

# Pacific Journal of Mathematics

**ON THE ORDER OF THE RECIPROCAL SET OF A BASIC SET  
OF POLYNOMIALS**

M. N. MIKHAIL

# ON THE ORDER OF THE RECIPROCAL SET OF A BASIC SET OF POLYNOMIALS

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**1. Introduction.** For the general terminology used in this paper the reader is referred to J. M. Whittaker [2], [3]. Let

$$p_n(z) = \sum_i p_{ni} z^i$$

be a basic set, and let

$$z^n = \sum_{i=0}^{D_n} \pi_{ni} p_i(z).$$

The order  $\omega$  and type  $\gamma$  of  $\{p_n(z)\}$  are defined as follows. Let  $M_i(R)$  be the maximum modulus of  $p_i(z)$  in  $|z| \leq R$ . Let

$$(1) \quad \omega_n(R) = \sum_i |\pi_{ni}| M_i(R),$$

$$(2) \quad \omega(R) = \limsup_{n \rightarrow \infty} \frac{\log \omega_n(R)}{n \log n},$$

$$(3) \quad \omega = \lim_{R \rightarrow \infty} \omega(R);$$

and, for  $0 < \omega < \infty$ , let

$$(4) \quad \gamma(R) = \limsup_{n \rightarrow \infty} \{\omega_n(R)\}^{1/(n\omega)} e^{-(n\omega)},$$

$$(5) \quad \gamma = \lim_{R \rightarrow \infty} \gamma(R).$$

If

$$P_n(z) = \sum_i \pi_{ni} z^i,$$

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then  $\{P_n(z)\}$  is called the reciprocal set of  $\{p_n(z)\}$ . We shall establish for certain basic sets new formulas expressing upper bounds of the order of the reciprocal set in terms of the data of the original set.

2. **Theorem.** The following theorem holds only if an infinity of  $\pi_{nn} \neq 0$ ; then the whole proof should be carried out for those values of  $n$  for which  $\pi_{nn} \neq 0$ . This is a genuine restriction since there are basic sets such that  $\pi_{nn} = 0$  for all  $n$ ; for example, for  $h = 0, 1, 2, \dots$ , let

$$p_{3h}(z) = -\frac{1}{2} z^{3h} + \frac{1}{2} z^{3h+1} + \frac{1}{2} z^{3h+2},$$

$$p_{3h+1}(z) = \frac{1}{2} z^{3h} - \frac{1}{2} z^{3h+1} + \frac{1}{2} z^{3h+2},$$

$$p_{3h+2}(z) = \frac{1}{2} z^{3h} + \frac{1}{2} z^{3h+1} - \frac{1}{2} z^{3h+2}.$$

NOTATION. For a fixed  $n$ , let  $p_{nh}'$  be the set of all nonzero elements  $p_{nh}$ , and let

$$\min_{h'} p_{nh}' = p_n'.$$

THEOREM 1. Let  $\{p_n(z)\}$  be a basic set of polynomials, such that

$$\limsup_{n \rightarrow \infty} \frac{D_n}{n} = a \quad (a \geq 1),$$

and of increase less than order  $\omega$  and type  $\gamma$ , and suppose that

$$\kappa = \liminf_{n \rightarrow \infty} \frac{\log |\pi_{nn}|}{n \log n}$$

and

$$k = \liminf_{n \rightarrow \infty} \frac{\log |p_n'|}{n \log n}.$$

Then its reciprocal set is of order  $\Omega$ , where

i) if  $k > \omega$ , then  $\Omega \leq \omega - \kappa$ ;

ii) if  $k \leq \omega$ , then  $\Omega \leq 2\omega - \kappa - k$ .

*Proof.* Let  $\gamma_1 > \gamma$ ; then in view of (4) we have

$$(6) \quad \omega_n(R) \leq \left( \frac{n\omega\gamma_1}{e} \right)^{n\omega}$$

for values of  $n > n_0$  and for sufficiently large values of  $R > R_0 > 1$ . From (1), we have

$$|\pi_{nn}| M_n(R) \leq \omega_n(R).$$

Then

$$|\pi_{nn}| |p_{ni}| R^i \leq \omega_n(R);$$

that is

$$(7) \quad |p_{ni}| \leq \frac{\omega_n(R)}{|\pi_{nn}|}.$$

Also

$$|\pi_{ij}| M_j(R) \leq \omega_i(R);$$

then

$$(8) \quad |\pi_{ij}| \leq \frac{\omega_i(R)}{M_j(R)} \leq \frac{\omega_i(R)}{\min_{h'} |p_{ih'}|} = \frac{\omega_i(R)}{|p_{i'}|}.$$

