

# Pacific Journal of Mathematics

**EIGENVALUES OF CIRCULANT MATRICES**

RICHARD STEVEN VARGA

# EIGENVALUES OF CIRCULANT MATRICES

RICHARD S. VARGA

**1. Introduction.** The integral equations

$$(1) \quad u(z_j) = \lambda \oint_C A(z, z_j) u(z) dq + \Phi(z_j),$$

where  $C$  is a smooth closed curve, and

$$A(z, z_j) = d \arg(z - z_j)/dq,$$

has many important applications. Thus [6], iteration of (1) gives a solution for the conformal mapping problem for the interior and exterior of  $C$ .

In numerical work, the rate of convergence of such iterations depends on the eigenvalues of the integral operator  $A(z, z_j)$ . It is known that the absolute values of the nontrivial<sup>1</sup> eigenvalues of the integral operator  $A(z, z_j)$  are less than one. A recent paper [1] gives a sharper bound to the eigenvalues.

However, in numerical computation, equation (1) must be replaced [6] by a discrete equation of the form

$$(2) \quad u_{r+1}(z_j) = \lambda \sum_{k=1}^N A_{jk} u_r(z_j) + \Phi(z_j).$$

This makes it important to know the relation between the eigenvalues of  $A(z, z_j)$  and those of the matrix  $A_{jk}$ .

We determine this relation below in the special case that  $C$  is an ellipse. In particular, we show that the eigenvalues of  $A_{jk}$  approach  $N/2$  times those of  $A(z, z_j)$  with exponential convergence. Since trapezoidal integration based on trigonometric interpolation gives exponential accuracy, this fact is probably

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<sup>1</sup>It is easy to verify that for the eigenfunction  $u(z) \equiv 1$ , we have the simple eigenvalue unity. By the nontrivial eigenvalues of  $A(z, z_j)$ , we mean all other eigenvalues.

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true for any analytic curve. However, it seemed most interesting to get quantitative bounds in the special case of ellipses.

**2. Circulant matrices.** For the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

it is known [1] that

$$A(z, z_j) = \pi^{-1} \frac{ab}{(a^2 + b^2) - (a^2 - b^2) \cos(q + q_j)}.$$

It follows that the associated matrix

$$A_{j,k} = \left\| \pi^{-1} \frac{ab}{(a^2 + b^2) - (a^2 - b^2) \cos(q_k + q_j)} \right\| = \|a_{j,k}\|$$

is a *circulant* matrix, in the usual sense that

$$(3) \quad a_{i+h, j-h} = a_{i,j}$$

for all integers  $h$ , where subscripts are taken mod  $N$ . We first show how to compute the eigenvalues of a circulant matrix in a way which seems somewhat more simple and perspicuous than that given in the literature [7].

Following the notation of [5], let  $\vec{\epsilon}_1, \dots, \vec{\epsilon}_n$  denote the unit vectors in  $V_n(C)$ , and let

$$\vec{\epsilon}_i \longrightarrow \sum a_{ij} \vec{\epsilon}_j$$

denote the linear transformation associated with the matrix  $A_{jk}$ . It is convenient to introduce the new basis

$$\vec{\alpha}_1, \dots, \vec{\alpha}_n \text{ defined by } \vec{\alpha}_l = \sum \omega^{lk} \vec{\epsilon}_k,$$

where  $\omega = e^{i2\pi/n}$  is a primitive  $n$ th root of unity. The matrix

$$\Omega = \|\omega^{ik}\|$$

is closely related to that used in Lagrangian resolvents; it is symmetric, and  $n^{-1/2} \Omega$  is unitary.

Relative to the basis  $\vec{\alpha}_1, \dots, \vec{\alpha}_n$ , cyclic matrices [2, p.124] are diagonalized, while circulant matrices (whose squares are cyclic matrices) reduce to monomial matrices which are reducible into  $2 \times 2$  components. Specifically, easy computations show that the basic transposition

$$R_m : \vec{\epsilon}_k \longrightarrow \vec{\epsilon}_{m-k} \quad (m = 0, 1, \dots, n-1),$$

corresponding to a circulant matrix with ones on a reversed diagonal:  $i + j \equiv m \pmod n$ , carries  $\vec{\alpha}_i$  into  $\omega^{im} \vec{\alpha}_{n-i}$ . Hence, a general circulant matrix  $\sum c_m R_m$  carries  $\vec{\alpha}_i$  into

$$(\sum c_m \omega^{im}) \vec{\alpha}_{n-i}.$$

Thus, in general, a pair of eigenvalues is associated with each subspace spanned by  $\vec{\alpha}_i$  and  $\vec{\alpha}_{n-i}$  (we have an exception when  $i = n$ , and, if  $n$  is even, when  $i = n/2$ ). On this subspace,  $A$  is similar to

$$\begin{pmatrix} 0 & c_m \omega^{im} \\ c_m \omega^{-im} & 0 \end{pmatrix}.$$

Hence, the eigenvalues  $\lambda_i, \lambda_{n-i}$  are the distinct roots of:

$$(4) \quad \lambda^2 = (\sum c_m \omega^{im}) (\sum c_m \omega^{-im}) = \left( \sum c_m \cos \frac{2\pi im}{n} \right)^2 + \left( \sum c_m \sin \frac{2\pi im}{n} \right)^2.$$

