

# Pacific Journal of Mathematics

**ON A THEOREM OF BEURLING AND KAPLANSKY**

MISCHA COTLAR

# ON A THEOREM OF BEURLING AND KAPLANSKY

M. COTLAR

**1. Introduction.** The object of this paper is to remark that a natural and simple proof of the theorem of Beurling and Kaplansky (Theorem 1 below) can be obtained by adapting to general groups a classical proof already given in the books of Wiener [8] and Zygmund [9]. In fact, Theorem 1 is an immediate consequence of a lemma (Lemma 1 below) which was proved by these authors in the case when the group is the integers or the real numbers. An easy generalization of Lemma 1 (Lemma 2 below) yields immediately the generalization of the Beurling and Kaplansky theorem stated as Theorem 2 below. For the history of the development of this theorem, see [3, p. 149] and [5]; the book [3] did not appear until the present paper had been submitted, but it seemed wise to add the reference.

**2. Statement of results.** Let  $A = \{a, b, \dots\}$  be a locally compact abelian group and  $X = \{x, y, \dots\}$  the dual group (the group operations will be written multiplicatively). Let

$$L^1(A) = \{f, g, h, p, \dots\}$$

denote the set of all integrable functions with respect to the Haar measure of  $A$ ,

$$\|f\| = \|f\|_1$$

the  $L^1$ -norm of  $f$ ,  $\hat{f}(x)$  the Fourier transform of  $f(a)$ ,

$$f_1 * f_2$$

the product of convolution (that is, the product in the group algebra),

$$f_1 f_2 = f_1(a) f_2(a)$$

the ordinary product of functions, and

$$(x, a) = x(a) = a(x)$$

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the value of the character  $x \in X$  at the point  $a \in A$ . Subsets of  $A$  will be denoted by  $C, D, \dots$ , subsets of  $X$  by  $P, Q, S, \dots$ , and subsets of  $L^1(A)$  by  $I, J, \dots$ .

The spectrum  $S(f)$  of a function  $f \in L^1(A)$  is the set of the points  $x \in X$  such that  $\hat{f}(x) = 0$ , and the spectrum  $S(I)$  of a set  $I \subset L^1(A)$  is the set of the points  $x \in X$  such that  $\hat{f}(x) = 0$  for all  $f \in I$ .

We suppose known the following Tauberian theorem of Segal and Godement (see [1] or [4]).

**THEOREM A.** *If  $I$  is a closed ideal of  $L^1(A)$ , and  $f \in L^1(A)$  is such that  $S(I)$  is interior to  $S(f)$ , then  $f \in I$ .*

Theorem A is a consequence of the regularity (in the sense of Silov) of the algebra  $L^1(A)$ , and the following Lemma A (see [7], [1], or [4]).

**LEMMA A.** *Given  $f \in L^1(A)$  and  $\epsilon > 0$ , there is a function  $g \in L^1(A)$  with the following properties:*

- (i)  $\hat{f}(x) = 0$  implies  $\hat{g}(x) = 0$ ; that is,  $S(f) \subset S(g)$ .
- (ii) If  $h = f - g$ , then  $\hat{h}(x)$  vanishes in a neighborhood of the point  $\omega$  (that is outside of a compact set  $P \subset X$ ).
- (iii)  $\|g\| \leq \epsilon$ .

It is known [6] that Theorem A is not true if  $S(f)$  is merely contained in but not interior to  $S(f)$ ; however, if  $S(I)$  consists of a single point, the following theorem is true:

**THEOREM 1** (Beurling and Kaplansky). *If  $I$  is a closed ideal such that  $S(I)$  consists of a single point  $x_0$ , then  $S(f) \supset S(I)$  implies  $f \in I$ .*

This is a special case of the following:

**THEOREM 2.** *Let  $I$  be a closed ideal such that the boundary  $P$  of  $S(I)$  is a reducible set (or that the intersection of  $P$  with the boundary of  $S(f)$  is a reducible set). Then  $S(f) \supset S(I)$  implies  $f \in I$ .*

A set is said to be reducible if it contains no nonvoid perfect subsets.

Theorem 1 was proved by Beurling in the case when  $A$  consists of the real numbers, using complex-variable methods. Kaplansky proved the theorem in the general case using the structure theory of groups. A direct and simple proof of Theorem 1 is given in a recent paper of Helson [2], and in the same paper is given a complete proof of Theorem 2.

We want to show that a still more natural and simple proof of Theorems 1 and 2 can be obtained as follows.

**2. Proofs.** We first reduce Theorem 1 to the following Lemma 1 (observe that Lemma A is obtained from Lemma 1 by replacing the point  $x_0$  by  $\infty$ ).

**LEMMA 1.** *Given a point  $x_0 \in S(f)$ ,  $f \in L^1(A)$ , and  $\epsilon > 0$ , there is a function  $g \in L^1(A)$  with the following properties:*

- (i)  $S(f) \subset S(g)$ ;
- (ii) if  $h = f - g$ , then  $\hat{h}(x)$  vanishes in a neighborhood  $U(x_0)$  of the point  $x_0$ ;
- (iii)  $\|g\| \leq \epsilon$ .

