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1. Introduction. Let $\mathfrak{E}(\mathfrak{X})$ be the Banach algebra of all bounded linear transformations defined on an infinite-dimensional Banach space $\mathfrak{X}$ and with range in $\mathfrak{X}$. Let $\mathfrak{K}(\mathfrak{X})$ be the set of completely continuous transformations contained in $\mathfrak{E}(\mathfrak{X})$. It is well known that $\mathfrak{K}(\mathfrak{X})$ is a closed two-sided ideal in $\mathfrak{E}(\mathfrak{X})$. Thus, under the usual definitions, the difference algebra $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ is again a Banach algebra. Let π be the canonical homomorphism of $\mathfrak{E}(\mathfrak{X})$ onto $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$.

The algebraic nature of $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ differs from that of $\mathfrak{E}(\mathfrak{X})$. In particular $\mathfrak{E}(\mathfrak{X})$ is semi-simple while $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ need not be semi-simple. An example of this is provided by taking for $\mathfrak{X}$ the Banach space $L(S)$ of Lebesgue-integrable numerical functions defined on, say, the unit interval S. If T and U are in $\mathfrak{E}(\mathfrak{X})$ and are weakly completely continuous then TU is completely continuous as shown by Dunford and Pettis [**5**, p. 370]. From this it follows readily that the image of the set of weakly completely continuous transformations in $\mathfrak{E}(\mathfrak{X})$ under π is contained in the radical $\mathfrak{P}_1$ of $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$. Hence $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ is not semi-simple for this $\mathfrak{X}$. On the other hand if $\mathfrak{X}$ is (separable) Hilbert space, then $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ is semi-simple.

In this paper we begin an investigation of the algebra $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$. In particular its radical and its set of regular elements are examined. This turns out to be useful in the study of certain properties of transformations in $\mathfrak{E}(\mathfrak{X})$.

In § 3 the inverse image $\pi^{-1}(\mathfrak{P}_1)$ of the radical is characterized. One formulation for this is that $\pi^{-1}(\mathfrak{P}_1)$ is the set of all $U \in \mathfrak{E}(\mathfrak{X})$ such that $(T+U)(\mathfrak{X})$ is closed and $(T+U)^{-1}(0)$ is finite-dimensional for all T which are regular in $\mathfrak{E}(\mathfrak{X})$.

A well-known result of Schauder [**13**] asserts that if I is the identity in $\mathfrak{E}(\mathfrak{X})$, and $U \in \mathfrak{K}(\mathfrak{X})$, then $I + U$ and its adjoint $I^* + U^*$ have the same (finite) nullity. In § 4 we obtain a generalization of this result as a reflection of the internal structure of $\mathfrak{E}(\mathfrak{X}) - \pi^{-1}(\mathfrak{P}_1)$. Let $\mathfrak{S}$ be any subset of $\mathfrak{E}(\mathfrak{X})$ containing

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I such that (1) $\pi(\mathfrak{C})$ is a multiplicative group and (2) the closure of the component of $\mathfrak{C}$ containing I intersects $\pi^{-1}(\mathfrak{P}_1)$. Then there is a subring $\mathfrak{C}_1$ of $\mathfrak{E}(\mathfrak{X})$ where the images of $\mathfrak{C}$ and $\mathfrak{C}_1$ are geometrically related in $\mathfrak{E}(\mathfrak{X}) - \pi^{-1}(\mathfrak{P}_1)$ such that (a) $\mathfrak{S}_1 \supset \pi^{-1}(\mathfrak{P}_1)$, (b) $\pi(\mathfrak{S}_1)$ is a group under the circle operation (see § 4) where for each $T \in \mathfrak{S}$, $U \in \mathfrak{S}_1$ the quantities nul T, nul T^*, and nul $(T + U)$, nul $(T^* + U^*)$ are all finite and

$$\text{nul}\,(T^* + U^*) - \text{nul}\,(T + U) = \text{nul}\,(T^*) - \text{nul}\,(T).$$

For $\mathfrak{S}$ the set of nonzero scalar multiples of I this result already improves Schauder's, for there

$$\mathfrak{S}_1 = \pi^{-1}(\mathfrak{P}_1);$$

and since

$$\text{nul}\,(I) = \text{nul}\,(I^*) = 0$$

we have

$$\text{nul}\,(I^* + U^*) = \text{nul}\,(I + U)$$

for every $U \in \pi^{-1}(\mathfrak{P}_1)$.

Let

$$f(T) = \text{nul}\,(T^*) - \text{nul}\,(T).$$

This is known [**1**, **15**] to be defined (finite) for the inverse image under π of the set of regular elements of $\mathfrak{E}(\mathfrak{X}) - \mathfrak{R}(\mathfrak{X})$. Atkinson [**1**] has shown that the equation $f(TU) = f(T) + f(U)$ is satisfied. In § 5 this is obtained as an application of the theory of functionals on an abstract semi-group. These considerations lead in § 6 to a detailed study of the relation of the sets in $\mathfrak{E}(\mathfrak{X})$ of elements with a one-sided or two-sided inverse to the corresponding sets, in $\mathfrak{E}(\mathfrak{X}) - \mathfrak{R}(\mathfrak{X})$.

2. Notation and preliminaries. Let $\mathfrak{X}$ be an infinite-dimensional Banach space and let $\mathfrak{E}(\mathfrak{X})$ be the algebra of all bounded linear transformations defined on $\mathfrak{X}$ into $\mathfrak{X}$ made into a Banach algebra by the usual definition of the norm of a transformation [**7**, p. 32] and with identity I. Let $\mathfrak{R}(\mathfrak{X})$ be the subset of $\mathfrak{E}(\mathfrak{X})$ consisting of the completely continuous transformations in $\mathfrak{E}(\mathfrak{X})$. It is well known [**2**, p. 96] that $\mathfrak{R}(\mathfrak{X})$ is a closed two-sided ideal in $\mathfrak{E}(\mathfrak{X})$. Thus under the usual definitions [**7**, p. 472] the difference algebra $\mathfrak{E}(\mathfrak{X}) - \mathfrak{R}(\mathfrak{X})$ is a

Banach algebra. Let π be the canonical homomorphism of $\mathfrak{E}(\mathfrak{X})$ into $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$. Let $\mathfrak{P}_1$ be the radical of $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ [7, p. 476], and let $\mathfrak{M}(\mathfrak{X})$ be any closed two-sided ideal of $\mathfrak{E}(\mathfrak{X})$ contained in $\pi^{-1}(\mathfrak{P}_1)$ and containing $\mathfrak{K}(\mathfrak{X})$. Let τ be the canonical homomorphism of $\mathfrak{E}(\mathfrak{X})$ onto $\mathfrak{E}(\mathfrak{X}) - \mathfrak{M}(\mathfrak{X})$.

2.1. LEMMA. *$T \in \mathfrak{E}(\mathfrak{X})$ has a left (right) inverse modulo $\mathfrak{M}(\mathfrak{X})$ if and only if T has a left (right) inverse modulo $\mathfrak{K}(\mathfrak{X})$.*

Proof. Suppose that T has a left inverse modulo $\mathfrak{M}(\mathfrak{X})$. Thus there exists $U \in \mathfrak{E}(\mathfrak{X})$, $V \in \mathfrak{M}(\mathfrak{X})$ such that $UT = I + V$. Now $V \in \pi^{-1}(\mathfrak{P}_1)$ so that $I + V$ has a two-sided inverse W modulo $\mathfrak{K}(\mathfrak{X})$. Hence WU is the desired left inverse of T modulo $\mathfrak{K}(\mathfrak{X})$.

It may be noted that since $\mathfrak{P}_1$ is closed in $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ then $\pi^{-1}(\mathfrak{P}_1)$ is a closed two-sided ideal in $\mathfrak{E}(\mathfrak{X})$.