For  $i = n$ , and  $i = n/2$  for  $n$  even, we have, similarly, the respective eigenvalues:

$$\lambda_n = \sum c_m; \quad \lambda_{n/2} = \sum_{m=0}^{n-1} (-1)^m c_m.$$

If the coefficients  $c_m$  are real, then it follows from (4) that all the eigenvalues are real. Furthermore, if we have an evenness-property for  $c_m$ 's, that is,  $c_k = c_{n-k}$ , then

$$\sum c_r \sin \frac{2\pi kr}{n} = 0,$$

which implies

$$\lambda_k = + \sum c_r \cos \frac{2\pi kr}{n} ; \lambda_{n-k} = - \sum c_r \cos \frac{2\pi kr}{n} .$$

If  $c_k = -c_{n-k}$ , then

$$\sum c_r \cos \frac{2\pi kr}{n} = 0 ,$$

which implies

$$\lambda_k = + \sum_{r=0}^{n-1} c_r \sin \frac{2\pi kr}{n} ; \lambda_{n-k} = - \sum_{k=0}^{n-1} c_r \sin \frac{2\pi kr}{n} .$$

The eigenvalues in the real or complex case can be conveniently calculated by the formulas

$$(5) \quad b_k = \sum_{j=0}^{n-1} c_{j+k} c_j ; \nu_j = \sum_{k=0}^{n-1} b_k \cos \frac{2\pi kj}{n} ,$$

where  $\lambda_i, \lambda_{n-i}$  are the distinct roots of

$$\lambda^2 = \nu_i ; \lambda_0 = + \sqrt{\nu_0} ; \lambda_{n/2} = + \sqrt{\nu_{n/2}} .$$

This involves about fifty per cent fewer steps than that usually given.

**3. Discrete approximation to eigenvalues.** For the circulant matrix  $A_{jk}$ , associated with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ,$$

$a > b > 0$ , we have the real coefficients

$$c_j = \frac{ab}{(a^2 + b^2) - (a^2 - b^2) \cos (2\pi j/N)} \quad (j = 0, 1, \dots, N-1) .$$

Since  $c_j = c_{N-j}$ , we have then as the positive eigenvalues:

$$(6) \quad \lambda_k(N) = + \sum_{r=0}^{N-1} c_r \cos \frac{2\pi kr}{N} = ab \sum_{r=0}^{N-1} \frac{\cos (2\pi kr/N)}{(a^2 + b^2) - (a^2 - b^2) \cos (2\pi r/N)}$$

$$(k = 0, 1, \dots, [N/2]).$$

Now

$$\lim_{N \rightarrow \infty} \frac{2\pi}{N} \lambda_k(N) = ab \int_{-\pi}^{+\pi} \frac{\cos k\theta d\theta}{(a^2 + b^2) - (a^2 - b^2) \cos \theta} = G(k).$$

But  $G(k)$  is tabulated [4, Table 65, no. 3]:

$$G(k) = \pi \left( \frac{a-b}{a+b} \right)^k.$$

Hence, from (6), it follows that

$$(7) \quad \lambda_k(N) \sim \frac{N}{2} \left( \frac{a-b}{a+b} \right)^k \quad (k = 0, 1, 2, \dots, [N/2]),$$

which gives us an asymptotic approximation to the eigenvalues of the matrix  $A_{jk}$ . The eigenvalues of  $A(z, z_j)$  can be shown, by means of [3], to be:

$$\left( \frac{a-b}{a+b} \right)^k \quad (k = 0, 1, 2, \dots).$$

**4. Error estimates.** We define  $E(m, N)$ , the error, by

$$(8) \quad \int_0^{2\pi} \frac{\cos m\theta d\theta}{(a^2 + b^2) - (a^2 - b^2) \cos \theta} + E(m, N)$$

$$= \frac{2\pi}{N} \sum_{k=0}^{N-1} \frac{\cos (2\pi km/N)}{(a^2 + b^2) - (a^2 - b^2) \cos (2\pi k/N)}.$$

