

Pacific Journal of Mathematics

**ON THE ORDER OF THE RECIPROCAL SET OF A BASIC SET
OF POLYNOMIALS**

M. N. MIKHAIL

ON THE ORDER OF THE RECIPROCAL SET OF A BASIC SET OF POLYNOMIALS

M. N. MIKHAIL

1. Introduction. For the general terminology used in this paper the reader is referred to J. M. Whittaker [2], [3]. Let

$$p_n(z) = \sum_i p_{ni} z^i$$

be a basic set, and let

$$z^n = \sum_{i=0}^{D_n} \pi_{ni} p_i(z).$$

The order ω and type γ of $\{p_n(z)\}$ are defined as follows. Let $M_i(R)$ be the maximum modulus of $p_i(z)$ in $|z| \leq R$. Let

$$(1) \quad \omega_n(R) = \sum_i |\pi_{ni}| M_i(R),$$

$$(2) \quad \omega(R) = \limsup_{n \rightarrow \infty} \frac{\log \omega_n(R)}{n \log n},$$

$$(3) \quad \omega = \lim_{R \rightarrow \infty} \omega(R);$$

and, for $0 < \omega < \infty$, let

$$(4) \quad \gamma(R) = \limsup_{n \rightarrow \infty} \{\omega_n(R)\}^{1/(n\omega)} e^{-(n\omega)},$$

$$(5) \quad \gamma = \lim_{R \rightarrow \infty} \gamma(R).$$

If

$$P_n(z) = \sum_i \pi_{ni} z^i,$$

Received July 2, 1952.

Pacific J. Math. 3 (1953), 617-623

then $\{P_n(z)\}$ is called the reciprocal set of $\{p_n(z)\}$. We shall establish for certain basic sets new formulas expressing upper bounds of the order of the reciprocal set in terms of the data of the original set.

2. **Theorem.** The following theorem holds only if an infinity of $\pi_{nn} \neq 0$; then the whole proof should be carried out for those values of n for which $\pi_{nn} \neq 0$. This is a genuine restriction since there are basic sets such that $\pi_{nn} = 0$ for all n ; for example, for $h = 0, 1, 2, \dots$, let

$$p_{3h}(z) = -\frac{1}{2} z^{3h} + \frac{1}{2} z^{3h+1} + \frac{1}{2} z^{3h+2},$$

$$p_{3h+1}(z) = \frac{1}{2} z^{3h} - \frac{1}{2} z^{3h+1} + \frac{1}{2} z^{3h+2},$$

$$p_{3h+2}(z) = \frac{1}{2} z^{3h} + \frac{1}{2} z^{3h+1} - \frac{1}{2} z^{3h+2}.$$

NOTATION. For a fixed n , let p_{nh}' be the set of all nonzero elements p_{nh} , and let

$$\min_{h'} p_{nh}' = p_n'.$$

THEOREM 1. Let $\{p_n(z)\}$ be a basic set of polynomials, such that

$$\limsup_{n \rightarrow \infty} \frac{D_n}{n} = a \quad (a \geq 1),$$

and of increase less than order ω and type γ , and suppose that

$$\kappa = \liminf_{n \rightarrow \infty} \frac{\log |\pi_{nn}|}{n \log n}$$

and

$$k = \liminf_{n \rightarrow \infty} \frac{\log |p_n'|}{n \log n}.$$

Then its reciprocal set is of order Ω , where

i) if $k > \omega$, then $\Omega \leq \omega - \kappa$;

ii) if $k \leq \omega$, then $\Omega \leq 2\omega - \kappa - k$.

Proof. Let $\gamma_1 > \gamma$; then in view of (4) we have

$$(6) \quad \omega_n(R) \leq \left(\frac{n \omega \gamma_1}{e} \right)^{n\omega}$$

for values of $n > n_0$ and for sufficiently large values of $R > R_0 > 1$. From (1), we have

$$|\pi_{nn}| M_n(R) \leq \omega_n(R).$$

Then

$$|\pi_{nn}| |p_{ni}| R^i \leq \omega_n(R);$$

that is

$$(7) \quad |p_{ni}| \leq \frac{\omega_n(R)}{|\pi_{nn}|}.$$

Also

$$|\pi_{ij}| M_j(R) \leq \omega_i(R);$$

then

$$(8) \quad |\pi_{ij}| \leq \frac{\omega_i(R)}{M_j(R)} \leq \frac{\omega_i(R)}{\min_{h'} |p_{ih'}|} = \frac{\omega_i(R)}{|p_{i'}|}.$$

