

Pacific Journal of Mathematics

**ON THE UNIQUE DETERMINATION OF SOLUTIONS OF THE
HEAT EQUATION**

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ON THE UNIQUE DETERMINATION OF SOLUTIONS OF THE HEAT EQUATION

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1. Introduction. Recently it has been shown independently by Hartman and Wintner [5] and by the present author [4] that if $u(x, t)$ has continuous derivatives u_{xx} and u_t , and is a nonnegative solution of the heat equation

$$(1) \quad u_{xx}(x, t) - u_t(x, t) = 0$$

in a rectangle $R: \{0 < x < 1; 0 < t < k \leq \infty\}$, then $u(x, t)$ can be represented in the form

$$(2) \quad u(x, t) = \int_{0+}^{1-0} G(x, t; y, 0) dA(y) \\ + \int_0^t G_y(x, t; 0, s) dB(s) - \int_0^t G_y(x, t; 1, s) dC(s),$$

where

$$(3) \quad G(x, t; y, s) = \frac{1}{2} \left[\vartheta_3 \left(\frac{x-y}{2}, t-s \right) - \vartheta_3 \left(\frac{x+y}{2}, t-s \right) \right],$$

and where ϑ_3 is the Jacobi theta function. The integrals are Riemann-Stieltjes integrals with nondecreasing integrator functions, A , B , and C . The first integral may be improper but is absolutely convergent. It was further shown (see [5] and [3]) that

$$(4) \quad u(x, 0+) = A'(x)$$

and

$$(5) \quad u(0+, t) = B'(t-0); \quad u(1-0, t) = C'(t-0)$$

at every point where the derivatives in question exist.

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2. Theorem. As to the question of the extent to which (4) and (5) uniquely determine $u(x, t)$, it is clear that they do not do so completely, for the singular solution $G_y(x, t; 0, 0)$, called a heat explosion by Doetsch [2], has normal boundary values identically zero on the three boundaries $x = 0$, $x = 1$, and $t = 0$ of R . Yet A , B , C , through formula (2), do uniquely determine u ; hence one might expect that by proper choice of the path of approach to the boundary, zero boundary values would assure the vanishing of u . In particular, because of the central role played by G and G_y in the representation (2), one might expect those paths to be the curves along which these functions become unbounded. This leads us to the following:

THEOREM. Suppose

(a) $u(x, t)$ is a nonnegative solution of (1) in R ;

(b) u_{xx} and u_t are continuous in R ;

(c) $u(x, 0+) = 0$ ($0 < x < 1$);

(d) for every s ($0 \leq s < k$), $\lim u(x, t) = 0$ as (x, t) tends to $(0, s)$ along some parabolic arc of the form $t - s = ax^2$, $a > 0$, and $\lim u(x, t) = 0$ as (x, t) tends to $(1, s)$ along some parabolic arc of the form $t - s = a(x - 1)^2$, $a > 0$.

Then $u(x, t) \equiv 0$ in R .

3. Proof. As we remarked in the first sentence, conditions (a) and (b) permit representation of u in the form (2). From the formula

$$(6) \quad \vartheta_3(x/2, t) = (\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp \left[\frac{-(x + 2n)^2}{4t} \right],$$

which can be found in [2], it is easily seen that for $0 < x < 1$ the two latter integrals in formula (2) $\rightarrow 0$ as $t \rightarrow 0+$. Furthermore,

$$\begin{aligned} \int_{0+}^{1-0} G(x, t; y, 0) dA(y) &= \int_{0+}^{\delta} G(x, t; y, 0) dA(y) \\ &+ \int_{\delta}^{1-\delta} G(x, t; y, 0) dA(y) + \int_{1-\delta}^{1-0} G(x, t; y, 0) dA(y), \end{aligned}$$

where $\delta < (1/2) \min [x, 1 - x]$ and is taken so small that, given $\epsilon > 0$,

$$\left| \int_{0+}^{\delta} G(x, t; y, 0) dA(y) \right| < \epsilon \quad \text{and} \quad \left| \int_{1-\delta}^{1-0} G(x, t; y, 0) dA(y) \right| < \epsilon$$

uniformly in t , for $0 < t \leq t_0$ for some t_0 . Possibility to do this is ensured by [5,

Lemma 2, p. 385]. Now

$$\begin{aligned} \int_{\delta}^{1-\delta} G(x, t; y, 0) dA(y) &= \int_{\delta}^{1-\delta} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y) \\ &+ \int_{\delta}^{1-\delta} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y+2n)^2}{4t}\right] dA(y) \\ &- \int_{\delta}^{1-\delta} \sum_{n=-\infty}^{\infty} (4\pi t)^{-1/2} \exp\left[\frac{-(x+y+2n)^2}{4t}\right] dA(y). \end{aligned}$$

The two latter integrals are easily seen to vanish with t . Since also the left side of (2) $\rightarrow 0$ as $t \rightarrow 0$, it follows that, if $\delta' < \delta$,

$$\begin{aligned} \overline{\lim}_{t \rightarrow 0+} \int_{\delta'}^{1-\delta'} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y) \\ \leq \overline{\lim}_{t \rightarrow 0+} \int_{\delta}^{1-\delta} (4\pi t)^{-1/2} \exp\left[\frac{-(x-y)^2}{4t}\right] dA(y) \leq 2\epsilon. \end{aligned}$$

Let $\epsilon \rightarrow 0$ and obtain

$$\lim_{t \rightarrow 0+} \int_{\delta'}^{1-\delta'} (4\pi t)^{-1/2} \exp\left[\frac{-(y-x)^2}{4t}\right] dA(y) = 0.$$

By [6, Th. 7], we see that $A(y)$ is constant between δ' , and $1 - \delta'$. Let $\delta' \rightarrow 0$. This ensures the vanishing of the first integral of (2).