From the definition of a reciprocal set, and in view of (1), we get

$$\Omega_n(R) \leq \sum_{i=0}^{D_n} |p_{ni}| \sum_j |\pi_{ij}| R^j \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \sum_{i=0}^{D_n} \sum_j |\pi_{ij}|$$

by (7); that is, by (8),

$$\Omega_n(R) \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \sum_{i=0}^{D_n} N_i \frac{\omega_i(R)}{|p_{i'}|}.$$

Then

$$\begin{aligned}\Omega_n(R) &\leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + \sum_{i=n_0+1}^{D_n} \frac{\omega_i(R)}{|P_i'|} \right\} \\ &\leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + \sum_{i=n_0+1}^{D_n} \frac{(i\omega\gamma_1)^{i\omega}}{|P_i'|} \right\} \quad \text{by (6),}\end{aligned}$$

where  $F(R)$  is a function independent of  $n$ .

Then for sufficiently large values of  $n > n_0$  and  $R > R_0$ , we get

$$\Omega_n(R) \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + D_n \left( \frac{n\omega\gamma_1}{n^{k_1/\omega}} \right)^{n\omega} \right\} \quad (\text{where } k_1 \geq k).$$

Hence:

i) If  $k > \omega$  (this implies  $k_1 > \omega$ ), then  $(n\omega\gamma_1/n^{k_1/\omega})^{n\omega}$ , for values of  $n > n_0$ , will be a small quantity compared to  $F(R)$ . Therefore,

$$\begin{aligned}&\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\log \Omega_n(R)}{n \log n} \\ &\leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\{ \frac{\log \omega_n(R)}{n \log n} + \frac{D_n \log R}{n \log n} - \frac{\log |\pi_{nn}|}{n \log n} + \frac{\log D_n}{n \log n} + \frac{\log F(R)}{n \log n} \right\},\end{aligned}$$

in view of (2) and (3); then

$$\Omega \leq \omega - \kappa.$$

ii) If  $k \leq \omega$ , then as  $k_1$  approaches  $k$  we find that  $F(R)$  will be very small compared to  $\{n\omega\gamma_1/n^{k_1/\omega}\}^{n\omega}$  for  $n > n_0$ . Therefore,

$$\begin{aligned}\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\log \omega_n(R)}{n \log n} &\leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\{ \frac{\log \omega_n(R)}{n \log n} - \frac{\log |\pi_{nn}|}{n \log n} \right. \\ &\quad \left. + \frac{D_n \log R + 2 \log D_n}{n \log n} + \frac{n\omega \left(1 - \frac{k_1}{\omega}\right) \log n}{n \log n} + \frac{n\omega \log \omega \gamma_1}{n \log n} \right\}\end{aligned}$$

in view of (2) and (3); then

$$\Omega \leq \omega - \kappa + \omega - k = 2\omega - \kappa - k.$$

N. B. *In the case of simple sets, the restriction mentioned above for  $\pi_{nn}$  is satisfied. In this case we have*

$$-\kappa = \limsup_{n \rightarrow \infty} \frac{\log |p_{nn}|}{n \log n}.$$

COROLLARY. *If  $\{p_n(z)\}$  is a simple set of polynomials,*

$$\left. \begin{array}{l} \text{i) if } k > \omega, \text{ then } \Omega \leq \omega - \kappa \\ \text{ii) if } k \leq \omega, \text{ then } \Omega \leq 2\omega - \kappa - k \end{array} \right\} \text{ where } \kappa = - \limsup_{n \rightarrow \infty} \frac{\log |p_{nn}|}{n \log n}.$$

3. **Examples.** We shall look at four examples.

$$\begin{aligned} \text{i) Let } p_n(z) &= n^{3n} z^n - n^{2n} z^{n-1} - n^{3n} z^{n+1} && (n \text{ odd}), \\ p_n(z) &= n^{2n} z^n - n^{3n} && (n \text{ even}), \\ p_0(z) &= 1. \end{aligned}$$

then

$$\begin{aligned} z^n &= n^{-3n} p_n(z) + n^{-n} (n-1)^{-2(n-1)} p_{n-1}(z) + (n+1)^{-2(n+1)} p_{n+1}(z) \\ &\quad + \{(n-1)^{(n-1)} n^{-n} + (n+1)^{(n+1)}\} p_0(z) \quad (n \text{ odd}), \\ z^n &= n^{-2n} p_n(z) + n^n p_0(z) \quad (n \text{ even}). \end{aligned}$$

By Theorem (1) of [1], we get  $\omega = 1$ . Since  $\kappa = -3$ ,  $k = 2$ , we get, according to case i) of the theorem,  $\Omega \leq 1 + 3 = 4$ . This is true because  $\Omega = 4$  by Corollary (1.1) of [1].