From the definition of a reciprocal set, and in view of (1), we get

$$\Omega_n(R) \leq \sum_{i=0}^{D_n} |p_{ni}| \sum_j |\pi_{ij}| R^j \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \sum_{i=0}^{D_n} \sum_j |\pi_{ij}|$$

by (7); that is, by (8),

$$\Omega_n(R) \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \sum_{i=0}^{D_n} N_i \frac{\omega_i(R)}{|p_{i'}|}.$$

Then

$$\begin{aligned} \Omega_n(R) &\leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + \sum_{i=n_0+1}^{D_n} \frac{\omega_i(R)}{|p_i'|} \right\} \\ &\leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + \sum_{i=n_0+1}^{D_n} \frac{(i\omega\gamma_1)^{i\omega}}{|p_i'|} \right\} \quad \text{by (6),} \end{aligned}$$

where $F(R)$ is a function independent of n .

Then for sufficiently large values of $n > n_0$ and $R > R_0$, we get

$$\Omega_n(R) \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + D_n \left(\frac{n\omega\gamma_1}{n^{k_1/\omega}} \right)^{n\omega} \right\} \quad (\text{where } k_1 \geq k).$$

Hence:

i) If $k > \omega$ (this implies $k_1 > \omega$), then $(n\omega\gamma_1/n^{k_1/\omega})^{n\omega}$, for values of $n > n_0$, will be a small quantity compared to $F(R)$. Therefore,

$$\begin{aligned} &\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\log \Omega_n(R)}{n \log n} \\ &\leq \lim_{R \rightarrow \infty} \left\{ \limsup_{n \rightarrow \infty} \left[\frac{\log \omega_n(R)}{n \log n} + \frac{D_n \log R}{n \log n} - \frac{\log |\pi_{nn}|}{n \log n} + \frac{\log D_n}{n \log n} + \frac{\log F(R)}{n \log n} \right] \right\}, \end{aligned}$$

in view of (2) and (3); then

$$\Omega \leq \omega - \kappa.$$

ii) If $k \leq \omega$, then as k_1 approaches k we find that $F(R)$ will be very small compared to $\{n\omega\gamma_1/n^{k_1/\omega}\}^{n\omega}$ for $n > n_0$. Therefore,

$$\begin{aligned} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\log \omega_n(R)}{n \log n} &\leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\{ \frac{\log \omega_n(R)}{n \log n} - \frac{\log |\pi_{nn}|}{n \log n} \right. \\ &\quad \left. + \frac{D_n \log R + 2 \log D_n}{n \log n} + \frac{n\omega \left(1 - \frac{k_1}{\omega}\right) \log n}{n \log n} + \frac{n\omega \log \omega \gamma_1}{n \log n} \right\} \end{aligned}$$

in view of (2) and (3); then

$$\Omega \leq \omega - \kappa + \omega - k = 2\omega - \kappa - k.$$

N. B. *In the case of simple sets, the restriction mentioned above for π_{nn} is satisfied. In this case we have*

$$-\kappa = \limsup_{n \rightarrow \infty} \frac{\log |p_{nn}|}{n \log n}.$$

COROLLARY. *If $\{p_n(z)\}$ is a simple set of polynomials,*

$$\left. \begin{array}{l} \text{i) if } k > \omega, \text{ then } \Omega \leq \omega - \kappa \\ \text{ii) if } k \leq \omega, \text{ then } \Omega \leq 2\omega - \kappa - k \end{array} \right\} \text{ where } \kappa = - \limsup_{n \rightarrow \infty} \frac{\log |p_{nn}|}{n \log n}.$$