2.2. LEMMA. *$T \in \mathfrak{E}(\mathfrak{X})$ has the properties that $T(\mathfrak{X})$ is closed and its null-space is finite-dimensional if and only if T takes each bounded set which is not conditionally compact onto a set which is not conditionally compact.*

Lemma 2.2 is a rewording of [**15**, Lemma 3.1].

If the null-space of T is finite-dimensional, its dimension is designated by nul T. A transformation with the properties of Lemma 2.2 is said in [**15**] to have *property A*.

2.3. LEMMA. *$T \in \mathfrak{E}(\mathfrak{X})$ has a two-sided inverse modulo $\mathfrak{M}(\mathfrak{X})$ if and only if both T and T^* have property A.*

Proof. By Lemma 2.1 we may take $\mathfrak{K}(\mathfrak{X})$ for $\mathfrak{M}(\mathfrak{X})$. The result then follows immediately from the results of [**15**, § 5] (see also [**1**, Theorem 1] and [**6**]).

If both T and T^* have property A we define

$$f(T) = \text{nul } T^* - \text{nul } T.$$

Here T^* is the adjoint of T Let $\mathfrak{H}$ be the set of all such transformations. By Lemma 2.3, $\mathfrak{H}$ is a semi-group.

2.4. LEMMA. *The function $f(T)$ is a continuous function on $\mathfrak{H}$. If T and U lie in the same component of $\mathfrak{H}$, then $f(T) = f(U)$.*

Proof. The continuity of f follows from the work of Dieudonne′ [**4**, proposition 4]; see also [**15**, Theorem 3.8] and [**1**, Theorem 4]. Since f is integer-valued, the second statement follows.

2.5. LEMMA. *If* $T \in \mathfrak{H}$ *and if* $U - T \in \mathfrak{W}(\mathfrak{X})$ *then* $U \in \mathfrak{H}$ *and* $f(T) = f(U)$.

Proof. It is clear that U has a two-sided inverse modulo $\mathfrak{K}(\mathfrak{X})$ if T does, by Lemma 2.1. That $f(T) = f(U)$ follows from Lemma 2.4 since the set $T + \mathfrak{W}(\mathfrak{X})$ is a connected subset of $\mathfrak{H}$.

We adopt the following notation used by Rickart [**12**] for a Banach algebra. An element is *left* (*right*) *regular* provided that it possesses a left (right) inverse in the algebra. If the element is both left and right regular then it possesses a unique two-sided inverse and is said to be regular. For $\mathfrak{E}(\mathfrak{X})$ we designate the sets of left regular, right regular, and regular elements by $\mathfrak{G}^l$, $\mathfrak{G}^r$, and $\mathfrak{G}$, respectively. The corresponding sets in $\mathfrak{E}(\mathfrak{X}) - \mathfrak{W}(\mathfrak{X})$ are designated by $\mathfrak{G}_1^l$, $\mathfrak{G}_1^r$, and $\mathfrak{G}_1$, respectively. In the foregoing notation, $\mathfrak{H} = \tau^{-1}(\mathfrak{G}_1)$.

Thus, by Lemmas 2.3 and 2.4, f defines a mapping of $\mathfrak{G}_1$ into the set of integers. This mapping will also be designated by f.

2.6. LEMMA. *Let* $T \in \mathfrak{H}$, $f(T) = 0$. *Then* T *can be expressed as the sum* $U + V$ *where* $U \in \mathfrak{G}$, $V \in \mathfrak{K}(\mathfrak{X})$.

Proof. This is given in [**15**, Corollary 3.11].

3. On the radical of $\mathfrak{E}(\mathfrak{X}) - \mathfrak{W}(\mathfrak{X})$. In view of Lemma 2.1 and the definition of the radical of $\mathfrak{E}(\mathfrak{X}) - \mathfrak{W}(\mathfrak{X})$, the inverse image under τ of the radical of $\mathfrak{E}(\mathfrak{X}) - \mathfrak{W}(\mathfrak{X})$ is the same set as $\pi^{-1}(\mathfrak{P}_1)$, where $\mathfrak{P}_1$ is the radical of $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$. In this section we determine the nature of $\pi^{-1}(\mathfrak{P}_1)$.

3.1. LEMMA. *Let* $T \in \mathfrak{E}(\mathfrak{X})$ *be an isomorphism between* $\mathfrak{X}$ *and a proper closed linear manifold of* $\mathfrak{X}$. *Then there exists a sphere in* $\mathfrak{E}(\mathfrak{X})$ *with center* T *each of whose elements have this property.*

Proof. By [**4**, proposition 1] there is a sphere $\mathfrak{S}$ about T such that for all U in $\mathfrak{S}$, U is bi-continuous. But T is in the interior of the set of elements of $\mathfrak{E}(\mathfrak{X})$ which are not regular [**14**, Corollary 2.2]. Hence for each $U \in \mathfrak{S}$ there is a proper closed linear manifold $\mathfrak{N}$ of $\mathfrak{X}$ such that U is an isomorphism of $\mathfrak{X}$ onto $\mathfrak{N}$ if the radius of $\mathfrak{S}$ is sufficiently small.

3.2. LEMMA. *Let* $T \in \mathfrak{E}(\mathfrak{X})$ *have range* $\mathfrak{X}$ *where* T *is not one-to-one. Then* *there is a sphere in* $\mathfrak{E}(\mathfrak{X})$ *with center* T *each of whose elements has these properties.*

Proof. This is shown in the same way by use of [**4**, Theorem 1] and [**14**, Corollary 3.12].

3.3. LEMMA. *Let* $T \in \mathfrak{E}(\mathfrak{X})$. *Suppose that* $T(T^*)$ *has property* A *while* $T^*(T)$ *does not. Then* T *can be expressed in the form* $T_1 + V$ *where* $V \in \mathfrak{K}(\mathfrak{X})$ *and* T_1 *is bi-continuous* $(T_1(\mathfrak{X}) = \mathfrak{X})$.

Proof. This is contained in [**15**, Theorem 3.13].

3.4. THEOREM. *Let* $T \in \mathfrak{G}$. *Suppose that for each* α $(0 < \alpha \leq 1)$ *either* $T + \alpha U$ *or* $T^* + \alpha U^*$ *has property* A. *Then* $T + \alpha U \in \mathfrak{H}$ $(0 \leq \alpha \leq 1)$ *and* $f(T + U) = 0$.

Proof. Note that $f(T) = 0$. The set $\mathfrak{H} = \pi^{-1}(\mathfrak{G}_1)$ is open in $\mathfrak{E}(\mathfrak{X})$. Thus either all the $T + \alpha U$ $(0 \leq \alpha \leq 1)$ are in $\mathfrak{H}$ or there is a smallest number β $(0 < \beta \leq 1)$ such that $T + \beta U \notin \mathfrak{H}$. In the latter case one of $T + \beta U$, $T^* + \beta U^*$ has property A but not the other. Suppose that $T + \beta U$ has property A. Then, by Lemma 3.3, $T + \beta U$ can be written in the form $T_1 + V$, where $T_1 \in \mathfrak{E}(\mathfrak{X})$ is bi-continuous and $V \in \mathfrak{K}(\mathfrak{X})$. If $T_1(\mathfrak{X}) = \mathfrak{X}$ then $T_1 \in \mathfrak{G}$ and thus $T + \beta U \in \mathfrak{H}$, contrary to the above. Thus $T_1 = T + \beta U - V$ an isomorphism between $\mathfrak{X}$ and a proper closed linear manifold of $\mathfrak{X}$. Consequently, by Lemma 3.1, if $0 < \alpha < \beta$, and $\beta - \alpha$ is sufficiently small, then $T + \alpha U - V$ has this property. But for such α, $T + \alpha U \in \mathfrak{H}$. Also, by Lemma 2.5, $T + \alpha U - V \in \mathfrak{H}$ and

$$f(T + \alpha U) = f(T + \alpha U - V).$$

Since

$$\operatorname{nul}(T + \alpha U - V) = 0, \quad \operatorname{nul}(T^* + \alpha U^* - V^*) > 0,$$

then

$$f(T + \alpha U) > 0.$$

However, since $f(T) = 0$, by Lemma 2.4 we have

$$f(T + \alpha U) = 0.$$

This contradiction establishes the result if $T + \beta U$ has property A. If $T^* + \beta U^*$ has property A then we proceed in a same way using dual results (Lemmas 3.2

and 3.3) to see that for $\alpha < \beta$ and close to β,

$$f(T+\alpha U)=0, \quad f(T+\alpha U)<0.$$

Thus we conclude that $T+\alpha U \in \mathfrak{H}$ $(0 \leq \alpha \leq 1)$. That

$$f(T+U)=f(T)=0$$

follows from Lemma 2.4.

