MAPPING PROPERTIES OF CESÀRO SUMS OF ORDER TWO OF THE GEOMETRIC SERIES

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1. Introduction. Previous investigations of the mappings

$$w = S_n^{(k)}(z)$$

of the unit circle $|z| \leq 1$, where

$$S_n^{(k)}(z) = \binom{n+k}{k} + \binom{n+k-1}{k} z + \cdots + \binom{k}{k} z^n$$

denotes the nth Cesàro sum of order k of the geometric series, have been made by Fejér, Schweitzer, Sidon, and Szegö. Knowledge of the properties of the sums $S_n^{(k)}(z)$ is valuable in the study of power series having coefficients monotonic of order $k + 1$.

The present article provides additional asymptotic properties for

$$S_n^{(2)}(e^{i\phi}) = x_n(\phi) + iy_n(\phi).$$

The following results are established:

THEOREM 1. *For n sufficiently large, an α_n exists such that $y_n(\phi)$ is increasing for $0 < \phi < \alpha_n$ and decreasing for $\alpha_n < \phi < \pi$. Furthermore,*

$$\alpha_n = \alpha/n + O(n^{-2}), \quad \text{where} \quad \pi < \alpha < 3\pi/2.$$

THEOREM 2. *For n sufficiently large, a β_n exists such that*

$$x_n'(\phi) \begin{cases} \leq 0, \ 0 < \phi < \beta_n, \ n \equiv 0 \,(\text{mod } 3) \\ < 0, \ 0 < \phi < \beta_n, \ n \equiv 1 \,(\text{mod } 3) \\ < 0, \ 0 < \phi < \beta_n, \ n \equiv 2 \,(\text{mod } 3) \end{cases}$$

where

$$\beta_n = \frac{2\pi}{3} + \frac{\beta}{n} - O(n^{-3/2}),$$

and $\beta = 2\pi$, $4\pi/3$, $2\pi/3$ *for* $n \equiv 0, 1, 2 \pmod 3$, *respectively.*

THEOREM 3. *For n sufficiently large, the mapping of* $|z| = 1$ *by*

$$w = S_n^{(2)}(e^{i\phi})$$

is convex for $0 < \phi < \gamma_n$, *where* γ_n *is the maximum angle for which convexity holds, and* $\gamma_n = \gamma/n + O(n^{-2})$ *where* $2\pi < \gamma < 3\pi$.

2. Proof of Theorem 1.

2.1. A closed expression for $\gamma_n'(\phi)$ has been presented by Szegö [10]:

$$(2.1) \quad \gamma_n'(\phi) =$$

$$\frac{1}{8 \sin^2 \phi/2} \left\{ -(n^2 + 3n + 3) - n \cdot \frac{\sin(n + 3/2)\phi}{\sin \phi/2} + 3 \frac{\sin^2(n + 1)\phi/2}{\sin^2 \phi/2} \right\}.$$

The inequality $\gamma_n'(\phi) < 0$ is satisfied if

$$(2.2) \qquad n^2 + 3n + 3 > -n \cdot \frac{\sin(n + 3/2)\phi}{\sin \phi/2} + 3 \frac{\sin^2(n + 1)\phi/2}{\sin^2 \phi/2},$$

or

$$(2.3) \qquad\qquad n^2 + 3n + 3 > n \cdot \csc \phi/2 + 3 \cdot \csc^2 \phi/2.$$

Let δ be fixed, $\delta > 0$, and consider the restriction $\phi > \delta/n$. For n sufficiently large, $\sin(\delta/2n) > \delta/\pi n$, and the previous inequality is maintained if δ is chosen so that

$$n^2 + 3n + 3 > n^2\pi/\delta + 3\pi^2 n^2/\delta^2, \quad \text{or} \quad \pi/\delta + 3\pi^2/\delta^2 < 1.$$

It is sufficient for the present problem to define $\delta = 3\pi$. Hence, if $\phi \geq 3\pi/n$, then $\gamma_n'(\phi) < 0$ and $3\pi/n > \alpha_n$. Since

$$\gamma_n'(\phi) = \sum_{m=1}^{n} a_m \cos m\phi,$$

where

$$a_m = m \binom{n + 2 - m}{2},$$

it at once follows that $\gamma_n'(\phi) > 0$ for $0 \leq \phi \leq \pi/2n$.