It is easy to see that Theorem 1 is an immediate consequence of Lemma 1 and Theorem A. In fact, if  $S(I)$  consists of a single point  $x_0 \in S(f)$ , then by Lemma 1 there is a function  $h$  such that  $\|f - h\| < \epsilon$ , and  $x_0$  is interior to  $S(h)$ ; hence, by Theorem A,  $h \in I$ . Since  $\epsilon$  is arbitrary and  $\|f - h\| \leq \epsilon$ , it follows that  $f \in I$ , and this proves Theorem 1.

Similarly it is easy to see that Theorem 2 is an immediate consequence of Theorem A, Lemma A, and the following Lemma 2.

**LEMMA 2.** *Given a compact reducible set  $Q \subset S(f)$ ,  $f \in L^1(A)$ , and  $\epsilon > 0$ , there is a function  $g \in L^1(A)$  with the following properties:*

- (i)  $S(f) \subset S(g)$ ;
- (ii) if  $h = f - g$ , then  $\hat{h}(x)$  vanishes in a neighborhood  $U(Q)$  of the set  $Q$ ;
- (iii)  $\|g\| \leq \epsilon$ .

Hence Theorems 1 and 2 will be proved if we prove Lemmas 1 and 2.

**3. Proof of Lemma 1.** Without loss of generality we may suppose  $x_0 = 1 = \text{unit of } X$ . Then by hypothesis

$$\hat{f}(x_0) = \int_A f(a) da = 0.$$

Given  $\epsilon > 0$ , there is a compact set  $C \subset A$  such that

$$(1) \quad \int_{A-C} |f(a)| da < \epsilon/4,$$

hence also

$$(2) \quad \left| \int_C f(a) da \right| = \left| \int_{A-C} f(a) da \right| < \epsilon/4.$$

If  $p(a)$  is any function from  $L^1(A)$ , and  $g = p * f$ , we have

$$g(a) = \int_A f(b) p(ab^{-1}) db = \int_C + \int_{A-C} f(b) p(ab^{-1}) db,$$

$$(3) \quad \|g\| \leq \int_A \left| \int_C f(b) p(ab^{-1}) db \right| da \\ + \int_A \left| \int_{A-C} f(b) p(ab^{-1}) db \right| da = M + N.$$

Using (1) and (2), and denoting the characteristic function of the set  $C' = A - C$  by  $\phi_{C'}$ , we have

$$(3a) \quad N = \int_A \left| \int_A f(b) \phi_{C'}(b) p(ab^{-1}) db \right| da \\ = \|(f \phi_{C'}) * p\| \leq \|f \phi_{C'}\| \cdot \|p\| \\ = \|p\| \cdot \int_{C'} |f(a)| da \leq \epsilon/4 \cdot \|p\|,$$

$$(3b) \quad M \leq \int_A \left| \int_C f(b) [p(ab^{-1}) - p(a)] db \right| da \\ + \int_A \left| \int_C f(b) db \right| |p(a)| da \\ \leq \left\{ \sup_{b \in C} \int_A |p(ab^{-1}) - p(a)| da \right\} \|f\| + \epsilon/4 \|p\|.$$

Let us denote  $p(ab^{-1})$  by  $p^b(a)$ ; then

$$(4) \quad \|g\| \leq \epsilon/2 \|p\| + \|f\| \sup_{b \in C} \|p^b - p\|.$$

Since

$$\hat{g}(x) = \hat{f}(x) \hat{p}(x),$$

$\hat{f}(x) = 0$  implies  $\hat{g}(x) = 0$ , and inequality (4) shows that Lemma 1 will be proved if we prove the following proposition.

PROPOSITION A. *Given  $\epsilon > 0$  and a compact set  $C \subset A$ , there is a function  $p(a)$  such that:*

- a)  $p \in L^1(A)$  and  $\|p\| \leq 2$ ;
- b) there is a neighborhood  $U(1)$  of the point  $1 \in X$  such that  $\hat{p}(x) = 1$  for  $x \in U(1)$ ;
- e)  $\|p^b - p\| < \epsilon$  for  $b$  in the compact set  $C$ .

*Proof of Proposition A.* Take two compact neighborhoods  $V$  and  $V'$  of the  $1 \in X$ , of measures  $\eta$  and  $\eta'$ , and such that

$$(5) \quad \bar{V} \subset V'; \quad \eta' \leq 4\eta,$$

and define

$$(6) \quad \hat{p}(x) = 1/\eta \{ \hat{\phi}_V * \hat{\phi}_{V'} \} = 1/\eta \{ \hat{\phi} * \hat{\phi}' \},$$

