $C(x')$ would separate x_1 and x_2 since

$$\text{length } A \leqq \pi d/6.$$

But this contradicts the fact that $S \subset C(x')$. For any $x \in D$, we have $z = y(x)$ if and only if $x = y(z)$. Hence every $x \in D$ is a point of intersection of at least two circles of Σ. These facts imply that $Y(x) \equiv \{x_1, x_2\}$.

Define

$$H \equiv \prod_{x \in D} K(x),$$

where $K(x)$ is the closed circular disk with center x and with radius d. Then each $x \in D$ lies in the interior of all $K(x) \subset H$ except $K(x_1)$ and $K(x_2)$, where $Y(x) = \{x_1, x_2\}$. Hence x is a corner-point of the boundary of H. As above, let A_1 and A_2 be the smallest arcs of $C(x_1)$ and $C(x_2)$ containing $Y(x_1)$ and $Y(x_2)$, respectively. We have shown that $A_1 \cdot A_2 = \{x\}$; and A_1 and A_2 are in the boundary of H. Thus x is an isolated corner of the boundary of H. Hence D contains a finite number of points, and by definition the boundary of H is the boundary of the d-convex hull of D. It is clearly a Reuleaux polygon, that is, a convex circular polygon whose arcs have radii d, and whose vertices are the centers of these arcs [2, pp. 130-131].

Finally, each of the circles in Σ contains S, and hence $S \subset H$.

COROLLARY 3. *Let S be a set satisfying the conditions of Theorem 8. Then $Y(x) \subseteq D$ for each $x \in S$.*

This is an immediate consequence of the fact that D consists of the vertices of H.

THEOREM 9. *Let $S \subset E_2$ be a compact set such that $Y(x)$ has at least two elements for each $x \in S$. Then S lies in the union of a finite number of line-segments. Moreover, if $Y(x)$ has exactly two elements for each $x \in S$, then S cannot be connected.*

Proof. Since $Y(x) \subseteq D$ for each $x \in S$, the fact that $Y(x)$ has a least two elements implies that x lies on the perpendicular bisector of the line joining two elements of D. Thus S is a subset of the set obtained by taking the union of the intersections of these perpendicular bisectors with H.

Since the set H has at least three corners x_1, x_2 and x_3, let S_i $(i = 1, 3)$ consist of those $x \in S$ such that $\{x_i, x_2\} \subseteq Y(x)$. Each set S_i is nonempty since S_i contains the center of the smaller arc of H joining x_i and x_2. From the continuity of $d(x, y(x))$, it follows that S_i is closed. Hence if S is connected, then $S_1 \cdot S_2 \neq 0$ (since S is compact), and thus there exists an $x' \in S$ such that $Y(x') \supseteq \{x_1, x_2, x_3\}$. This establishes the theorem.

We also obtain the following result due to Bing [1].

COROLLARY 4. *Let S be a bounded set in E_2 containing at least two points, and having the property that with every two points $x \in S$, $y \in S$ there exists a $z \in S$ such that the triangle $x\ y\ z$ is equilateral. Then S is the set of vertices of an equilateral triangle.*

Proof. The closure $\overline{S}$ of S must also satisfy the hypothesis stated. Consider the set D of Theorem 8 relative to $\overline{S}$. If $x \in D$, and $\{y, z\} \subseteq Y(x)$, then $d(y, z) = d$, so that x, y, z form the vertices of a Reuleaux polygon, and therefore by Theorem 8 we have $D = \{x, y, z\}$. Now let u be the centroid of the triangle x, y, z. By Theorem 9, S is contained in the segments xu, yu, and zu. Suppose $v \in (S \cdot xu - x)$; then $Y(v) = \{y, z\}$. But v, y, z is not equilateral; hence $S \cdot xu = x$. Similarly, $S \cdot yu = y$, $S \cdot zu = z$. Consequently, $S = \{x, y, z\}$.

4. Remarks and problems. Several questions are raised by our theorems.

(1) If we try to characterize disconnected sets in E_2 for which $y(x)$ is single-valued, we see that this condition is not very restrictive. In fact, given any set S which contains at least one point of the boundary of its r-convex hull H for some radius r, we can adjoin a single point z to S, such that z lies on an interior normal to H at a point of $H \cdot S$, and such that $y(x)$ relative to $S + \{z\}$ is single-valued on $S + \{z\}$.

(2) The characterization of connected sets S in E_n $(n > 2)$ for which $y(x)$ is single-valued on S offers considerable difficulties. The mapping $y(x)$ still yields a continuous map of S into the boundary of its convex hull, but it need no longer be an onto mapping. For example, the torus, both the solid and its surface, will have single-valued $y(x)$ for suitable ratios of the two radii. The argument that a nontrivial compact S which contains no indecomposable continua cannot be simply-connected holds, however, regardless of dimension, since every continuous mapping of such a simply-connected set S into itself has fixed points [4].

(3) The generalization of the discussion of multivalued $y(x)$ suggests the

following problem: Let S be a compact set in E_n such that $Y(x)$ has at least k elements for $x \in S$. Does it follow that S lies in the union of a finite number of $(n-k+1)$-dimensional planes? (Note that in the case $k = 1$ this is no restriction, while for $k > n+1$ there would be no sets S.) Are there any sets for which $k = n+1$?

It seems likely that this generalization is false, since the argument which proved the finiteness of the set D in Theorem 8 fails for $n > 2$.

In the case $k \geq n$, all points of D are vertices of their d-conves hull. Thus in this case D must surely be denumerable.

(4) Is it possible to generalize Corollary 4, as follows:

If the bounded set S in E_n contains at least two points; and if, for some $k \geq 2$, with every two points $x, y \in S$ there are $k-1$ points in S which together with x, y form the vertices of a regular k-simplex, does it follow that S is the set of vertices of a regular l-simplex, where $k \leq l \leq n$?

(5) Another question raised by Corollary 4 is the following:

What are the sets (bounded sets, compact sets) S in E_2 which have the property that with every $x, y \in S$ there is a $z \in S$ such that $x\ y\ z$ is an isosceles triangle with vertex z and prescribed verticle angle α?

For $\alpha < \pi/3$, a nontrivial set with the stated property obviously cannot be bounded. For $\alpha = \pi/3$, the question for the bounded case is answered by Corollary 4. For $\alpha > \pi/3$, there is a considerable variety of bounded sets, although none of them can be finite. In fact, for $\alpha > \pi/3$ every S must be dense in itself; and thus, if closed, it must be perfect. The case $\alpha = \pi$ has been discussed by J. W. Green and W. Gustin [3]; for closed sets S, this case characterizes convexity.

An easy argument shows that for compact S, and $\pi/3 < \alpha \leq \pi/2$, the entire line-segment joining two farthest points of S must be contained in S.

It may also be worth remarking that if S has the foregoing property for an angle α, then its complement has the same property for the angle $\pi - \alpha$. Thus the case $\alpha = \pi/2$ is especially noteworthy, since in this case the class of all S with the stated property is closed under the operation of taking complements.

(6) Finally, one should compare the theorems about $Y(x)$ with those for $M(x)$, where $M(x)$ denotes the subset of S whose points have minimum distance from x. In particular the theorem of Motzkin [6, 7] (see also Jessen [5]) states that a closed set S is convex if and only if $M(x)$ is a single point for all x. This theorem does not correspond to any of the results on $Y(x)$ in §1. In fact, the

analogous assumption, concerning a (not necessarily closed) set S in E_n, that $y(x)$ be single-valued *for all* x, is satisfied if and only if S consists of a single point.

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