2.2. In the next section it is shown that in the interval $\pi/2n < \phi < 3\pi/n$ there is exactly one $\phi = \alpha_n$ such that $\gamma_n'(\phi) = 0$ if n is sufficiently large. More precisely, for the $\phi = \alpha_n$ the second derivative does not vanish and $\alpha_n \sim \alpha/n$, $\alpha > 0$, where $\pi < \alpha < 3\pi/2$. The magnitude of α is defined as the root of a transcedental equation.

It is possible to express (2.1) in the following form:

(2.4) $$\gamma_n'(\phi) = n^2/8 \cdot g_n(\phi) \cdot \csc^2 \phi/2,$$

where the function $g_n(\phi)$ is defined as

$$g_n(\phi) = -1 - \frac{1}{n} \cdot \frac{\sin(n + 3/2)\phi}{\sin \phi/2} + \frac{3}{n^2} \cdot \frac{\sin^2(n + 1)\phi/2}{\sin^2 \phi/2} - \left(\frac{3}{n} + \frac{3}{n^2}\right).$$

Let $\phi = c/n$, $\pi/2 < c < 3\pi$, and then $g_n(\phi)$ becomes a function of c, denoted by $G_n(c)$. In addition,

$$\lim_{n \to \infty} G_n(c) = -1/c^2 \cdot f(c),$$

where

$$f(c) = 2c \cdot \sin c + 6 \cdot \cos c + (c^2 - 6).$$

Furthermore, $G_n(c)$ converges uniformly to this limit for arbitrary values of c in the interval. It is sufficient to show that the function $f(c)$ has a unique simple zero in the interval $\pi/2 < c < 3\pi/2$ to assure that $g_n(c)$ has a simple zero in the same interval if n is sufficiently large.

An easy calculation yields

$$f'(c) = 8 \cdot \cos^2(c/2) \cdot (c/2 - \tan c/2).$$

Thus it is seen that $f(0) = 0$, $f'(c) < 0$ for $0 < c < \pi$; $f(\pi) = -(12 - \pi^2)$ < 0, $f'(c) > 0$ for $\pi < c < 2\pi$; $f(2\pi) > 0$ and if $c > 2\pi$ then $f(c) > (c-1)^2$ $- 13 > 0$. Since $f(3\pi/2) > 0$, there is a simple positive zero, $c = \alpha$, of the function $f(c)$, $\pi < \alpha < 3\pi/2$. In conclusion, $\alpha_n \sim \alpha/n$, $\pi < \alpha < 3\pi/2$, for n sufficiently large.

2.3. It is not difficult to find a more precise asymptotic expression for α_n. For this purpose let $\alpha_n = c/n$, where $c = \alpha + a/n$ and a is a bounded, real constant. Let $h_n(a)$ denote $g_n(c)$ when the latter is regarded as a function of a. Let $\phi = c/n$; a simplification yields

$$-n^2 h_n(a) = n^2(2c \cdot \sin c + 6 \cos c + c^2 - 6)/c^2 + n(3 + 3 \cos c - 6/c \cdot \sin c)$$

$$+ (5/2 - 13c/6 \cdot \sin c - 5/2 \cdot \cos c) + O(1/n).$$

If

$$h(c) = 3c^2[1 + \cos c - 2/c \cdot \sin c]$$

and

$$k(c) = c^2[5/2 - 13c/6 \cdot \sin c - 5/2 \cdot \cos c],$$

then it is possible to rewrite the previous expression in the form

$$-n^2 c^2 h_n(a) = n^2 \cdot f(c) + n \cdot h(c) + k(c) + O(1/n).$$

Let the functions $f(c)$, $h(c)$, $k(c)$ be expanded by Taylor's formula for values of c near α. Then the previous equality becomes

$$-n^2 c^2 h_n(a) = n[a \cdot f'(\alpha) + h(\alpha)] + a^2/2 \cdot f''(\alpha) + a \cdot h'(\alpha) + k(\alpha) + O(1/n).$$

