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EXTENSION OF A RENEWAL THEOREM

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1. Introduction. A chance variable x will be called a d -lattice variable if

$$(1) \quad \sum_{n=-\infty}^{\infty} \Pr\{x = nd\} = 1$$

and

$$(2) \quad d \text{ is the largest number for which (1) holds.}$$

If x is not a d -lattice variable for any d , x will be called a *nonlattice variable*. The main purpose of this paper is to give a proof of:

THEOREM 1. *Let x_1, x_2, \dots be independent identically distributed chance variables with $E(x_1) = m > 0$ (the case $m = +\infty$ is not excluded); let $S_n = x_1 + \dots + x_n$; and, for any $h > 0$, let $U(a, h)$ be the expected number of integers $n \geq 0$ for which $a \leq S_n < a + h$. If the x_n are nonlattice variables, then*

$$U(a, h) \rightarrow \frac{h}{m}, 0 \quad \text{as } a \rightarrow +\infty, -\infty.$$

If the x_n are d -lattice variables, then

$$U(a, d) \rightarrow \frac{d}{m}, 0 \quad \text{as } a \rightarrow +\infty, -\infty.$$

(If $m = +\infty$, h/m and d/m are interpreted as zero.)

This theorem has been proved (A) for nonnegative d -lattice variables by Kolmogorov [5] and by Erdős, Feller, and Pollard [4]; (B) for nonnegative nonlattice variables by the writer [1], using the methods of [4]; (C) for d -lattice variables by Chung and Wolfowitz [3]; (D) for nonlattice variables such that the distribution of some S_n has an absolutely continuous part and $m < \infty$ by Chung

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and Pollard [2], using a purely analytical method; and (E) in the form given here by Harris (unpublished). Harris' proof does not essentially use the results of the special cases (A), (B), (C), (D); the proof given here obtains Theorem 1 almost directly from the special cases (A) and (B) by way of an integral identity and an equation of Wald.

2. An integral identity. Let N_1 be the smallest n for which $S_n > 0$, and write $z_1 = S_{N_1}$; let N_2 be the smallest $n > 0$, for which $S_{N_1+n} - S_{N_1} > 0$, and write $z_2 = S_{N_1+N_2} - S_{N_1}$, and so on. Continuing in this way, we obtain sequences $N_1, N_2, \dots; z_1, z_2, \dots$ of independent, positive, identically distributed chance variables such that

$$S_{N_1+\dots+N_K} = z_1 + \dots + z_K.$$

Let $V(t), R(t)$ denote the expected number of integers $n \geq 0$ for which

$$T_n = z_1 + \dots + z_n \leq t \text{ and } -t \leq S_n \leq 0,$$

$n < N_1$, respectively. That $V(t) < \infty$ follows from a theorem of Stein [6], and that $R(t) < \infty$ follows from $E(N_1) < \infty$, which we show in the next section. The integral identity is:

$$\text{THEOREM 2. } U(a, h) = \int_0^\infty [R(t-a) - R(t-a-h)] dV(t).$$

Proof. If n_K is the number of integers n with

$$N_1 + \dots + N_K \leq n < N_1 + \dots + N_{K+1} \text{ and } a \leq S_n < a + h,$$

we have

$$E(n_K | T_K = t) = R(t-a) - R(t-a-h),$$

so that

$$E(n_K) = \int_0^\infty [R(t-a) - R(t-a-h)] dF_K(t),$$

where $F_K(t) = \Pr\{T_K \leq t\}$. Summing over $K = 0, 1, 2, \dots$, and using the fact that

$$V(t) = \sum_{K=0}^{\infty} F_K(t),$$

we obtain the theorem.

3. Wald's equation. The main purpose of this section is to note that $E(N_1)$ is finite, so that an equation of Wald [7, p. 142] holds.

THEOREM 3. $E(N_1) < \infty$ and $mE(N_1) = E(z_1)$, so that $m, E(z_1)$ are both finite or both infinite.

Proof. In showing $E(N_1)$ finite, we may suppose $\{x_n\}$ bounded above; for defining $x_n^* = \min\{s_n, M\}$ yields an $N_1^* \geq N_1$; choosing M sufficiently large makes $E(x_n^*) > 0$, and $E(N_1^*) < \infty$ implies $E(N_1) < \infty$. Since

$$\frac{T_K}{K} = \frac{S_{N_1 + \dots + N_K}}{N_1 + \dots + N_K} \cdot \frac{N_1 + \dots + N_K}{K},$$

we obtain, letting $K \rightarrow \infty$ and using the strong law of large numbers, first that $E(z_1) = mE(N_1)$ and next since if $\{x_n\}$ is bounded above and $\{z_n\}$ is bounded, that $E(N_1)$ is finite in this case and consequently in general.