N. B. *This example and the following examples show that the values given in the conclusion of the above theorem are "best possible."*

$$\begin{aligned} \text{ii) Let } p_n(z) &= n^{2n} z^n - n^{3n/2} z^{2n} - n^{2n} && (n \text{ odd}), \\ p_n(z) &= \left(\frac{n}{2}\right)^{3n/2} z^n - \left(\frac{n}{2}\right)^{2n}, \text{ with } p_0(z) = 1 && (n \text{ even}), \end{aligned}$$

Then

$$z^n = n^{-2n} p_n(z) + n^{-7n/2} p_{2n}(z) + (1 + n^{n/2}) p_0(z) \quad (n \text{ odd}),$$

$$z^n = \left(\frac{n}{2}\right)^{-3n/2} p_n(z) + \left(\frac{n}{2}\right)^{n/2} p_0(z) \quad (n \text{ even}),$$

Applying theorem (1) of [1], we get  $\omega = 1/2$ . Now  $\kappa = -2$ ,  $k = 3/2$ . Then according to case i), of the theorem, we get

$$\Omega \leq \frac{1}{2} + 2.$$

This is true because  $\Omega = 5/2$  by Corollary (1.1) of [1].

$$\text{iii) Let } p_n(z) = n^n z^n - n^{n/2} z^{n-1} - n^{3n/2} \quad (n \text{ odd}),$$

$$p_n(z) = (n+1)^{(n+1)} z^n - (n+1)^{2(n+1)} z^{(n+1)} - (n+1)^{5(n+1)/2} \quad (n \text{ even}),$$

$$p_0(z) = 1.$$

Then

$$z^n = \frac{1}{1 - n^{n/2}} \left\{ n^{-n} p_n(z) + n^{-3n/2} p_{n-1}(z) + (n^{n/2} + n^n) p_0(z) \right\} \quad (n \text{ odd}),$$

$$z^n = \frac{1}{1 - (n+1)^{(n+1)/2}} \left\{ (n+1)^{-(n+1)} p_n(z) + p_{n+1}(z) + 2(n+1)^{3(n+1)/2} p_0(z) \right\} \quad (n \text{ even}).$$

Applying theorem (1) of [1], we get  $\omega = 1$ . Now  $\kappa = -1$ ,  $k = 1/2$ . Then according to case ii) of the theorem, we get

$$\Omega \leq 2 + 1 - \frac{1}{2} = \frac{5}{2}.$$

This is true because  $\Omega = 5/2$  by Corollary (1.1) of [1].

$$\text{iv) Let } p_n(z) = \frac{2^{(n-1)}}{2^{(n-1)} n^{2n} + (n-1)^{3(n-1)}} z^n + \frac{2^{(n-1)} n^n}{2^{(n-1)} n^{2n} + (n-1)^{3(n-1)}} \\ + \frac{2^{2(n-1)} (n-1)^{(n-1)}}{2^{(n-1)} n^{2n} + (n-1)^{3(n-1)}} z^{n-1} \quad (n \text{ odd}),$$

$$p_n(z) = \frac{2^{2n} (n+1)^{2(n+1)}}{2^n n^n (n+1)^{2(n+1)} + n^{4n}} z^n - \frac{n^n}{2^n (n+1)^{2(n+1)} + n^{3n}} z^{n+1} \\ - \frac{n^n (n+1)^{(n+1)}}{2^n (n+1)^{2(n+1)} + n^{3n}} z^{2n+2} \quad (n \text{ even}),$$

$$p_0(z) = 1.$$

Then

$$z^n = n^{2n} p_n(z) - n^{3n} p_{2n}(z) \\ - (n-1)^{2(n-1)} p_{n-1}(z) - n^{5n} p_{2n+1}(z) \quad (n \text{ odd}),$$

$$z^n = \left(\frac{1}{2} n\right)^n p_n(z) + \left(\frac{1}{2} n\right)^{2n} p_{n+1}(z) \quad (n \text{ even}).$$

Applying theorem (1) of [1], we get  $\omega = 1$ . Now  $\kappa = 2, k = -3$ . Then according to case ii) of the theorem, we get

$$\Omega \leq 2 - 2 + 3 = 3.$$

This is true because  $\Omega = 3$  by Corollary (1.1) of [1].

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