Thus one obtains

$$\lim_{n \to \infty} [-n \cdot c^2 h_n(a)] = a \cdot f'(\alpha) + h(\alpha) \text{ and } f'(\alpha) \neq 0.$$

Obviously the limit has a zero for the value $a = -h(\alpha)/f'(\alpha)$, or

$$a = -3\alpha^2/8 \cdot (1 + \cos \alpha - 2/\alpha \cdot \sin \alpha) \cdot \sec^2 \alpha/2 \cdot (\alpha/2 - \tan \alpha/2)^{-1},$$

and α is the simple zero of the function

$$f(c) = 2c \cdot \sin c + 6 \cdot \cos c + c^2 - 6$$

in the interval $\pi < \alpha < 3\pi/2$.

This shows that for n sufficiently large, $y_n'(\phi) = 0$ for

$$\phi = \alpha_n = \alpha/n + (a + \epsilon_n)/n^2,$$

where $\epsilon_n \longrightarrow 0$. Thus the assertion of Theorem 1 has been verified.

3. Proof of Theorem 2.

3.1. In the article by Szegö [10], a closed expression for $x_n'(\phi)$ is presented:

$$(3.1) \quad x_n'(\phi)$$

$$= \frac{\cos \phi/2}{8 \sin^3 \phi/2} \left[-(2n+3) - (n+3/2) \frac{\cos(n+3/2)\phi}{\cos \phi/2} + \frac{3}{2} \cdot \frac{\sin(n+3/2)\phi}{\sin \phi/2} \right].$$

It immediately follows that $x_n'(\phi)$ is negative if

$$(3.2) \quad [3/(2n+3)]^2 \cdot \csc^2 \phi/2 + \sec^2 \phi/2 < 4; \quad \cot \phi/2 > 0.$$

Let $0 < \phi \le \pi/n$. As

$$x_n(\phi) = \sum_{m=1}^{n} b_m \cos m\phi,$$

where

$$b_m = \binom{n+2-m}{2},$$

then $x_n'(\phi) < 0$. Next consider the interval $\pi/n \le \phi \le 2\pi/3 - c/n$, where c is fixed, $c > 0$. Since

$$[3/(2n+3)]^2 \cdot \csc^2 \phi/2 + [1 - \sin^2 \phi/2]^{-1},$$

as a function of $\sin^2 \phi/2$, is convex from below, it obtains its maximum at one or both end-points of the interval. Thus in order to prove the inequality (3.2) it is sufficient to consider only the end-point values of $\pi/n \le \phi \le 2\pi/3 - c/n$. It easily follows that (3.2) is satisfied by $\phi = \pi/n$. Now study $\phi = 2\pi/3 - c/n$. Since

$$\sin^{-2} \phi/2 = O(1), \quad \cos^2 \phi/2 = 1/4 \cdot (1 + \sqrt{3} \cdot c/n) + O(1/n^2),$$

the left side of (3.2) then can be written as

$$[3/(2n+3)]^2 \cdot O(1) + 4[1 + \sqrt{3} \cdot c/n + O(1/n^2)]^{-1}$$

$$= 4(1 - \sqrt{3} \cdot c/n) + O(1/n^2),$$

which indeed is less than 4 provided n is sufficiently large. The minimum value of n is a function of c. Thus it now is established that $x_n''(\phi) < 0$ for $0 < \phi \le 2\pi/3 - c/n$, if n is sufficiently large, $n \ge n_1(c)$, where c is an arbitrary positive fixed magnitude.