3.5. THEOREM. *The following formulas for $\pi^{-1}(\mathfrak{P}_1)$ hold*:

(a) $\pi^{-1}(\mathfrak{P}_1)=\{U \in \mathfrak{B}(\mathfrak{X}) \mid$ *for each* $T \in \mathfrak{G}$ *either* $T+U$ *or* T^*+U^* *has property* $A\}$;

(b) $\pi^{-1}(\mathfrak{P}_1)=\{U \in \mathfrak{B}(\mathfrak{X}) \mid T+U$ *has property* A *for each* $T \in \mathfrak{G}\}$;

(c) $\pi^{-1}(\mathfrak{P}_1)=\{U \in \mathfrak{B}(\mathfrak{X}) \mid T^*+U^*$ *has property* A *for each* $T \in \mathfrak{G}\}$.

Proof. If $T \in \mathfrak{G}$ and $U \in \pi^{-1}(\mathfrak{P}_1)$ then $\pi(T) \in \mathfrak{G}_1$ and

$$\pi(T+U)=\pi(T)+\pi(U) \in \mathfrak{G}_1,$$

by the definition of $\mathfrak{P}_1$. Then $T+U \in \mathfrak{H}$ and it follows that $\pi^{-1}(\mathfrak{P}_1)$ is contained in each of the sets on the right.

Let the set on the right side of (a) be denoted by $\mathfrak{S}$. Then if $T \in \mathfrak{G}$, $U \in \mathfrak{S}$, $\alpha \neq 0$ a scalar, then $\alpha T+U$ or αT^*+U^* has property A. Hence, for each scalar α, $T+\alpha U$ or $T^*+\alpha U^*$ has property A. Theorem 3.4 shows that $T+\alpha U \in \mathfrak{H}$ for all scalars α. Next we show that if $W \in \mathfrak{G}$, $U \in \mathfrak{S}$ then $UW \in \mathfrak{S}$. Both W and W^* have property A. Hence, by the nature of $\mathfrak{S}$ and [**15**, Theorem 3.4], for each $T \in \mathfrak{G}$ either

$$(TW^{-1}+U)W=T+UW$$

has property A or

$$W^*[(TW^{-1})^*+U^*]=T^*+(UW)^*$$

has property A. Hence $UW \in \mathfrak{S}$.

Next let $U_i \in \mathfrak{S}$, $i=1,2$. For each $T \in \mathfrak{G}$, by the above $T+\alpha U_1 \in \mathfrak{H}$ for $0 \leq \alpha \leq 1$ and, by Theorem 3.4, $f(T+U_1)=0$. By Lemma 2.6, $T+U_1$ can be expressed in the form T_1+V, where $T_1 \in \mathfrak{G}$ and $V \in \mathfrak{K}(\mathfrak{X})$. Likewise $T_1+U_2 \in \mathfrak{H}$ and so, by Lemma 2.5,

$$T_1 + U_2 + V = T + (U_1 + U_2)$$

is in $\mathfrak{H}$. This shows that $\mathfrak{S}$ is a linear manifold in $\mathfrak{E}(\mathfrak{X})$ with the further property that if $T_1, T_2 \in \mathfrak{G}$ and $U \in \mathfrak{S}$ then $U(T_1 - T_2) \in \mathfrak{S}$. However, since $\mathfrak{E}(\mathfrak{X})$ is a Banach algebra, an arbitrary element $W \in \mathfrak{E}(\mathfrak{X})$ can be expressed as the difference of two regular elements. Thus $\mathfrak{S}$ is a right ideal in $\mathfrak{E}(\mathfrak{X})$. Consequently $\pi(\mathfrak{S})$ has the property that, for each $\pi(U) \in \pi(\mathfrak{S})$ and each V in $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$, $\pi(I) + \pi(T)V \in \mathfrak{G}_1$. Thus $\pi(\mathfrak{S}) \subset \mathfrak{P}_1$. This completes the proof for formula (a).

The same argument shows that the right sides of (b) and (c) are contained in $\pi^{-1}(\mathfrak{P}_1)$.

3.6. Corollary. *Let $\mathfrak{L}$ be a (left or right) ideal in $\mathfrak{E}(\mathfrak{X})$. Suppose that for each $T \in \mathfrak{L}$, either $I + T$ or $I^* + T^*$ has property A. Then for each $T \in \mathfrak{L}$,* nul $(I + T)$ *and* nul $(I^* + T^*)$ *are finite and equal.*

Proof. By Theorem 3.5, $\mathfrak{L} \subset \pi^{-1}(\mathfrak{P}_1)$. Thus $I + T \in \mathfrak{H}$ for each $T \in \mathfrak{L}$. Since $\mathfrak{L}$ is a linear manifold,

$$f(I + T) = f(I) = 0$$

by Lemma 2.4.

This is a direct generalization of Schauder's well-known result [**13**, p. 189] that if U is completely continuous then

$$\text{nul } (I + U) = \text{nul } (I^* + U^*)$$

since the two-sided ideal $\mathfrak{K}(\mathfrak{X})$ fulfills the conditions of Corollary 3.6.

3.7. Corollary. *The following statements are equivalent:*

(1) $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ *is semi-simple;*

(2) *for $U \in \mathfrak{E}(\mathfrak{X})$, $U \in \mathfrak{K}(\mathfrak{X})$ if and only if $(T + U)(\mathfrak{X})$ is closed in $\mathfrak{X}$ and either* nul $(T + U)$ *or* nul $(T^* + U^*)$ *is finite for each T regular in $\mathfrak{E}(\mathfrak{X})$.*

Proof. Note that $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ is semi-simple if and only if $\pi^{-1}(\mathfrak{P}_1) = \mathfrak{K}(\mathfrak{X})$. Also $(T + U)(\mathfrak{X})$ is closed if and only if $(T^* + U^*)(\mathfrak{X}^*)$ is closed in $\mathfrak{X}^*$ [**2**, Chapt. 10]. Then Corollary 3.7 follows from Theorem 3.5 and Lemma 2.3.

If $\mathfrak{X}$ is a separable Hilbert space then since, as shown by Calkin [**3**, Theorem 1.4], $\mathfrak{K}(\mathfrak{X})$ is a maximal, two-sided ideal in $\mathfrak{E}(\mathfrak{X})$, (1) holds. For spaces satisfying (1), (2) gives a necessary and sufficient condition for complete continuity which seems to be new (for sufficiency) even in the Hilbert space case.

4. A generalized Schauder nullity theorem. We give here the result (Theorem 4.5) discussed in § 1. The preliminary material, it is felt, is of independent interest and is presented in greater generality than is absolutely necessary for our purposes.