We shall assume that  $N > 2m$ , and that  $N$  is even. We have:

$$\int_0^{2\pi} \frac{\cos m\theta d\theta}{(a^2 + b^2) - (a^2 - b^2) \cos \theta} = \frac{1}{a^2 + b^2} \int_0^{2\pi} \frac{\cos m\theta d\theta}{1 - \gamma \cos \theta},$$

where

$$\gamma = \frac{a^2 - b^2}{a^2 + b^2} < 1.$$

Since  $\gamma \cos \theta < 1$  for all values of  $\theta$ , we can write:

$$\begin{aligned} \int_0^{2\pi} \frac{\cos m\theta \, d\theta}{(a^2 + b^2) - (a^2 - b^2) \cos \theta} &= \frac{1}{a^2 + b^2} \int_0^{2\pi} \cos m\theta \left( \sum_{k=0}^{\infty} \gamma^k \cos^k \theta \right) d\theta \\ &= \frac{1}{a^2 + b^2} \sum_{k=0}^{\infty} \gamma^k \int_0^{2\pi} \cos^k \theta \cos m\theta \, d\theta, \end{aligned}$$

since the series converges uniformly and absolutely. Now

$$\cos^k \theta = \frac{1}{2} \beta_0^k + \sum_{p=1}^k \beta_p^k \cos p\theta,$$

where the Fourier coefficients are given by

$$(9) \quad \beta_p^k = \frac{1}{\pi} \int_0^{2\pi} \cos^k \theta \cos p\theta \, d\theta.$$

Rewriting, we have

$$\begin{aligned} &\int_0^{2\pi} \frac{\cos m\theta \, d\theta}{(a^2 + b^2) - (a^2 - b^2) \cos \theta} \\ &= \frac{1}{a^2 + b^2} \sum_{k=0}^{\infty} \gamma^k \left\{ \frac{1}{2} \beta_0^k \int_0^{2\pi} \cos m\theta \, d\theta + \sum_{p=1}^k \beta_p^k \int_0^{2\pi} \cos m\theta \cos p\theta \, d\theta \right\}. \end{aligned}$$

Using the orthogonality of the cosines in the interval  $[0, 2\pi]$ , we obtain:

$$(10) \quad \int_0^{2\pi} \frac{\cos m\theta \, d\theta}{(a^2 + b^2) - (a^2 - b^2) \cos \theta} = \frac{\pi}{a^2 + b^2} \sum_{k=m}^{\infty} \gamma^k \beta_m^k.$$

We shall now obtain a similar expression for the sum in (8):

$$\begin{aligned} & \frac{2\pi}{N} \sum_{k=0}^{N-1} \frac{\cos(2\pi km/N)}{(a^2 + b^2) - (a^2 - b^2) \cos(2\pi k/N)} \\ &= \frac{2\pi}{N(a^2 + b^2)} \sum_{k=0}^{N-1} \cos \frac{2\pi km}{N} \sum_{j=0}^{\infty} \gamma^j \cos^j \frac{2\pi k}{N}. \end{aligned}$$

Since  $\gamma < 1$ , the sum is absolutely convergent, and we have

$$\begin{aligned} & \frac{2\pi}{N} \sum_{k=0}^{N-1} \frac{\cos(2\pi km/N)}{(a^2 + b^2) - (a^2 - b^2) \cos(2\pi k/N)} \\ &= \frac{2\pi}{N(a^2 + b^2)} \sum_{j=0}^{\infty} \gamma^j \left\{ \sum_{k=0}^{N-1} \cos \frac{2\pi km}{N} \cos^j \frac{2\pi k}{N} \right\}. \end{aligned}$$

Now,

$$\sum_{k=0}^{N-1} \cos \frac{2\pi km}{N} \cos^j \frac{2\pi k}{N} = \sum_{k=0}^{N-1} \cos \frac{2\pi km}{N} \left\{ \frac{1}{2} \beta_0^j + \sum_{p=1}^j \beta_p^j \cos \frac{2\pi kp}{N} \right\}.$$

Since this is a finite sum, then

$$\begin{aligned} (11) \quad & \frac{2\pi}{N} \sum_{k=0}^{N-1} \frac{\cos(2\pi km/N)}{(a^2 + b^2) - (a^2 - b^2) \cos(2\pi k/N)} \\ &= \frac{2\pi}{N(a^2 + b^2)} \sum_{j=0}^{\infty} \gamma^j \left\{ \frac{1}{2} \beta_0^j \sum_{k=0}^{N-1} \cos \frac{2\pi km}{N} + \sum_{p=1}^j \beta_p^j \sum_{k=0}^{N-1} \cos \frac{2\pi km}{N} \cos \frac{2\pi kp}{N} \right\}. \end{aligned}$$