3. Examples. We shall look at four examples.

$$\begin{aligned} \text{i) Let } p_n(z) &= n^{3n} z^n - n^{2n} z^{n-1} - n^{3n} z^{n+1} && (n \text{ odd}), \\ p_n(z) &= n^{2n} z^n - n^{3n} && (n \text{ even}), \\ p_0(z) &= 1. \end{aligned}$$

then

$$\begin{aligned} z^n &= n^{-3n} p_n(z) + n^{-n} (n-1)^{-2(n-1)} p_{n-1}(z) + (n+1)^{-2(n+1)} p_{n+1}(z) \\ &\quad + \{(n-1)^{(n-1)} n^{-n} + (n+1)^{(n+1)}\} p_0(z) \quad (n \text{ odd}), \end{aligned}$$

$$z^n = n^{-2n} p_n(z) + n^n p_0(z) \quad (n \text{ even}).$$

By Theorem (1) of [1], we get $\omega = 1$. Since $\kappa = -3$, $k = 2$, we get, according to case i) of the theorem, $\Omega \leq 1 + 3 = 4$. This is true because $\Omega = 4$ by Corollary (1.1) of [1].

N. B. *This example and the following examples show that the values given in the conclusion of the above theorem are "best possible."*

$$\begin{aligned} \text{ii) Let } p_n(z) &= n^{2n} z^n - n^{3n/2} z^{2n} - n^{2n} && (n \text{ odd}), \\ p_n(z) &= \left(\frac{n}{2}\right)^{3n/2} z^n - \left(\frac{n}{2}\right)^{2n}, \text{ with } p_0(z) = 1 && (n \text{ even}), \end{aligned}$$

Then

$$z^n = n^{-2n} p_n(z) + n^{-7n/2} p_{2n}(z) + (1 + n^{n/2}) p_0(z) \quad (n \text{ odd}),$$

$$z^n = \left(\frac{n}{2}\right)^{-3n/2} p_n(z) + \left(\frac{n}{2}\right)^{n/2} p_0(z) \quad (n \text{ even}),$$

Applying theorem (1) of [1], we get $\omega = 1/2$. Now $\kappa = -2$, $k = 3/2$. Then according to case i), of the theorem, we get

$$\Omega \leq \frac{1}{2} + 2.$$

This is true because $\Omega = 5/2$ by Corollary (1.1) of [1].

$$\text{iii) Let } p_n(z) = n^n z^n - n^{n/2} z^{n-1} - n^{3n/2} \quad (n \text{ odd}),$$

$$p_n(z) = (n+1)^{(n+1)} z^n - (n+1)^{2(n+1)} z^{(n+1)} - (n+1)^{5(n+1)/2} \quad (n \text{ even}),$$

$$p_0(z) = 1.$$

Then

$$z^n = \frac{1}{1 - n^{n/2}} \left\{ n^{-n} p_n(z) + n^{-3n/2} p_{n-1}(z) + (n^{n/2} + n^n) p_0(z) \right\} \quad (n \text{ odd}),$$

$$z^n = \frac{1}{1 - (n+1)^{(n+1)/2}} \left\{ (n+1)^{-(n+1)} p_n(z) + p_{n+1}(z) + 2(n+1)^{3(n+1)/2} p_0(z) \right\} \quad (n \text{ even}).$$

Applying theorem (1) of [1], we get $\omega = 1$. Now $\kappa = -1$, $k = 1/2$. Then according to case ii) of the theorem, we get

$$\Omega \leq 2 + 1 - \frac{1}{2} = \frac{5}{2}.$$

This is true because $\Omega = 5/2$ by Corollary (1.1) of [1].

$$\text{iv) Let } p_n(z) = \frac{2^{(n-1)}}{2^{(n-1)} n^{2n} + (n-1)^{3(n-1)}} z^n + \frac{2^{(n-1)} n^n}{2^{(n-1)} n^{2n} + (n-1)^{3(n-1)}} z^{n-1} + \frac{2^{2(n-1)} (n-1)^{(n-1)}}{2^{(n-1)} n^{2n} + (n-1)^{3(n-1)}} z^{n-1} \quad (n \text{ odd}),$$

$$p_n(z) = \frac{2^{2n} (n+1)^{2(n+1)}}{2^n n^n (n+1)^{2(n+1)} + n^{4n}} z^n - \frac{n^n}{2^n (n+1)^{2(n+1)} + n^{3n}} z^{n+1} - \frac{n^n (n+1)^{(n+1)}}{2^n (n+1)^{2(n+1)} + n^{3n}} z^{2n+2} \quad (n \text{ even}),$$

$$p_0(z) = 1.$$

Then

$$z^n = n^{2n} p_n(z) - n^{3n} p_{2n}(z) - (n-1)^{2(n-1)} p_{n-1}(z) - n^{5n} p_{2n+1}(z) \quad (n \text{ odd}),$$

$$z^n = \left(\frac{1}{2} n\right)^n p_n(z) + \left(\frac{1}{2} n\right)^{2n} p_{n+1}(z) \quad (n \text{ even}).$$