We adopt the following notation. B is a ring with an identity element e. G is the set of regular elements of B (the elements with a two-sided inverse). For each subgroup G_0 of G let $\mathfrak{J}(G_0)$ be the set of "invariant translations" of G_0, namely the set of $x \in B$ such that $G_0 + x = G_0$. It is clear that

$$\mathfrak{J}(G_0) = \{x \in B \mid y \pm x \in G_0 \text{ for every } y \in G_0\}.$$

In the ring B we consider along with the usual algebraic operations also the "circle operation"

$$x \circ y = x + y - xy.$$

For information on this operation see [7, Chapter 22]. It is evident that $G_0 \cap \mathfrak{J}(G_0)$ is empty.

4.1. THEOREM. *For any subgroup G_0 of G, $\mathfrak{J}(G_0)$ is a subring of B which is a group under the circle operation. Conversely if R is a subring of B which is a group under the circle operation then there exists a subgroup G_0 of G such that $R = \mathfrak{J}(G_0)$. If B is a Banach algebra then $\mathfrak{J}(G)$ is the radical of B.*

Proof. It is clear that if $x \in \mathfrak{J}(G_0)$ then so does $-x$. Thus if x_1 and x_2 lie in $\mathfrak{J}(G_0)$, and $y \in G_0$, then both

$$(y + x_1) + x_2 \quad \text{and} \quad (y - x_1) - x_2$$

lie in G_0, so that $x_1 + x_2 \in \mathfrak{J}(G_0)$. Next we show if $x \in \mathfrak{J}(G_0)$, $y \in G_0$, then $yx \in \mathfrak{J}(G_0)$. For let $z \in G_0$. Then

$$z \pm yz = y(y^{-1}z \pm x) \in G_0.$$

Similarly $xy \in \mathfrak{J}(G_0)$. Since

$$y \pm x_1x_2 = (y + x_1)(e \pm x_2) \mp yx_2 - x_1$$

it follows from the above that $x_1x_2 \in \mathfrak{J}(G_0)$ if x_1 and $x_2 \in \mathfrak{J}(G_0)$. Thus $\mathfrak{J}(G_0)$ is a subring of B.

To see that $\mathfrak{I}(G_0)$ is a group under the circle operation note first that for $x_1, x_2 \in \mathfrak{I}(G_0)$ we have

$$x_1 \circ x_2 = x_1 + x_2 - x_1 x_2 \in \mathfrak{I}(G_0).$$

Now the set of all elements of B with an inverse under the circle operation is a group with the zero element θ of B as the identity element [7, p. 456]. Thus it is sufficient to show that x_1 has an inverse in $\mathfrak{I}(G_0)$ under this operation. Since $e - x_1 \in G_0 \subset G$ there exists an element $w \in B$ such that

$$(e - x_1)(e - w) = (e - w)(e - x_1) = e.$$

Then clearly w is the inverse of x_1 under this operation. Let $y \in G_0$. Then, since

$$x_1 w = w x_1 = x_1 + w$$

we have that

$$(y \pm w)(e - x_1) = y \pm w - y x_1 \mp w x_1 = y(e - x_1) \mp x_1$$

is an element of G_0. Since $(e - x_1) \in G_0$ it follows that $w \in \mathfrak{I}(G_0)$.

Next consider a subring R which is a group under the circle operation. Let G_0 be the set of all elements of the form $e - x$, $x \in R$. If $x_1, x_2 \in R$ then

$$(e - x_1)(e - x_2) = e - x_1 \circ x_2 \in G_0.$$

There exists $z \in R$ such that

$$x_1 \circ z = z \circ x_1 = \theta.$$

Then

$$(e - x_1)(e - z) = (e - z)(e - x_1) = e$$

so that G_0 is a group. We show that $\mathfrak{I}(G_0) = R$. Take $x \in \mathfrak{I}(G_0)$. Then $e - x \in G_0$, and, by the definition of G_0, $x \in R$. On the other hand if $x \in R$, $y \in G_0$ then we may write $y = e - x_1$, where

$$x_1 \in R \text{ and } y \pm x = e - x_1 \pm x \in G_0$$

since R is a ring. Thus $x \in \mathfrak{I}(G_0)$ and $\mathfrak{I}(G_0) = R$.

Finally let B be a Banach algebra. If z is an arbitrary element of B then since, for a sufficiently small scalar λ,

$$e - \lambda z = w \in G\cdot$$

we may write z as the sum of two elements in G. By the above we see that for $x \in \mathfrak{J}(G)$, we have $zx \in \mathfrak{J}(G)$ and thus $e - zx \in G$. Hence x lies in the (Jacobson) radical Q of B. Conversely if $x \in Q$, then for each $w \in G$,

$$w \pm x = w(e \pm w^{-1}x) \in G,$$

so that $x \in \mathfrak{J}(G)$. This completes the proof.

4.2. Corollary. *In the notation of Theorem 4.1, $\mathfrak{J}(G_0)$ is a two-sided ideal in the subring $R(G_0)$ of B generated by G_0 and lies in the radical Q of $R(G_0)$. Examples exist for which $\mathfrak{J}(G_0) = Q$ and also for which $\mathfrak{J}(G_0) \neq Q$.*

Proof. By the arguments of Theorem 4.1, if $y \in R(G_0)$ then yx, $xy \in \mathfrak{J}(G_0)$ for each $x \in \mathfrak{J}(G_0)$ so that $\mathfrak{J}(G_0)$ is a two-sided ideal of $R(G_0)$. Since $e - yx \in G_0$ for every $y \in R(G_0)$, and G_0 is contained in the set of regular elements of $R(G_0)$, $\mathfrak{J}(G_0) \subset Q$. By Theorem 4.1, if B is a Banach algebra then $\mathfrak{J}(G) = Q$. Take next for B the ring of integers modulo 9. For G_0 take the set consisting of 1 and 8. Here $R(G_0) = B$ and the radical Q of B is the set $\{0, 3, 6\}$. On the other hand $\mathfrak{J}(G_0)$ consists of the zero element alone.

Following Kaplansky [8, p. 153] we call B a *metric ring* if to each element x there is associated a real number $|x|$ such that

$$|\theta| = 0,\ |x| > 0 \text{ if } x \neq \theta,\ |-x| = |x|,\ |x+y| \leq |x| + |y|,\ |xy| \leq |x|\,|y|.$$

Here $|x - y|$ is the metric of B. In this context the sets $\mathfrak{J}(G_0)$ possesses certain topological properties. (The metric ring to which the theory is applied is $\mathfrak{C}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$).

4.3. Lemma. *If G_0 is open then $\mathfrak{J}(G_0)$ is closed. The following statements are equivalent.*

(1) $\mathfrak{J}(G_0) \subset \overline{G_0}$.

(2) $0 = \inf |y|$, $y \in G_0$.

(3) $\mathfrak{J}(G_0) \cap \overline{G_0}$ *is nonempty.*

Proof. Let G_0 be open. Suppose that $x_n \in \mathfrak{J}(G_0)$ $(n = 1, 2, 3, \cdots)$ and that $x_n \longrightarrow x$. Given any $y \in G_0$ there exists a sphere S of radius, say, $r > 0$ about y such that $S \subset G_0$. Consequently $S \pm x_n \subset G_0$ for each n. Take n so large that $|x - x_n| < r$. Then for such an integer n, $y \pm (x - x_n) \in S$ and thus

$$y \pm x = y \pm (x - x_n) \mp x_n \in G_0 .$$

Hence $x \in \mathfrak{Z}(G_0)$.