From [ 8, p. 212 ], we have the result that

$$\sum_{j=0}^{N-1} \cos \frac{2\pi kj}{N} \cos \frac{2\pi lj}{N} = \begin{cases} N & \text{for } k = 0, N, 2N, \dots, \text{ if } l = 0; \text{ zero otherwise} \\ \frac{N}{2} & \text{for } k = l, N - l, N + l, 2N - l, \dots, \text{ if } l \neq 0; \\ \text{otherwise.} & \end{cases}$$

Thus, in the case that  $m \neq 0$ , we have, for example



$$\sum_{p=0}^j \beta_p^j \sum_{k=0}^{N-1} \cos \frac{2\pi km}{N} \cos \frac{2\pi pk}{N} = \frac{N}{2} \{ \beta_m^j + \beta_{N-m}^j + \dots \\ + \beta_{(r_j-1)N+m}^j + \beta_{r_j N-m}^j \},$$

where

$$r_j = \left[ \frac{j+m}{N} \right].$$

Thus, we obtain, for  $m \neq 0$

$$(12) \quad \frac{2\pi}{N} \sum_{k=0}^{N-1} \frac{\cos(2\pi km/N)}{(a^2 + b^2) - (a^2 - b^2) \cos(2\pi k/N)} \\ = \frac{\pi}{a^2 + b^2} \sum_{j=m}^{\infty} \gamma^j \{ \beta_m^j + \dots + \beta_{r_j N-m}^j \}.$$

From our original definition, we have

$$(13) \quad E(m, N) = \frac{\pi}{a^2 + b^2} \sum_{j=N-m}^{\infty} \gamma^j \{ \beta_{N-m}^j + \dots + \beta_{r_j N-m}^j \}, m \neq 0 \\ E(0, N) = \frac{\pi}{a^2 + b^2} \sum_{j=N}^{\infty} \gamma^j \{ \beta_N^j + \dots + \beta_{r_j N}^j \}.$$

We establish the following:

LEMMA.

$$\beta_j^l = \begin{cases} 0; & l-j \not\equiv 0 \pmod{2} \\ \frac{1}{2^l} (C_{(l-j)/2}^l + C_{(l+j)/2}^l); & l-j \equiv 0 \pmod{2}. \end{cases}$$

*Proof.* From (9), we have

$$\beta_j^l = \frac{1}{\pi} \int_0^{2\pi} \cos^l \theta \cos j\theta \, d\theta = \frac{1}{\pi} \oint \frac{(z + 1/z)^l}{2^l} \left( \frac{z^j + z^{-j}}{2} \right) \frac{dz}{zi},$$

where the path of integration is the circumference of the unit circle. This reduces to

$$\beta_j^l = \frac{1}{2\pi i} \cdot \frac{1}{2^l} \sum_{p=0}^l C_p^l \left\{ \oint \frac{z^{j+2p}}{z^{l+1}} \, dz + \oint \frac{z^{-j+2p}}{z^{l+1}} \, dz \right\}.$$

Applying Cauchy’s residue theorem, we have the desired result.

COROLLARY.

$$\frac{1}{2} \beta_0^l + \sum_{j=1}^l \beta_j^l = 1.$$

*Proof.* This is an immediate consequence of the Lemma. From the Lemma, we see that  $E(m, N)$  is nonnegative, since the terms in the sum in (6) are nonnegative. Furthermore, by the Corollary, it is clear that

$$\begin{aligned} (14) \quad E(m, N) &< \frac{\pi}{a^2 + b^2} \sum_{j=N-m}^{\infty} \gamma^j \\ &= \frac{\pi}{a^2 + b^2} \frac{\gamma^{N-m}}{1-\gamma} = \frac{\pi}{2b^2} \left( \frac{a^2 - b^2}{a^2 + b^2} \right)^{N-m}. \end{aligned}$$

In the particular case  $a = 3, b = 2$ , this reduces to

$$E(m, N) < \frac{\pi}{8} \left( \frac{5}{13} \right)^{N-m},$$

which is in good agreement with the numerical results in § 5.

**5. Numerical results.** For  $N = 16, a = 3, b = 2$ , the following numerical results were obtained:

Table 1

	Calculated	Approximated by (7) of § 3
1. $\sqrt{\nu_0}$	8.00000	8.00000
2. $\sqrt{\nu_1}$	1.60000	1.60000
3. $\sqrt{\nu_2}$	0.32000	0.32000
4. $\sqrt{\nu_3}$	0.06400	0.06400
5. $\sqrt{\nu_4}$	0.01279	0.01280
6. $\sqrt{\nu_5}$	0.00256	0.00256
7. $\sqrt{\nu_6}$	0.00051	0.00051
8. $\sqrt{\nu_7}$	0.00011	0.00010
9. $\sqrt{\nu_8}$	0.00003	0.00002

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