3.2. Next let $\phi = 2\pi/3$. By (3.1) it follows that

$$x_n''(2\pi/3) = -(2n+3)/6\sqrt{3} \cdot (1 - \cos 2\pi n/3) - 1/6 \cdot \sin 2\pi n/3.$$

Three possible cases for the n arise. For $n \equiv 0 \pmod 3$, $x_n''(2\pi/3) = 0$; whereas for $n \equiv 1, 2 \pmod 3$, $x_n''(2\pi/3) < 0$. Thus the behavior of $x_n''(\phi)$ in the neighborhood of $\phi = 2\pi/3$ must be examined more fully, $n \equiv 0 \pmod 3$. Let

$$x_n''(\phi) = r(\phi) \cdot s(\phi),$$

where

$$r(\phi) = 1/8 \cdot \cos \phi/2 \cdot \csc^3 \phi/2$$

and

$$s(\phi) = -(2n+3) - (n+3/2) \cdot \cos(n+3/2)\phi \cdot \sec \phi/2$$

$$+ 3/2 \cdot \sin(n+3/2)\phi \cdot \csc \phi/2.$$

As $s = 0$ for $\phi = 2\pi/3$, then

$$x_n''(2\pi/3) = r(2\pi/3) \cdot s'(2\pi/3).$$

Upon letting $N = n + 3/2$, we see that

$$s'(2\pi/3) = 0, \quad x_n''(2\pi/3) = 0.$$

An examination of the third derivative shows that

$$x_n'''(2\pi/3) = r(2\pi/3) \cdot s''(2\pi/3).$$

As $r(2\pi/3) > 0$, $\operatorname{sgn} x_n''(\phi) = \operatorname{sgn} s''(\phi)$. Since

$$s''(\phi) = N^3 \sec \phi/2 \cdot \cos N\phi + O(N^2),$$

then $s''(2\pi/3) = -2N^3 + O(N^2)$, and for n sufficiently large $s''(2\pi/3) < 0$. It is now known than $x_n'(\phi) < 0$ for $0 < \phi < 2\pi/3$ if n is sufficiently large.

3.3. This section extends the investigation beyond $\phi = 2\pi/3$. For this purpose let $\phi = 2\pi/3 + c/N$, where again $N = n + 3/2$. The substitution of this value of ϕ into (3.1) yields

$$(3.3) \quad \frac{8 \sin^3 \phi/2}{\cos \phi/2} \cdot x_n'(\phi) = -2N\left[1 - \frac{1}{2}\frac{\cos(2\pi n/3 + c)}{\cos \phi/2}\right] - \frac{3}{2} \cdot \frac{(2\pi n/3 + c)}{\sin \phi/2}.$$

Any easy calculation shows that

$$\sin \phi/2 = \sqrt{3}/2 + c/4N + c^2 \cdot O(1/N^2),$$

$$\cos \phi/2 = 1/2 - \sqrt{3} \cdot c/4N + c^2 \cdot O(1/N^2).$$

The remainder of the section will study the separate cases of $n \,(\operatorname{mod} 3)$.

$n \equiv 0 \,(\operatorname{mod} 3)$. Let us rewrite (3.3) as follows:

$$(3.4) \quad \frac{8 \sin^3 \phi/2 \cdot x_n'(\phi)}{\cos \phi/2 \cdot 2(1 - \cos c)}$$

$$= -N + \frac{\sqrt{3}}{2}\left[\frac{c \cdot \cos c - \sin c}{1 - \cos c}\right] + \frac{c^2}{1 - \cos c} \cdot O\left(\frac{1}{N}\right).$$