4. The d-lattice case. For d -lattice variables, Theorem 2 yields

$$(3) \quad U(nd, d) = \sum_{s=0}^{\infty} r(s-n) v(s) = \sum_{s=0}^{\infty} r(s) v(s+n),$$

where $r(s) = R(sd) - R([s-1]d)$ and $v(s) = V(sd) - V([s-1]d)$. Now

$$\sum_{s=0}^{\infty} r(s) = \lim_{t \rightarrow \infty} R(t) = E(N_1) < \infty.$$

Theorem (A) asserts that

$$v(n) \rightarrow \frac{d}{E(z_1)}, 0 \quad \text{as } n \rightarrow \infty, -\infty;$$

applying this to (1) yields

$$U(nd, d) \rightarrow \frac{dE(N_1)}{E(z_1)}, 0 \quad \text{as } n \rightarrow \infty, -\infty,$$

and Wald's equation yields Theorem 1 for d -lattice variables.

5. The nonlattice case. For nonlattice variables we have, rewriting Theorem

2 with a change of variable,

$$U(a, h) = \int_M^\infty [R(t) - R(t-h)] dV(t+a).$$

For any $M > 0$, write

$$U(a, h) = I_1(M, a, h) + I_2(M, a, h),$$

where

$$I_1 = \int_0^M [R(t) - R(t-h)] dV(t+a)$$

and

$$I_2 = \int_0^\infty [R(t) - R(t-h)] dV(t+a).$$

Theorem B applied to $\{z_n\}$ yields

$$V(t+h) - V(t) \rightarrow \frac{h}{E(z_1)}$$

for all $h > 0$ as $t \rightarrow \infty$, so that, since $R(t)$ is monotone,

$$\begin{aligned} I_1 &= \int_0^M R(t) dV(t+a) - \int_0^{M-h} R(t) dV(t+a+h) \\ &\rightarrow \frac{1}{E(z_1)} \cdot \int_{M-h}^M R(t) dt, \quad 0 \quad \text{as } a \rightarrow \infty, -\infty \end{aligned}$$

for fixed M, h . We now show that, for fixed $h, I_2(M, a, h) \rightarrow 0$ as $M \rightarrow \infty$ uniformly in a . We have

$$\begin{aligned} I_2 &= \sum_{n=0}^{\infty} \int_{M+nh}^{M+(n+1)h} [R(t) - R(t-h)] dV(t+a) \\ &\leq \sum_{n=0}^{\infty} R_1(M, n) [V(a+M+(n+1)h) - V(a+M+nh)], \end{aligned}$$

where

$$R_1(M, n) = \sup [R(t) - R(t-h)]$$

as t varies over the interval $(M + nh, M + (n + 1)h)$. Since, by Theorem (B),

$$V(b + h) - V(b) \rightarrow \frac{h}{E(z_1)} \quad \text{as } b \rightarrow \infty,$$

there is a constant c (for the given h) such that

$$I_2(M, a, h) \leq c \sum_{n=0}^{\infty} R_1(M, n) \quad \text{for all } M \text{ and } a.$$

Now

$$\sum_{n=0}^{\infty} R_1(M, 2n) \leq E(N_1) - R(M) \quad \text{and} \quad \sum_{n=0}^{\infty} R_1(M, 2n + 1) \leq E(N_1) - R(M),$$

and $R(M) \rightarrow E(N_1)$ as $M \rightarrow \infty$. Thus

$$|U(a, h) - I_1(M, a, h)| < \epsilon(M, h)$$

for all a , where $\epsilon(M, h) \rightarrow 0$ as $M \rightarrow \infty$ for fixed h . Then

$$\begin{aligned} \left| U(a, h - \frac{hE(N_1)}{E(z_1)}) \right| &\leq \epsilon(M, h) + \left| I_1(M, a, h) - \frac{1}{E(z_1)} \int_{M-h}^M R(t) dt \right| \\ &\quad + \left| \frac{1}{E(z_1)} \int_{M-h}^M R(t) dt - hE(N_1) \right|, \end{aligned}$$

so that

$$\begin{aligned} \limsup_{a \rightarrow \infty} \left| U(a, h) - \frac{hE(N_1)}{E(z_1)} \right| \\ \leq \epsilon(M, h) + \frac{1}{E(z_1)} \left| \int_{M-h}^M R(t) dt - hE(N_1) \right|. \end{aligned}$$

Letting $M \rightarrow \infty$ yields

$$U(a, h) \rightarrow \frac{hE(N_1)}{E(z_1)} \quad \text{as } a \rightarrow \infty,$$

and Wald's equation yields Theorem 1 for $a \rightarrow \infty$. Similarly,

$$U(a, h) \leq \epsilon(M, h) + |I_1(M, a, h)|$$

for all a , so that

$$\limsup_{a \rightarrow -\infty} U(a, h) \leq \epsilon(M, h)$$

and $U(a, h) \rightarrow 0$ as $a \rightarrow -\infty$. This completes the proof.

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