ON THE EXISTENCE PROBLEM OF LINEAR PROGRAMMING

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1. Notation and introduction. We shall consistently use Latin capital letters to represent real rectangular matrices, lower-case Latin letters for real column vectors, and lower-case Greek letters for real scalars. The appearance of a subscript on a symbol "lowers its classification one step"; thus: A will represent a matrix, A_j will represent the jth column of A (a column vector), A_{ij} will be the entry in the ith row and the jth column of A; b_i will be the ith component of the column vector b; and so on. The dimensions of the matrices and vectors will not always be mentioned, but it is of course implied that they are consistent in the sense that all indicated operations are meaningful; for instance, the appearance of the product Ax implies that the number of columns in the matrix A is equal to the number of components of x. Vector and matrix inequalities are based upon the following notations (where Γ represents either a matrix or a vector):

$$\Gamma = 0 \text{ means each entry in } \Gamma \text{ is zero,}$$

$$\Gamma \geqq 0 \text{ means each entry in } \Gamma \text{ is nonnegative,}$$

$$\Gamma \geq 0 \text{ means } \Gamma \geqq 0, \text{ but } \Gamma = 0 \text{ is false,}$$

$$\Gamma > 0 \text{ means each entry in } \Gamma \text{ is positive.}$$

Several of the proofs given below can be replaced by proofs based on the transposition theorem on linear inequalities [4].

We shall be concerned with the following problem: Given a matrix A and a vector b, does there exist a vector $x \geqq 0$, such that $Ax = b$? Otherwise expressed, we wish to consider the problem of whether the set

$$\{A; b\} = \{x \mid Ax = b, \ x \geqq 0\}$$

is nonempty.

In order to eliminate trivial cases, we assume that $b \neq 0$; and also make the obviously nonrestrictive assumption that no column of A is identically zero.

The problem described above is then the existence problem of linear programming, which arises from mathematical formulations of many problems of economics and of production planning and scheduling [2; see also articles too numerous to list, in *Econometrica*]. It is also possible, by the use of simple devices, to express any problem concerning the existence of a solution to a set of linear inequalities in the above form. We shall, however, impose a condition on A which will not ordinarily be met when a general linear inequality problem is so expressed, but which is satisfied in a great many applications (see §2). Under this condition, we shall show that a solution to the given problem exists if and only if the value of a certain game is zero; and that, in fact, if this is the case, then the optimal strategies for the second player in the game correspond in a very simple way to the solutions of the given problem (§4).

We shall also take a few words to describe a slightly novel geometrical interpretation of the problem under our restricting condition (§3).

2. The condition. Theorem (2.2), below, asserts the equivalence of the following four hypotheses on matrix A. The first form will be of interest to us in the remainder of the paper; while the others are of direct significance in economic and planning problems — (2.1′), for instance, says, essentially, that no "activity" is reversible (Cf. [2, Ch. III, Postulate A]). Additional conditions are given in [3] and [4].

(2.1)   HYPOTHESIS. *There exists w such that $w'A > 0$.*

(2.1′)   HYPOTHESIS. *There exists no $x \geq 0$ such that $Ax = 0$.*

(2.1″)   HYPOTHESIS. *For any b, the set $\{A; b\}$ is bounded.*

(2.1‴) HYPOTHESIS. *For any b such that $\{A; b\}$ is nonempty, and for any c, the minimum of $c'x$ over $\{A; b\}$ is attained in $\{A; b\}$.*

(2.2)   THEOREM. *Hypotheses (2.1), (2.1′), (2.1″), and (2.1‴) are equivalent.*

Proof of (2.2). If $w'A > 0$, $x \geq 0$, and $Ax = 0$, then $0 = w'(Ax) = (w'A)x > 0$, a contradiction which shows that (2.1) implies (2.1′).

Suppose (2.1′) holds, and that there exist vectors $y \geq 0$, $z \geq 0$, and number λ with $0 < \lambda < 1$, such that

$$A(\lambda y + (1 - \lambda)z) = 0.$$

Then

$$\lambda y + (1 - \lambda) z \geqq 0,$$

so by $(2.1')$ we have

$$\lambda y + (1 - \lambda) z = 0;$$

whence, in fact, $y = z = 0$. This shows that the point $u = 0$ is an extreme of the convex set

$$U = \{ u \mid \text{for some } x \geqq 0, \ u = Ax \}.$$

It follows [2, Ch. XVIII, Theorem 17] that there exists a vector w such that $w'(Ax) > 0$ for any point $u = Ax \neq 0$ of U. But since we have assumed that no column of A is zero, we need merely allow x to run through the vectors $(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$ to see that $w'A > 0$. This shows that $(2.1')$ implies (2.1).

It is clear, since $\{A; b\}$ is closed, that $(2.1'')$ and $(2.1''')$ are equivalent. Hence we shall complete the proof by showing that $(2.1')$ implies $(2.1'')$, and that $(2.1''')$ implies $(2.1')$.