If (1) holds then so does (2) since $\theta \in \mathfrak{Z}(G_0)$. If (2) holds then (3) is clear for the same reason. Suppose that (3) holds. Let

$$w \in \mathfrak{Z}(G_0) \cap \overline{G_0},\ w = \lim y_n,\ y_n \in G_0 .$$

By Theorem 4.1, $w \circ x \in \mathfrak{Z}(G_0)$ for each $x \in \mathfrak{Z}(G_0)$. But

$$w \circ x = \lim (x + y_n - y_n x),$$

and by Theorem 4.1, $y_n + x - y_n x \in G_0$. Hence $w \circ x \in \overline{G_0}$. By Theorem 4.1 again there exists an element z in $\mathfrak{Z}(G_0)$ such that $w \circ z = \theta$. Inasmuch as $z \circ x \in \mathfrak{Z}(G_0)$, by the above

$$w \circ (z \circ x) = (w \circ z) \circ x = x$$

lies in $\overline{G_0}$.

For the group G_0 in the metric ring B let G_{0p} be the *principal component*, that is, that which contains e. Arguments of Hille [7, p. 93] show that G_{0p} is a subgroup of G_0.

4.4. LEMMA. *If* $\mathfrak{Z}(G_{0p}) \subset \overline{G}_{0p}$ *then* $\mathfrak{Z}(G_0)$ *is connected and* $\mathfrak{Z}(G_0) \subset \overline{G}_{0p}$. *If* $\mathfrak{Z}(G_0)$ *is connected, then* $\mathfrak{Z}(G_0) \subset \mathfrak{Z}(G_{0p})$.

Proof. Suppose that $\mathfrak{Z}(G_{0p}) \subset \overline{G}_{0p}$. Then by Lemma 4.3, $\theta \in \overline{G}_{0p}$. Take $x \in \mathfrak{Z}(G_0)$. The set xG_{0p}, being a continuous image of a connected set, is connected; moreover, xG_{0p} lies in $\mathfrak{Z}(G_0)$ by Corollary 4.2. Since θ lies in the closure of xG_{0p}, the set

$$F = xG_{0p} \cup \{\theta\}$$

is a connected subset of $\mathfrak{Z}(G_0)$ which contains x and θ. Hence each element of $\mathfrak{Z}(G_0)$ lies in a connected subset containing θ. Thus $\mathfrak{Z}(G_0)$ is connected.

Suppose that $\mathfrak{Z}(G_0)$ is connected. Then for each $z \in G_{0p}$, $z + \mathfrak{Z}(G_0)$ is a connected subset of G_0 containing z. Hence

$$z + \mathfrak{J}(G_0) \subset G_{0p} \quad \text{and} \quad \mathfrak{J}(G_0) \subset \mathfrak{J}(G_{0p}).$$

In the statement of the following theorem, the group to which the symbol $\mathfrak{J}$ is applied lies in the Banach algebra $\mathfrak{E}(\mathfrak{X}) - \pi^{-1}(\mathfrak{P}_1)$.

4.5. THEOREM. *Let $\mathfrak{S}$ be any set in $\mathfrak{E}(\mathfrak{X})$ containing the identity I. Let π and τ be the canonical homomorphisms of $\mathfrak{E}(\mathfrak{X})$ onto $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ and $\mathfrak{E}(\mathfrak{X}) - \pi^{-1}(\mathfrak{P}_1)$, respectively. Suppose that $\pi(\mathfrak{S})$ is a multiplicative group in $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ and that the closure of the component of $\mathfrak{S}$ containing I contains an element of $\pi^{-1}(\mathfrak{P}_1)$. Then for each $T \in \mathfrak{S}$, $U \in \tau^{-1}\, \mathfrak{J}[\tau(\mathfrak{S})]$ we have*

$$f(T) = f(T + U).$$

Furthermore, $\tau^{-1}\, \mathfrak{J}[\tau(\mathfrak{S})] \supset \pi^{-1}(\mathfrak{P}_1)$, and is the inverse image under π of a subring of $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ which is a group under the circle operation.

Proof. Consider $\tau(\mathfrak{S})$. By Lemma 2.1 it is a subgroup of the set of regular elements $\mathfrak{G}_1$ of $\mathfrak{E}(\mathfrak{X}) - \pi^{-1}(\mathfrak{P}_1)$. Since τ is continuous, by our hypothesis the principal component of $\tau(\mathfrak{S})$ contains the zero element of $\mathfrak{E}(\mathfrak{X}) - \pi^{-1}(\mathfrak{P}_1)$ in its closure. Hence in this algebra, by Lemmas 4.3 and 4.4, $\mathfrak{J}[\tau(\mathfrak{S})]$ is connected. By Lemma 2.4, f is continuous on $\mathfrak{G}_1$; and if $T_1 \in \tau(\mathfrak{S})$, $U_1 \in \mathfrak{J}[\tau(\mathfrak{S})]$ then since T_1 and $T_1 + U_1$ lie in the same component of $\mathfrak{G}_1$, we have

$$f(T_1 + U_1) = f(T_1).$$

Thus $f(T + U) = f(T)$ if $T \in \mathfrak{S}$ and $U \in \tau^{-1}[\mathfrak{J}(\tau(\mathfrak{S})]$.

Let

$$\tau^{-1}\, \mathfrak{J}[\tau(\mathfrak{S})] = \mathfrak{S}_1 \quad \text{and} \quad \pi(\mathfrak{S}_1) = \mathfrak{S}_2.$$

Clearly $\pi^{-1}(\mathfrak{S}_2) = \mathfrak{S}_1$ since $\mathfrak{S}_1 \supset \mathfrak{K}(\mathfrak{X})$ which is the kernel of π. By Theorem 4.1, $\mathfrak{J}[\tau(\mathfrak{S})]$ is a subring of $\mathfrak{E}(\mathfrak{X}) - \pi^{-1}(\mathfrak{P}_1)$ which is a group under the circle Operation. Then $\mathfrak{S}_1$ is a subring of $\mathfrak{E}(\mathfrak{X})$, and $\mathfrak{S}_2$ a subring of $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$. We next show that $\mathfrak{S}_2$ is a group under the circle operation. As $\mathfrak{S}_2$ is a subring, it is closed under that operation. Let $T_1 \in \mathfrak{S}_2$, $T_1 = \pi(T)$, $T \in \mathfrak{S}_1$. Then there exists $V \in \mathfrak{S}_1$ such that

$$[\tau(I) - \tau(V)][\tau(I) - \tau(T)] = [\tau(I) - \tau(T)][\tau(I) - \tau(V)] = \tau(I).$$

Then by Lemma 2.1, $I - T$ has a two-sided inverse $I - W$ modulo $\mathfrak{K}(\mathfrak{X})$. Since

$$T_1 \circ \pi(W) = \pi(W) \circ T_1 = 0$$

it suffices to show that $\pi(W) \in \mathfrak{S}_2$. Now $\tau(W) = \tau(V)$ since the two-sided inverse of $\tau(I-T)$ in $\mathfrak{E}(\mathfrak{X}) - \pi^{-1}(\mathfrak{P}_1)$ is unique. Therefore $W \in \mathfrak{S}_1$ and thus $\pi(W) \in \mathfrak{S}_2$.

5. Functionals on semi-groups. Atkinson [1] has shown that on $\mathfrak{H}$ the equation

$$f(TU) = f(T) + f(U)$$

is valid. By an entirely different analysis we show how such functionals can be obtained in a semi-group and then apply the results to $\mathfrak{H}$.