Now let us turn to the boundary $x = 0$. Suppose that for some t_0 the boundary function $B(s)$ is not continuous. If σ is the jump (positive since $B(s)$ is increasing) in $B(s)$ at $s = t_0$, then for $t > t_0$, since $G_y(x, t; 0, s) \geq 0$ (see [5, p. 370]).

$$\begin{aligned} u(x, t) &\geq \int_0^t G_y(x, t; 0, s) dB(s) \geq \sigma G_y(x, t; 0, t_0) \\ &= \frac{1}{2} \sigma x \pi^{-1/2} (t - t_0)^{-3/2} \exp\left[\frac{-x^2}{4(t - t_0)}\right] \\ &+ \frac{1}{2} \sigma \pi^{-1/2} (t - t_0)^{-3/2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (2n + x) \exp\left[\frac{-(2n + x)^2}{4(t - t_0)}\right]. \end{aligned}$$

Since $u(x, t) \rightarrow 0$ as $(x, t) \rightarrow (0, t_0)$ along $t - t_0 = ax^2$ for some $a > 0$, we have

$$u(x, t) \geq \frac{1}{2} \sigma \pi^{-1/2} x^{-2} a^{-3/2} \exp \left[\frac{-1}{4a} \right] \\ + \frac{1}{2} \sigma \pi^{-1/2} a^{-3/2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2n+x}{x^3} \exp \left[\frac{-(2n+x)^2}{4ax^2} \right],$$

As $x \rightarrow 0+$, the sum clearly $\rightarrow 0$; but

$$\lim_{(x, t) \rightarrow (0, t_0)} u(x, t) = 0 \geq \lim_{x \rightarrow 0} \frac{1}{2} \sigma \pi^{-1/2} x^{-2} a^{-3/2} \exp \left[\frac{-1}{4a} \right] = \infty.$$

This is a contradiction. Hence $\sigma = 0$, and $B(s)$ is continuous for $0 \leq s < k$.

Now let $t = t_0 + ax^2$. Then

$$u(x, t) \geq \int_{t_0}^{t_0+ax^2/2} G_y(x, t; 0, s) dB(s) \\ = \int_{t_0}^{t_0+ax^2/2} \frac{1}{2} x \pi^{-1/2} (t-s)^{-3/2} \exp \left[\frac{-x^2}{4(t-s)} \right] dB(s) \\ + \int_{t_0}^{t_0+ax^2/2} \frac{1}{2} \pi^{-1/2} (t-s)^{-3/2} Q(x, t; s) dB(s),$$

where

$$Q(x, t; s) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (2n+x) \exp \left[\frac{-(2n+x)^2}{4(t-s)} \right]$$

Clearly the latter integral vanishes with x , Since in the interval of integration we have

$$\exp \left[\frac{-x^2}{4(t-s)} \right] \geq \exp \left[\frac{-x^2}{4(ax^2/2)} \right] = \exp \left[\frac{-1}{2a} \right]$$

and

$$t - s \leq ax^2,$$

it follows that

$$\begin{aligned} u(x, t) &\geq \frac{1}{2} \pi^{-1/2} a^{-3/2} x^{-2} \exp \left[\frac{-1}{2a} \right] \left[B \left(t_0 + \frac{ax^2}{2} \right) - B(t_0) \right] + o(1) \\ &\geq K \frac{B(t_0 + ax^2/2) - B(t_0)}{ax^2/2} + o(1), \end{aligned}$$

where K is a positive constant. Letting $x \rightarrow 0$, we obtain

$$0 \geq \lim_{x \rightarrow 0} \frac{B(t_0 + ax^2/2) - B(t_0)}{ax^2/2} = D^+[B(t_0)].$$

Hence, by [1, p.580], $B(s)$ is a monotone decreasing function. Since it is non-decreasing, it must be constant. Similarly it can be shown that $C(s)$ is constant. This completes the proof.

It seems probable that conditions (b), (c) and (d) would ensure the vanishing of $u(x, t)$ if it were represented by (2) with A, B, C of bounded variation, but the proof eludes the author.

REFERENCES

1. C. Caratheodory, *Vorlesungen Über Reelle Funktionen*, Berlin, 1927.
2. G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, New York, 1943.
3. W. Fulks, *On the boundary values of solutions of the heat equation*, Pacific J. Math. 2 (1952), 141-145.
4. ———, *On integral representations and uniqueness of solutions of the heat equation*, University of Minnesota Thesis, June, 1949.
5. P. Hartman and A. Wintner, *On the solutions of the equation of heat conduction*, American J. Math. 72 (1950), 367-395.
6. D. V. Widder, *Positive temperatures on an infinite rod*, Trans. Amer. Math. Soc. 55 (1944), 85-95.

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