If $(2.1'')$ is false, we may choose $x_i, \ i = 1, 2, \cdots$, in $\{A; b\}$, such that the distance from some point x_0 in $\{A; b\}$ to x_i approaches infinity as i approaches infinity. The segments $[x_0, x_i]$ are contained in the convex set $\{A; b\}$, and it is easily shown by topological methods that some subset of this collection of segments converges to a set of form

$$\{x_0 + \mu x \mid 0 \leqq \mu < \infty \} \subset \{A; b\},$$

where $x \neq 0$. Since $A(x_0 + \mu x) = b$, it follows that $Ax = 0$; while since $x_0 + \mu x \geqq 0$, it follows that $x \geqq 0$. Hence $(2.1')$ is false, so $(2.1')$ implies $(2.1'')$.

(A one-line proof of the last result can be based upon the unpublished generalized simplex method of computation, due to George Dantzig.)

Finally, if $\{A; b\}$ is nonempty, choose $x_0 \geqq 0$ such that $Ax_0 = b$. If $x \geqq 0$ and $Ax = 0$, then let $c = -x$. Plainly $c'(x_0 + \mu x)$ approaches minus infinity as μ approaches infinity, while $x_0 + \mu x$ is in $\{A; b\}$ for all nonnegative μ. This shows that $(2.1''')$ implies $(2.1')$.

The following theorem shows that if we are assuming (2.1), then there is no real loss of generality in assuming that $w'b > 0$.

(2.3) THEOREM. *If $w'A > 0$ and $\{A; b\}$ is nonempty, then $w'b > 0$.*

Proof of (2.3). If $Ax = b$ and $x \geq 0$, then, since $b \neq 0$, we must have $x \geq 0$. Then

$$w'b = w'(Ax) = (w'A)x > 0.$$

3. A geometrical interpretation. We assume in this section that Hypothesis (2.1) is satisfied, and also that $w'b > 0$.

Simply multiplying each column of A by an appropriate positive factor, we may obtain a matrix B such that the inner product of w with any column of B is equal to $w'b$. That is to say, the vectors B_j and b all lie on the hyperplane $w'z = w'b$ in the euclidean space whose dimension is that of b. Then also, for x in $\{B; b\}$,

$$1 = \frac{w'(Ax)}{w'b} = \frac{(w'A)x}{w'b} = (1, \cdots, 1)x.$$

That is, the sum of the components of x is one. Thus x provides nonnegative "weights" to be attached to the vectors B_j in order that their center of gravity will be b.

It is obvious that $\{A; b\}$ and $\{B; b\}$ are either empty or nonempty together (and in fact, that there is a very simple relation between their points). Thus the following almost trivial theorem, which follows immediately from the above remarks, is of interest, because its hypothesis is not essentially more restrictive than Hypothesis (2.1).

(3.1) THEOREM. *Suppose the columns of B and the vector b all lie in some hyperplane not containing the origin. Then $\{B; b\}$ is nonempty if and only if b is in the convex closure of the columns of B.*

This simple result, which has previously been noted by A. Charnes, is directly related to the fact that (for absolutely general B and b), $\{B; b\}$ is nonempty if and only if b is contained in the cone with vertex 0 generated by the columns of B.

4. A relationship with game theory. We shall assume now that the columns of A and the vector b all lie in some hyperplane not containing the origin. From the preceding section, we see that this is not really a more restrictive assumption than Hypothesis (2.1); and, in fact, that if the vector w of Hypothesis (2.1) is known, then the amount of computation required to put the problem in

the form demanded here is very small.

Let $A_{k_1}, \cdots, A_{k_m}$ be any maximal linearly independent subset of the columns of A, and write:

$$A_j = \sum_{i=1}^{m} D_{ij} A_{k_i},$$

$$b = \sum_{i=1}^{m} e_i A_{k_i}.$$

If it is not possible to express b in this form, then it is clear that $\{A; b\}$ is empty, and we are finished; in practical problems, the columns of A span the euclidean space containing them.

Now form the matrix $D = (D_{ij})$ (the index i varying over rows) and the column vector $e = (e_i)$. The transformation $A_j \longrightarrow D_j$, $b \longrightarrow e$, is simply the one-to-one linear transformation of the space spanned by the A_j onto m-dimensional euclidean space, carrying the A_{k_i} to the unit vectors on the coordinate axes. From the linearity, it is immediate that $\{A; b\} = \{D; e\}$.

It is easily seen (as we saw that the sum of the components of the vector x in § 3 was unity) that

$$\sum_{i=1}^{m} D_{ij} = 1 = \sum_{i=1}^{m} e_i.$$

Thus, if we let $H_{ij} = D_{ij} - e_i$, and form the matrix $H = (H_{ij})$ (the index i varying over rows); then H is a matrix such that the sum of the elements in any column is zero.

Moreover, since if x is in $\{D; e\}$, the sum of the components of x must be unity, it is easily seen that

$$\{D; e\} = \{x \mid Hx = 0, \sum x_i = 1, x \geq 0\}.$$

Now think of H as the matrix of a zero-sum two-person game, H_{ij} being the amount paid by the second (minimizing) player to the first (maximizing) player, in case the first player chooses row i and the second player chooses column j. Since the sum of the entries in each column is zero, it is clear that if the first player uses the mixed strategy $(1/m, \cdots, 1/m)$, then the payoff to him is zero.