5.1. NOTATION. Let S be any semi-group, the product of two elements x, y in S being denoted by xy. Let g and g^* be real-valued functions defined on S, where

$$(1) \qquad \begin{aligned} g(x_2) &\le g(x_1 x_2) \le g(x_1) + g(x_2) \\ g^*(x_1) &\le g^*(x_1 x_2) \le g^*(x_1) + g^*(x_2) \end{aligned}$$

for all x_1, x_2 in S. Let

$$h(x) = g^*(x) - g(x),$$

and let S_+ (S_-) be the subset of S for which $h(x) \ge 0$ ($h(x) \le 0$). Suppose that there is a reflexive and symmetric relation $\sim$ on S defined for certain pairs of elements of S such that $x \sim y$ implies $h(x) = h(y)$, and where for each $x \in S$ there exists $y \in S$, $x \sim y$ with either $g(y) = 0$ or $g^*(y) = 0$, The relation $\sim$ need not be transitive. Since g and g^* are nonnegative on S it follows that the existence of y, $x \sim y$, where $g(y) = 0$ ($g^*(y) = 0$), is equivalent to $x \in S_+$ ($x \in S_-$).

5.2. THEOREM. *Suppose that, in the notation of* 5.1,

(a) $x_i \sim z_i$ $(i = 1, 2)$ *implies that* $h(x_1 x_2) = h(z_1 z_2)$ *holds. Then the formula*

$$(2) \qquad h(x_1 x_2) = h(x_1) + h(x_2)$$

is valid either for all $x_1 \in S_+$ *or for all* $x_2 \in S_-$. *If also*

(b) *there exist* y_1, y_2 *in* S, *where* $h(y_1) > 0$ *and* $h(y_2) < 0$, *then formula* (2) *is valid on* S.

Formula (2) *is valid on* S *if* (a) *holds and*

(c_1) *for each* $x \in S_+$ *there exists* $y \in S$ *such that* $xy \in S_-$,

(c_2) *for each* $x \in S_-$ *there exists* $y \in S$ *such that* $yx \in S_+$.

Proof. We remark that (a) is a necessary condition for (2) since, from (2),

$$h(x_1 x_2) = h(x_1) + h(x_2) = h(z_1) + h(z_2) = h(z_1 z_2).$$

From (1) we obtain

$$g^*(x_1) - g(x_1) - g(x_2) \leq g^*(x_1 x_2) - g(x_1 x_2) \leq g^*(x_1) + g^*(x_2) - g(x_2)$$

or

$$h(x_1) - g(x_2) \leq h(x_1 x_2) \leq h(x_2) + g^*(x_1) \tag{3}$$

Now suppose that (a) holds. Then

$$h(x_1) \leq h(x_1 x_2) \leq h(x_1) + h(x_2) \qquad x_1, x_2 \in S_+, \tag{4}$$

$$h(x_1) + h(x_2) \leq h(x_1 x_2) \leq h(x_2) \qquad x_1, x_2 \in S_-, \tag{5}$$

$$h(x_1 x_2) = h(x_1) + h(x_2) \qquad x_1 \in S_+,\ x_2 \in S_-. \tag{6}$$

To show (4) we may assume that

$$g(x_i) = 0,\ g^*(x_i) = h(x_i) \qquad (i = 1, 2).$$

Then (4) follows from (3). For (5) we may assume that

$$-g(x_i) = h(x_i),\ g^*(x_i) = 0 \qquad (i = 1, 2),$$

and again use (3). In the last situation, (3) yields

$$h(x_1) + h(x_2) \leq h(x_1 x_2) \leq h(x_1) + h(x_2).$$

Next we observe that (c_1) and (c_2) cannot both be false. If, for example, (c_1) is false then for some $x_1 \in S_+$ we have $x_1 y \in S_+$ for all $y \in S$, which yields (c_2).

Suppose now that (a) and (c_2) hold. We show that (2) holds for all x_1, x_2 where $x_2 \in S_-$. By (6) we may suppose that $x_1 \in S_-$. There exists $w \in S$ such that $h(wx_1) \geq 0$. For case 1 we take $w \in S_-$. Then by (5),

$$h(w) + h(x_1) \leq h(wx_1) \leq h(x_1) \leq 0.$$

This implies that $h(x_1) = 0$. Then (2) follows from (6). For case 2 we take $w \in S_+$. This gives, by (6),

$$h(wx_1) = h(w) + h(x_1), \tag{7}$$

$$h(wx_1 x_2) = h(wx_1) + h(x_2). \tag{8}$$

Now (5) shows that $x_1 x_2 \in S_-$. Then, by (6),

$$h(wx_1 x_2) = h(w) + h(x_1 x_2). \tag{9}$$

A combination of (7), (8), and (9) yields (2).

Suppose next that (a) and (c_1) hold. Entirely analogous arguments using (4) in place of (5) show that (2) holds for all x_1, x_2 where $x_1 \in S_+$.

Now assume (a) and (b). We show that (c_1) and (c_2) hold. If (c_1) does not hold then (c_2) must hold and there exists $x \in S_+$ such that $xy \in S_+$ for all $y \in S$. Select y such that $h(y) < 0$. By (a) and (c_2) and the above, $h(y^n) = n\, h(y)$ for any positive integer n and thus $y^n \in S_-$. Also

$$0 \leq h(xy^n) = h(x) + n\, h(y).$$

This is impossible if n is chosen sufficiently large. Thus (c_1) holds. Similarly (c_2) holds.

To conclude the proof we show that (a), (c_1), and (c_2) imply (2). By the above our assumptions give the validity of (2) for any pair x_1, x_2 where either $x_1 \in S_+$ or $x_2 \in S_-$. The remaining case involves $x_1 \in S_-$ and $x_2 \in S_+$. We may select, by (c_2), $w \in S$ such that $wx_1 \in S_+$. If $w \in S_-$ then, as shown above, $h(x_1) = 0$ so that (2) is valid for x_1, x_2. Supposing that $w \in S_+$, we obtain (7), (8), and (9), which again yield (2) for x_1, x_2.

We return to $\mathfrak{E}(\mathfrak{X})$ and start with the following simple result:

5.3. Lemma. *Let $T_i \in \mathfrak{E}(\mathfrak{X})$ $(i = 1, 2)$ have finite nullity. Then*

$$\operatorname{nul}(T_2) \leq \operatorname{nul}(T_1 T_2) \leq \operatorname{nul}(T_1) + \operatorname{nul}(T_2). \tag{10}$$

This follows from the fact, readily established, that

$$\operatorname{nul}(T_1 T_2) = \operatorname{nul}(T_2) + \dim [T_2(\mathfrak{X}) \cap T_1^{-1}(0)].$$

5.4. Lemma. *Suppose that $T \in \mathfrak{H}$ and $f(T) \geq 0$ (≤ 0). Then there exists $V \in \mathfrak{H}$ such that $V - T \in \mathfrak{K}(\mathfrak{X})$, $f(T) = f(V)$, and* $\operatorname{nul}(V) = 0$ $(\operatorname{nul}(V^*) = 0)$.

The existence of the transformation V with the indicated property of the nullity follows from [**15**, Theorem 3.13]. That $f(T) = f(V)$ follows from Lemma 2.5.

5.5. COROLLARY. *Let* $T_i \in \mathfrak{H}$ $(i = 1, 2)$. *Then* $f(T_1 T_2) = f(T_1) + f(T_2)$, *and* f *defines a homomorphism of the group of regular elements of* $\mathfrak{S}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ *into the additive group of integers.*

We show that this result of Atkinson follows from the above. In the notation of 5.1, set

$$S = \mathfrak{H},\ g^*(T) = \text{nul}\ (T^*),\quad g(T) = \text{nul}\ (T).$$

Since

$$(T_1 T_2)^* = T_2^* T_1^*,$$

Lemma 5.3 shows that formula (1) is valid. For the relation $T_1 \sim T_2$ we take $T_1 - T_2 \in \mathfrak{K}(\mathfrak{X})$. Lemmas 3.2, 2.4, and 5.4 and the relation

$$f(T) = \text{nul}\ (T^*) - \text{nul}\ (T)$$

show that Theorem 5.2 may be applied to give the first conclusion. The second conclusion is an immediate consequence.