Let

$$F(c) = [c \cdot \cos c - \sin c] \cdot [1 - \cos c]^{-1}.$$

Since

$$F'(c) = \sin c \cdot [\sin c - c][1 - \cos c]^{-2},$$

it is easily seen that $F(c)$ is decreasing for $0 < c < \pi$ and increasing for $\pi < c < 2\pi$. It follows that $x_n'(\phi) \leq 0$ for $0 < \phi \leq 2\pi/3 + c/N$, where $\pi < c < 2\pi - \epsilon$, ϵ a fixed positive number. Now

$$c = 2\pi - \delta/\sqrt{N},$$

δ a fixed positive number for n sufficiently large. Then

$$F(c) = 4\pi/\delta^2 \cdot N + O(1/N),$$

so that, for the above value of ϕ, (3.4) becomes

$$\frac{4\sin^3 \phi/2 \cdot x_n'(\phi)}{\cos \phi/2 \cdot (1 - \cos c)} = -N + 2\pi\sqrt{3} \cdot N/\delta^2 + O(1).$$

In addition,

$$(1 - \cos c)^{-1} = (1 - \cos \delta/\sqrt{N})^{-1} = O(N).$$

Thus

$$x_n'(\phi) < 0 \quad \text{if} \quad 2\pi\sqrt{3}/\delta^2 < 1,$$

and

$$x_n'(\phi) > 0 \quad \text{if} \quad 2\pi\sqrt{3}/\delta^2 > 1.$$

Thus

$$\delta = (2\pi)^{1/2} \cdot (3)^{1/4}$$

furnishes the critical value of ϕ. It has been shown that, for $n \equiv 0 \pmod 3$,

$$x_n'(\phi) < 0 \quad \text{for} \quad 0 < \phi < 2\pi/3$$

and

$$x_n'(\phi) \le 0 \quad \text{for} \quad 0 < \phi < 2\pi/3 + 2\pi/N - O(N^{-3/2}),$$

for n sufficiently large.

$n \equiv 1 \pmod 3$. It is possible to rewrite (3.3) so that the right side becomes

$$-2N[1 - \cos(c + 2\pi/3)]$$

$$+ \sqrt{3}[c \cdot \cos(c + 2\pi/3) - \sin(c + 2\pi/3)] + c^2 \cdot O(1/N).$$

By reasoning as in the previous case, one finds $x_n'(\phi) < 0$ for $0 \le c \le 4\pi/3 - \epsilon$, $\epsilon > 0$, for n sufficiently large. Let

$$c = 4\pi/3 - \delta/\sqrt{N}.$$

Then the right side of (3.3) reduces to

$$-\delta^2 + 4\pi/\sqrt{3} + O(1/\sqrt{N}).$$

Therefore

$$x_n''(\phi) < 0 \quad \text{if} \quad \delta > 2 \cdot \pi^{1/2} \cdot 3^{-1/4},$$

and

$$x_n''(\phi) > 0 \quad \text{if} \quad \delta < 2 \cdot \pi^{1/2} \cdot 3^{-1/4},$$

for n sufficiently large. It follows that $x_n''(\phi) < 0$ for $0 < \phi < \beta_n$, where

$$\beta_n = 2\pi/3 + 4\pi/3N - O(N^{-3/2}),$$

for n sufficiently large.

$n \equiv 2 \,(\text{mod } 3)$. In this case the right side of (3.3) becomes

$$-2N[1 - \cos(c + 4\pi/3)]$$

$$+ \sqrt{3}\,[c \cdot \cos(c + 4\pi/3) - \sin(c + 4\pi/3)] + O(1/N).$$

It follows that $x_n''(\phi) < 0$ for $0 \le c \le 2\pi/3 - \epsilon$, $\epsilon > 0$, for n sufficiently large. Let

$$c = 2\pi/3 - \delta/\sqrt{N}.$$

Then the right side of (3.3) is equivalent to

$$-\delta^2 + 2\pi/\sqrt{3} + O(N^{-1/2}).$$

Thus

$$x_n''(\phi) < 0 \quad \text{if} \quad \delta > (2\pi)^{1/2} \cdot 3^{1/4}$$

and

$$x_n''(\phi) > 0 \quad \text{if} \quad \delta < (2\pi)^{1/2} \cdot 3^{1/4},$$

for n sufficiently large. It has been shown that $x_n''(\phi) < 0$ for $0 < \phi < \beta_n$, where

$$\beta_n = 2\pi/3 + 2\pi/3N - O(N^{-3/2}),$$

for n sufficiently large.