Thus, the value of the game (to the first player) is nonnegative.

This value is actually zero if and only if there is a mixed strategy $x = (x_i)$ available to the second player, such that

$$\max_i \sum_j H_{ij} x_j = 0$$

[3, Theorem 2.9]. On the other hand, if $\max \sum H_{ij} x_j = 0$, then we have $\sum H_{ij}$ $x_j = 0$ for each i; otherwise the first player's strategy $(1/m, \cdots, 1/m)$ would yield a negative payoff. Hence the value of game H is zero if and only if there exists x with $Hx = 0$, $\sum x_i = 1$, $x \geq 0$. In this case it is clear that just such mixed strategies x are optimal for the second player. But it was remarked above that $\{A; b\} = \{D; e\}$, while $\{D; e\}$ is identical with the set of such strategies; so the following result is proved:

(4.1) THEOREM. *Let the columns of A and the vector b all lie in some hyperplane not containing the origin, and suppose $b = Ax$ is solvable. Form the matrix H as above. Then $\{A; b\}$ is nonempty if and only if the value of the game whose matrix is H is zero. If this value is zero, then $\{A; b\}$ is the set of optimal strategies for the second player of the game.*

5. Another transformation. Suppose again that there exists w such that $w'A > 0$ and $w'b > 0$. Multiplying negative components of w, along with corresponding rows of A and corresponding components of b, by -1; adding sufficiently small positive increments to the zero components of the vector so obtained from w; and multiplying this vector by a suitable positive constant; we get v, C, and e, such that

$$v > 0, \sum v_i = 1, v'C > 0, v'e > 0, \text{ and } \{C; e\} = \{A; b\}.$$

As before, we can easily modify C to obtain D such that for each column D_j of D, we have $v'D_j = v'e$, while not essentially changing the problem.

Now form a matrix H by subtracting e from each column of D. The following result is then proved in exactly the same way as the first part of (4.1). A statement corresponding to the second part of (4.1) could, of course, be added.

(5.1) THEOREM. *Suppose $w'A > 0$ and $w'b > 0$. Form matrix H as above. Then $\{A; b\}$ is nonempty if and only if the value of the game whose matrix is H is zero.*

A theorem analogous to the following can, of course, be proved relative to

the matrix H of § 4.

(5.2) THEOREM. *Under the hypothesis of* (5.1), *form H as above. Then* $\{A; b\}$ *is nonempty if and only if either one of the following* (*equivalent*) *conditions is fulfilled.*

(5.21)   *There exists no* $w \geq 0$ *such that* $w'H > 0$.

(5.22)   *The matrix H does not satisfy Hypothesis* (2.1).

Proof of (5.2). The equivalence of (5.21) with the nonemptiness of $\{A; b\}$ is immediate from (5.1) and [3, Theorem 2.9].

Also, from (5.1) and [3, Theorem 2.9], or directly from the proof of (5.1), we see that the nonemptiness of $\{A; b\}$ is equivalent to the existence of $x \geq 0$ such that $Hx = 0$. Then the equivalence of the emptiness of $\{A; b\}$ and Hypothesis (2.1) follows from (2.2).

It is worth noting that for the matrix H, there exists arbitrary w such that $w'H > 0$ if and only if there exists $w \geq 0$ such that $w'H > 0$, as is shown by the equivalence of (5.21) and (5.22).

6. Comments. Theorem (4.1) is of interest in spite of the greater simplicity of Theorem (5.1), because the value of the game obtained for one of these theorems may be "insensitive" to relevant features of the programming problem concerned. In particular, the value of the game of (5.1) would depend only slightly upon those rows corresponding to small components of v. Hence it is desirable to have more than one such result if applications involving approximations are to be made. Note also that in many practical problems, it is easy to find w, and a simple basis $A_{k_1}, \cdots, A_{k_m}$, so that the D_{ij} and the e_i of § 4 are easily obtained.

Necessary and sufficient conditions for the nonemptiness of $\{A; b\}$ are of interest because it is in general much easier to obtain reasonable estimates of the value of a game than to compute optimal strategies (or, equivalently, to find a vector x satisfying the conditions of the original problem). An approximation to the value of one of the games will not, of course, give a definite answer to the question of whether $\{A; b\}$ is nonempty; but in view of the fact that it is quite generally necessary in applications of mathematics to develop an understanding of what constitutes a "reasonable tolerance", this is not necessarily a serious disadvantage in regard to the use of the theorem for answering practical existence questions. The meaning of a given positive value for the game in terms of the *way* in which the given programming problem

problem is "infeasible" is easily phrased in intuitively satisfactory terms, so it should not be too difficult to make sense of the idea of a "reasonable tolerance."

It is also suggested by the theorems that it might be valuable to attack as a statistical hypothesis-testing problem the following questions: Is the value of the game with matrix H equal to zero? Is this value nonnegative? Is it positive?

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