Following ideas of Mackey [**10**, p. 171] we shall say that the Banach space $\mathfrak{X}$ is *stable* if there exists a continuous isomorphism of $\mathfrak{X}$ onto a closed subspace $\mathfrak{X}_1$ of deficiency one. We say that $\mathfrak{X}$ is *stable-like* if there exists a continuous isomorphism of $\mathfrak{X}$ onto a closed subspace $\mathfrak{X}_1$ of finite deficiency.

5.6. THEOREM. *The functional* f *is non-trivial if and only if* $\mathfrak{X}$ *is stable-like.*

Proof. If $\mathfrak{X}$ is stable-like, consider the isomorphism T of $\mathfrak{X}$ onto $\mathfrak{X}_1$ of deficiency n. Then nul $(T^*) = n$ and nul $(T) = 0$, so that $f(T) = n$.

Suppose that f is non-trivial. Then there exists $T \in \mathfrak{H}$ such that $f(T) \neq 0$. Since T has a two-sided inverse V modulo $\mathfrak{K}(\mathfrak{X})$, and $f(V) = -f(T)$ by Corollary 5.5, we may assume $f(T) = n > 0$. By Lemma 5.4, there exists a bi-continuous isomorphism U where nul $(U^*) = n$. Then $U(\mathfrak{X})$ is a closed subspace of deficiency n.

Whether or not every infinite-dimensional Banach space must be stable or even stable-like seems to be an open question (see [**10**, p. 205]). This subject is pursued a bit further in Theorem 6.7 and 6.9.

If $\mathfrak{X}$ is finite-dimensional then (10) can be replaced by the more specific rule, known as Sylvester's law of nullity [9, p. 11] which states that

$$\max[\operatorname{nul}(T_1), \operatorname{nul}(T_2)] \leq \operatorname{nul}(T_1 T_2) \leq \operatorname{nul}(T_1) + \operatorname{nul}(T_2).$$

We show that the validity of Sylvester's rule for all $T_i \in \mathfrak{H}$ where $\mathfrak{X}$ is infinite-dimensional implies that $\mathfrak{X}$ is not stable-like. For suppose otherwise. Consider

$$T_2 \in \mathfrak{H},\ f(T_2) = n > 0,\ \operatorname{nul}(T_2) = 0.$$

Then by [14, Theorem 3.15] there exists $T_1 \in \mathfrak{E}(\mathfrak{X})$ such that $T_1 T_2 = I$. Since I and $T_2 \in \mathfrak{H}$, by [15, Theorem 5.4] we see that $T_1 \in \mathfrak{H}$. By Sylvester's rule, $\operatorname{nul}(T_1) = 0$, so that T_1 is regular in $\mathfrak{E}(\mathfrak{X})$ and therefore so is T_2, which is a contradiction.

Another generalization of Schauder's theorem may be obtained as follows. Yosida and Kakutani [16] have considered the collection $\mathfrak{Q}(\mathfrak{X})$ of all *quasi-completely continuous* transformations in $\mathfrak{E}(\mathfrak{X})$ i.e. the class of all $T \in \mathfrak{E}(\mathfrak{X})$ such that there exists $V \in \mathfrak{K}(\mathfrak{X})$ and an integer n such that $\|T^n - V\| < 1$.

5.7. THEOREM. *Let $T \in \mathfrak{H}$, and let V be a two-sided inverse of T modulo $\mathfrak{K}(\mathfrak{X})$. Suppose that there exists $W \in \pi^{-1}(\mathfrak{P}_1)$ and an integer m such that $V^m U - W \in \mathfrak{Q}(\mathfrak{X})$. Then $T^m + U \in \mathfrak{H}$, and*

$$f(T^m + U) = mf(T).$$

Proof. Let $V^m U = R_1$ and $R_1 - W = R_2$. By hypothesis there is an integer n such that $I - R_2^n$ is of the form $S_1 + S_2$, where $S_1 \in \mathfrak{G}$ and $S_2 \in \mathfrak{K}(\mathfrak{X})$. Since $\pi^{-1}(\mathfrak{P}_1)$ is a two-sided ideal, there exists $S_3 \in \pi^{-1}(\mathfrak{P}_1)$ such that

$$I - R_1^n = S_1 + S_3.$$

But, by Lemma 2.5, $S_1 + S_3 \in \mathfrak{H}$. Therefore $I - R_1^n$ has a two-sided inverse modulo $\mathfrak{K}(\mathfrak{X})$. Since

$$I - R_1^n = (I - R_1)(I + R_1 + \cdots + R_1^{n-1}) = (I + R_1 + \cdots + R_1^{n-1})(I - R_1)$$

then $I - R_1 \in \mathfrak{H}$. Since the hypothesis on U is satisfied by all αU, $|\alpha| \leq 1$, it follows from Theorem 3.4 that

$$f(I - R_1) = f(I + R_1) = 0.$$

Applying Corollary 5.5, we obtain

$$f(T^m + U) = f[T^m(I + R_1)] = mf(T).$$

6. On the images of left and right regular elements. We make here a detailed study of the images of the sets $\mathfrak{G}$, $\mathfrak{G}^l$, and $\mathfrak{G}^r$ under π. In view of Lemma 2.1, the results also hold for the mapping τ. In particular, we show the following:

6.1. THEOREM. *The canonical homomorphism π has the following properties*:

(1) $\pi(\mathfrak{G}^l \cup \mathfrak{G}^r) = \pi(\mathfrak{G}^l) \cup \pi(\mathfrak{G}^r) = \mathfrak{G}_1^l \cup \mathfrak{G}_1^r$;

(2) $\pi(\mathfrak{G}) = \pi(\mathfrak{G}^l) \cap \pi(\mathfrak{G}^r)$;

(3) *the sets $\pi(\mathfrak{G})$, $\pi(\mathfrak{G}^l)$, and $\pi(\mathfrak{G}^r)$ are open and closed in the sets $\mathfrak{G}_1$, $\mathfrak{G}_1^l$, and $\mathfrak{G}_1^r$, respectively*;

(4) *$\pi(\mathfrak{G})$ is a normal subgroup of $\mathfrak{G}_1$; either $\pi(\mathfrak{G}) = \mathfrak{G}_1$ or $\mathfrak{G}_1/\pi(\mathfrak{G})$ is isomorphic, as a topological group, to the additive group of integers in the discrete topology.*

The interest of (1) lies in the fact that if $\mathfrak{X}$ is stable-like, then $\pi(\mathfrak{G}^l) \neq \mathfrak{G}_1^l$ and $\pi(\mathfrak{G}^r) \neq \mathfrak{G}_1^r$ (see Lemma 6.3). And for (2), even though $\mathfrak{G} = \mathfrak{G}^l \cap \mathfrak{G}^r$ this does not of itself imply that

$$\pi(\mathfrak{G}^l \cap \mathfrak{G}^r) = \pi(\mathfrak{G}^l) \cap \pi(\mathfrak{G}^r).$$

In the course of the proof the following notation is used. $\mathfrak{H}_0$ is the subset of $\mathfrak{H}$ consisting of those T for which $f(T) = 0$ and $\mathfrak{H}_+$ ($\mathfrak{H}_-$) of those T for which $f(T) > 0$ ($f(T) < 0$). The minus sign for sets in $\mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X})$ is used in the set-theoretic sense. From the definitions we have $\pi(\mathfrak{H}) = \mathfrak{G}_1$.

The following lemmas are part of the proof of Theorem 6.1.

6.2. LEMMA. $\pi(\mathfrak{G}) = \{T_1 \in \mathfrak{E}(\mathfrak{X}) - \mathfrak{K}(\mathfrak{X}) \mid \pi^{-1}(T_1) \subset \mathfrak{H}_0\}$, and $\pi(\mathfrak{G}) = \pi(\mathfrak{H}_0)$.