If $n + 3/2$ is substituted for N, then the results expressed in Theorem 2 are proved.

4. Proof of Theorem 3.

4.1 The Curvature of an image is defined to be

$$1/\rho = [1 + \Re\ z \cdot f''(z)/f'(z)] \cdot [\,|\,z \cdot f'(z)\,|\,]^{-1}.$$

If the point $w = f(z)$ traverses a closed, single-valued curve in a preassigned positive direction, then the curve is called convex if

(4.1) $1 + \Re[z \cdot f''(z)/f'(z)] > 0.$

Let us examine the inequality (4.1) for the function

$$f(e^{i\phi}) = s_n^2(e^{i\phi}) = x_n(\phi) + i \cdot y_n(\phi)$$

if $z = e^{i\phi}$. By the employment of differentiation and elementary algebraic steps after substituting the derivatives in the left side of (4.1), one obtains

$$1 + \Re[z \cdot f''(z)/f'(z)] = [x_n' \cdot y_n'' - x_n'' \cdot y_n'] \cdot [x_n'^2 + y_n'^2]^{-1}.$$

Thus the condition for the mapping to be convex is satisfied if

(4.2) $x_n' \cdot y_n'' - x_n'' \cdot y_n' > 0.$

4.2. The next section studies the previous condition of convexity for the function $w = s_n^2(z)$, $z = e^{i\phi}$, where $\phi = \gamma/n$, $\gamma > 0$, for n sufficiently large.

In the present case the expressions for $y_n''(\gamma/n)$ and $x_n''(\gamma/n)$, for which see (2.1) and (3.1), become

$$y_n''(\gamma/n) = n^4/\gamma^4 \left[-\gamma \sin\gamma - 3\cos\gamma + 3 - \gamma^2/2 + O(1/n)\right],$$

$$x_n''(\gamma/n) = n^4/\gamma^4 \left[-2\gamma - \gamma \cos\gamma + 3\sin\gamma + O(1/n)\right].$$

Substitution of the latter expressions into (4.2) yields directly

$$(2 \sin \gamma - \gamma \cos \gamma - \gamma)(-2\gamma - \gamma \cos \gamma + 3 \sin \gamma)$$

$$- (-2 + \gamma \sin \gamma + 2 \cos \gamma)(-\gamma \sin \gamma - 3 \cos \gamma + 3 - \gamma^2/2) + O(1/n) > 0.$$

Further simplification of the previous inequality, which establishes the requirement for convexity of the image of $|z| = 1$, leads to the convenient form

$$(4.3) \qquad \sin \gamma (\tan \gamma/2 - \gamma/2)(6 - \gamma^2/2 - 3\gamma \cot \gamma/2) + O(1/n) > 0.$$

The remainder of the section is devoted to determining the maximum value of $\phi = \gamma/n$ which satisfies (4.3). In particular it is shown that the maximum angle $\gamma_n = \gamma/n$ for which the mapping of $|z| = 1$ by $w = s_n^2(z)$ is convex, where $z = e^{i\phi}$, is determined by $2\pi < \gamma < 3\pi$, for n sufficiently large.