Applying theorem (1) of [1], we get $\omega = 1$. Now $\kappa = 2, k = -3$. Then according to case ii) of the theorem, we get

$$\Omega \leq 2 - 2 + 3 = 3.$$

This is true because $\Omega = 3$ by Corollary (1.1) of [1].

REFERENCES

1. M. N. Mikhail, *Basic sets of polynomials and their reciprocal, product and quotient sets*, Duke Math. J. (to appear).
2. J. M. Whittaker, *Interpolatory function theory*, Cambridge, England, 1935.
3. ———, *Sur les séries de base de polynomes quelconques*, Paris, 1949.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

R. M. ROBINSON
University of California
Berkeley 4, California

E. HEWITT
University of Washington
Seattle 5, Washington

R. P. DILWORTH
California Institute of Technology
Pasadena 4, California

E. F. BECKENBACH
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

H. BUSEMANN	P. R. HALMOS	BØRGE JESSEN	J. J. STOKER
HERBERT FEDERER	HEINZ HOPF	PAUL LÉVY	E. G. STRAUS
MARSHALL HALL	R. D. JAMES	GEORGE PÓLYA	KÔSAKU YOSIDA

SPONSORS

UNIVERSITY OF BRITISH COLUMBIA	UNIVERSITY OF SOUTHERN CALIFORNIA
CALIFORNIA INSTITUTE OF TECHNOLOGY	STANFORD RESEARCH INSTITUTE
UNIVERSITY OF CALIFORNIA, BERKELEY	STANFORD UNIVERSITY
UNIVERSITY OF CALIFORNIA, DAVIS	WASHINGTON STATE COLLEGE
UNIVERSITY OF CALIFORNIA, LOS ANGELES	UNIVERSITY OF WASHINGTON
UNIVERSITY OF CALIFORNIA, SANTA BARBARA	* * *
UNIVERSITY OF NEVADA	AMERICAN MATHEMATICAL SOCIETY
OREGON STATE COLLEGE	NATIONAL BUREAU OF STANDARDS,
UNIVERSITY OF OREGON	INSTITUTE FOR NUMERICAL ANALYSIS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors except Robinson, whose term expires with the completion of the present volume; they might also be sent to M. M. Schiffer, Stanford University, Stanford, California, who is succeeding Robinson. All other communications to the editors should be addressed to the managing editor, E. F. Beckenbach, at the address given above.

Authors are entitled to receive 100 free reprints of their published papers and may obtain additional copies at cost.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$8.00; single issues, \$2.50. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES

COPYRIGHT 1953 BY PACIFIC JOURNAL OF MATHEMATICS

L. Carlitz, <i>Some theorems on generalized Dedekind sums</i>	513
L. Carlitz, <i>The reciprocity theorem for Dedekind sums</i>	523
Edward Richard Fadell, <i>Identifications in singular homology theory</i>	529
Harley M. Flanders, <i>A method of general linear frames in Riemannian geometry. I</i>	551
Watson Bryan Fulks, <i>The Neumann problem for the heat equation</i>	567
Paul R. Garabedian, <i>Orthogonal harmonic polynomials</i>	585
R. E. Greenwood and Andrew Mattei Gleason, <i>Distribution of round-off errors for running averages</i>	605
Arthur Eugene Livingston, <i>The space H^p, $0 < p < 1$, is not normable</i>	613
M. N. Mikhail, <i>On the order of the reciprocal set of a basic set of polynomials</i>	617
Louis Joel Mordell, <i>On the linear independence of algebraic numbers</i>	625
Leo Sario, <i>Alternating method on arbitrary Riemann surfaces</i>	631
Harold Nathaniel Shapiro, <i>Iterates of arithmetic functions and a property of the sequence of primes</i>	647
H. Shniad, <i>Convexity properties of integral means of analytic functions</i>	657
Marlow C. Sholander, <i>Plane geometries from convex plates</i>	667