where  $\hat{\phi} = \hat{\phi}_V$  ( $\hat{\phi}' = \hat{\phi}_{V'}$ ) is the characteristic function of the set  $V$  ( $V'$ ). Since  $\hat{\phi}, \hat{\phi}' \in L^2(X)$ , by Plancherel's theorem  $\hat{p}(x)$  is the Fourier transform of a function  $p(a) \in L^1(A)$ . Since  $\bar{V} \subset V'$ , there is a neighborhood  $U = U(1)$  such that  $V \cdot U \subset V'$ , and from (6) it is clear that  $\hat{p}(x) = 1$  for  $x \in U$ . Using the Plancherel theorem it is easy to see that  $p(a)$  satisfies also the conditions a) and c), provided  $V'$  is taken small enough (cfr. [5]). For instance, let us prove condition c). Since the Fourier transform of  $\phi^b - \phi$  is  $\hat{\phi}(x) [(x, b) - 1]$ , and since  $\hat{\phi}(x) = 0$  outside of  $V' \cdot V'$ , it follows that if  $b \in C$ , and  $V'$  is small enough, then

$$\|\phi^b - \phi\|_2 = \|[x, b] - 1\| \hat{\phi} \|_2 \leq \epsilon_1 \|\hat{\phi}\|_2 = \epsilon_1 \eta^{1/2},$$

for every  $b \in C$ , where  $\epsilon_1 > 0$  is arbitrarily small. Since

$$p(a) = \phi(a) \phi'(a) / \eta,$$

by Plancherel's theorem,

$$\|p^b - p\|_1 = 1/\eta \|\phi \phi' - \phi^b \phi'^b\| \leq 1/\eta [\|\phi'(\phi - \phi^b)\| + \|\phi^b(\phi' - \phi'^b)\|]$$

$$\leq 1/\eta [ \|\phi'\|_2 \epsilon_1 \|\phi\|_2 + \|\phi\|_2 \epsilon_1 \|\phi'\|_2 ] \leq 2\epsilon_1 (\eta\eta')^{1/2}/\eta \leq 4\epsilon_1,$$

and this proves condition c).

REMARK. As we already mentioned, the foregoing proof of Lemma 1 is an adaptation of a proof given in Zygmund's book. Zygmund considers the particular case when  $A$  consists of the integers and  $X$  is the unit circle, so that the functions  $\hat{f}(x)$  are periodic functions with absolutely convergent Fourier series, and he takes for  $\hat{p}(x)$  the function

$$\hat{p}(x) = 1 \quad \text{if } |x| \leq \eta,$$

$$\hat{p}(x) = 0 \quad \text{if } |x| \geq 2\eta,$$

$$\hat{p}(x) \text{ linear if } \eta \leq |x| \leq 2\eta.$$

Then he proves that the total variation of the derivative of the function is bounded by a fixed number, and from this he deduces properties a), b), c) of the function  $p(a)$ . This is the only point in Zygmund's proof which does not apply to general groups; however, it is easy to see that the function  $\hat{p}$  used by Zygmund is exactly what formula (6) reduces to when  $V$  is taken to be an interval, and thus the proof can be adapted to the general case.

**4. Proof of Lemma 2.** Let  $Q \subset S(f)$  be a compact reducible set, and let  $Q^{(1)} = Q'$  be the set of the points  $x$  such that any neighborhood of  $x$  contains an infinite subset of  $Q$ . Define

$$Q^{(2)} = (Q^{(1)})',$$

and form in the usual way the sequence of derivative sets:

$$Q \supset Q^{(1)} \supset Q^{(2)} \supset \dots \supset Q^{(\alpha)} \supset \dots$$

Let  $w$  be such that

$$Q^{(w)} = Q^{(w+1)};$$

then  $Q^{(w)}$  is a perfect set; and since  $Q$  is reducible,  $Q^{(w)} = 0$ . If  $w = 1$ , then  $Q$  is a finite set and  $n$  successive applications of Lemma 1 yields Lemma 2 in this case. We will now prove Lemma 2 by induction on  $w$ .

Suppose that Lemma 2 is true if  $Q^{(w)} = 0$  for  $w < w_0$ ; we shall prove that

it is also true if  $Q^{(w)} = 0$  for  $w = w_0$ . Consider first the case when  $w_0 = w' + 1$ . Then  $Q^{(w')}$  is a finite set, and hence there is a function  $h \in L^1(A)$  such that

$$\|f - h\| \leq \epsilon/2, \quad S(f) \subset S(h),$$

and  $\hat{h}(x)$  vanishes on an open set  $U \supset Q^{(w')}$ . Since  $Q - U$  has the property

$$(Q - U)^{(w')} = 0,$$

and  $w' < w_0$ , by the inductive assumption there is a function  $h'$  such that

$$S(f) \subset S(h) \subset S(h'), \quad \|h - h'\| \leq \epsilon/2,$$

and  $\hat{h}'(x)$  vanishes on an open set  $U' \supset Q - U$ . Hence  $\hat{h}'(x)$  vanishes on  $U \cup U' \supset Q$ , and

$$\|f - h'\| \leq \|f - h\| + \|h - h'\| \leq 2\epsilon/2 = \epsilon.$$

If  $w_0$  is not of the form  $w' + 1$ , then by definition

$$Q^{(w_0)} = \bigcap_{w < w_0} Q^{(w)};$$

hence for some  $w' < w_0$  we must have  $Q^{(w')} = 0$ , and by the inductive assumption Lemma 2 is true in this case.

This proves Lemma 2.

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