4.3. Consider the elementary function

$$v(\gamma) = \sin \gamma [\tan(\gamma/2) - \gamma/2].$$

Define γ_0 by the equality $\tan(\gamma_0/2) = \gamma_0/2$. Then it is easily shown that

$$(4.4) \qquad v(\gamma) \begin{cases} > 0, \ 0 < \gamma < 2\pi, \\ < 0, \ 2\pi < \gamma < \gamma_0, \\ > 0, \ \gamma_0 < \gamma < 3\pi, \end{cases}$$

Let us define

$$f(\gamma) = 6 - \gamma^2/2 - 3\gamma \cot(\gamma/2).$$

Then the image of $|z| = 1$ is convex if

$$(4.5) \qquad f(\gamma) \begin{cases} > 0, \ 0 < \gamma < 2\pi, \\ < 0, \ 2\pi < \gamma < \gamma_0, \\ > 0, \ \gamma_0 < \gamma < 3\pi, \end{cases}$$

for n sufficiently large.

Next it is shown that the first two inequalities for $f(\gamma)$ in (4.5) are satisfied, however, for $\gamma_0 < \gamma < 3\rho$, one finds that $f(\gamma) < 0$. Since

$$f'(\gamma) \cdot \sin^2(\gamma/2) = -\gamma \sin^2(\gamma/2) - 3/2 \cdot \sin \gamma + 3\gamma/2,$$

by a further differentiation with respect to γ one can obtain

$$d/d\gamma\{f'(\gamma) \cdot \sin^2(\gamma/2)\} = 1 - \cos\gamma - \gamma/2 \sin\gamma,$$

$$= \sin\gamma\{\tan(\gamma/2) - \gamma/2\} = v(\gamma).$$

Consider the interval $0 < \gamma < 2\pi$. By (4.4), $v(\gamma) > 0$. Also $f'(\gamma) \cdot \sin^2(\gamma/2) = 0$ if $\gamma = 0$. Thus $f'(\gamma) \sin^2(\gamma/2) > 0$ for $0 < \gamma < 2\pi$, and consequently $f'(\gamma) > 0$ for the same interval. Finally,

$$f(0) = \lim_{\gamma \to 0^+} f(\gamma) = 0,$$

which establishes the fact that $f(\gamma) > 0$ in the interval $0 < \gamma < 2\pi$.

In the interval $2\pi < \gamma < \gamma_0$, $v(\gamma) < 0$; therefore the function $f'(\gamma) \sin^2(\gamma/2)$ is decreasing. It follows that

$$f'(\gamma_0) \cdot \sin^2(\gamma_0/2) = \gamma_0[3/2 - \sin^2(\gamma_0/2)] - 3/2 \cdot \sin\gamma_0,$$

and thus $f'(\gamma_0) \cdot \sin^2(\gamma_0/2) > 1/2(\gamma_0 - 3) > 0$. Consequently $f'(\gamma) \sin^2(\gamma/2)$ and also $f'(\gamma)$ are positive in the interval $2\pi < \gamma < \gamma_0$. Hence $f(\gamma)$ is increasing. As $f(\gamma)$ has no lower bound as γ approaches 2π from above, and

$$f(\gamma_0) = 6 - \gamma_0^2/2 - 3\gamma_0 \cot(\gamma_0/2) < -\gamma_0^2/2 < 0,$$

then it can be concluded that $f(\gamma) < 0$ for $2\pi < \gamma < \gamma_0$.

Finally consdier the interval $\gamma_0 < \gamma < 3\pi$. Since $v(\gamma) > 0$, $f'(\gamma_0) \sin^2(\gamma_0/2) > 0$, and thus $f'(\gamma_0) > 0$, then $f'(\gamma) > 0$ holds for $\gamma_0 < \gamma \; 3\pi$. Hence $f(\gamma)$ is increasing. But

$$f(3\pi) = 6 - 9\pi^2/2 < 0,$$

so that a γ exists such that $f(\gamma) < 0$ occurs in the interval.

It was shown in (4.5) that if the image of $|z| = 1$ was to be convex for $\gamma_0 < \gamma < 3\pi$, then $f(\gamma) > 0$. Thus the image is not convex for the complete interval, which completes the proof of Theorem 3.

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