Proof. The second statement follows immediately from the first. Suppose that $T_1 = \pi(T)$, $T \in \mathfrak{G}$. Then $\pi^{-1}(T_1) = T + \mathfrak{K}(\mathfrak{X})$, so that for each $U \in \pi^{-1}(T_1)$, $f(U) = f(T)$ by Lemma 2.5. Since $f(T) = 0$, we see that $\pi(\mathfrak{G})$ is contained in the right-hand set. Next assume that T_1 is in the right-hand set. Let $\pi(T) = T_1$. Then $T \in \mathfrak{H}_0$, and $f(T) = 0$. By Lemma 2.6 there exists $V \in \mathfrak{K}(\mathfrak{X})$ such that $T + V \in \mathfrak{G}$. But $\pi(T + V) = T_1$.

6.3. LEMMA. $\pi(\mathfrak{G}^l) = \mathfrak{G}_1^l - \pi(\mathfrak{H}_-)$.

Proof. Clearly $\pi(\mathfrak{G}^l) \subset \mathfrak{G}_1^l$. We shall show that $\pi(\mathfrak{G}^l) \cap \pi(\mathfrak{H}_-)$ is empty. Suppose contrariwise that $T_1 \in \pi(\mathfrak{G}^l) \cap \pi(\mathfrak{H}_-)$. Then there exists $T \in \mathfrak{G}^l$, $U \in \mathfrak{H}_-$ such that $\pi(T) = \pi(U) = T_1$. Then there exists $W \in \mathfrak{K}(\mathfrak{X})$ such that $T = U + W$. Hence, by Lemma 2.5, $f(T) = f(U) < 0$. But from the definition of f, nul $(T) > 0$. Therefore T cannot be one-to-one and this contradicts $T \in \mathfrak{G}^l$. We conclude that $\pi(\mathfrak{G}^l) \subset \mathfrak{G}_1^l - \pi(\mathfrak{H}_-)$.

Suppose that $T_1 \in \mathfrak{G}_1^l - \pi(\mathfrak{H}_-)$ and $\pi(T) = T_1$. By [**15**, Theorem 5.4], T has property A. Since $T \notin \mathfrak{H}_-$, either nul (T^*) is not finite or nul $(T^*) < \infty$ and $f(T) \geq 0$. Then by [**15**, Theorem 3.13] there exists $V \in \mathfrak{K}(\mathfrak{X})$ such that $T + V$ is a bi-continuous mapping of $\mathfrak{X}$ into $\mathfrak{X}$. Moreover, by [**15**, Theorems 5.3 and 5.4], there exists a projection of $\mathfrak{X}$ onto $(T + V)(\mathfrak{X})$. Therefore, by [**14**, Theorem 3.15], $T + V \in \mathfrak{G}^l$. However, $\pi(T + V) = \pi(T) = T_1$. Thus $\mathfrak{G}^l - \pi(\mathfrak{H}_-) \subset \pi(\mathfrak{G}^l)$.

6.4. LEMMA. $\pi(\mathfrak{G}^r) = \mathfrak{G}_1^r - \pi(\mathfrak{H}_+)$.

In references cited in the proof of Lemma 6.3, dual results exist to those used in 6.3 which enable one to conduct the proof in the same way.

6.5. LEMMA. $\pi(\mathfrak{H}_-) \subset \pi(\mathfrak{G}^r)$ *and* $\pi(\mathfrak{H}_+) \subset \pi(\mathfrak{G}^l)$.

Proof. Suppose that $T \in \mathfrak{H}_-$. By [**15**, Theorem 3.13] there exists $V \in \mathfrak{K}(\mathfrak{X})$ such that $(T + V)(\mathfrak{X}) = \mathfrak{X}$. Also, by Lemma 2.4, nul $(T + V) < \infty$. Hence [**14**, Theorem 3.18] shows that $T + V \in \mathfrak{G}^r$. However, $\pi(T + V) = \pi(T)$. The other statement is proved using dual results.

6.6. LEMMA. *$\mathfrak{H}_0$, $\mathfrak{H}_+$, $\mathfrak{H}_-$ are open and closed as subsets of $\mathfrak{H}$. These sets are disjoint.*

Proof. Since $f(T)$, by Lemma 2.5, is a continuous integral-valued function on $\mathfrak{H}$, the sets are open and closed subsets of $\mathfrak{H}$.

We turn now to the statements of Theorem 6.1.

Consider (1). By Lemmas 6.3 and 6.4,

$$\pi(\mathfrak{G}^l \cup \mathfrak{G}^r) = \pi(\mathfrak{G}^l) \cup \pi(\mathfrak{G}^r) = [\mathfrak{G}_1^l - \pi(\mathfrak{H}_-)] \cup [\mathfrak{G}_1^r - \pi(\mathfrak{H}_+)].$$

By Lemma 6.5,

$$\pi(\mathfrak{H}_-) \subset \pi(\mathfrak{G}^r),\ \pi(\mathfrak{H}_+) \subset \pi(\mathfrak{G}^l),$$

so that

$$\pi(\mathfrak{G}^l) \cup \pi(\mathfrak{G}^r) = \mathfrak{G}_1^l \cup \mathfrak{G}_1^r .$$

As for (2), note first that $\pi(\mathfrak{G}) = \pi(\mathfrak{H}_0)$ by Lemma 6.2. By Lemmas 6.3 and 6.4,

$$\pi(\mathfrak{G}^l) \cap \pi(\mathfrak{G}^r) = \mathfrak{G}_1^l \cap \mathfrak{G}_1^r - [\pi(\mathfrak{H}_-) \cup \pi(\mathfrak{H}_+)].$$

But $\mathfrak{G}_1^l \cap \mathfrak{G}_1^r = \mathfrak{G}_1 = \pi(\mathfrak{H})$. Also the sets $\pi(\mathfrak{H}_+)$, $\pi(\mathfrak{H}_-)$ and $\pi(\mathfrak{H}_0)$ are disjoint since if, for example, $T_1 \in \pi(\mathfrak{H}_+) \cap \pi(\mathfrak{H}_-)$, $T_1 = \pi(T)$, $T \in \mathfrak{H}_+$ and $T_1 = \pi(V)$, $V \in \mathfrak{H}_-$, then $\pi(T - V) = 0$ so that $T - V \in \mathfrak{R}(\mathfrak{X})$; whence, by Lemma 2.4, $f(T) = f(V)$ which is impossible. Hence

$$\pi(\mathfrak{G}^l) \cap \pi(\mathfrak{G}^r) = \pi(\mathfrak{H}) - [\pi(\mathfrak{H}_-) \cup \pi(\mathfrak{H}_+)] = \pi(\mathfrak{H}_0) = \pi(\mathfrak{G}).$$

The mapping π is a continuous linear mapping of the Banach algebra $\mathfrak{E}(\mathfrak{X})$ onto the Banach algebra $\mathfrak{E}(\mathfrak{X}) - \mathfrak{R}(\mathfrak{X})$. Consequently it takes open sets into open sets. Since $\mathfrak{G}$, $\mathfrak{G}^l$, $\mathfrak{G}^r$, $\mathfrak{G}_1$, $\mathfrak{G}_1^l$ and $\mathfrak{G}_1^r$ are open (see, for example, [**12**]) the statement of (3) on openness follows. Likewise, from Lemma 6.6, $\pi(\mathfrak{H}_-)$ is open in $\mathfrak{G}_1 \subset \mathfrak{G}_1^l$. Since

$$\pi(\mathfrak{G}^l) = \mathfrak{G}_1^l - \pi(\mathfrak{H}_-)$$

by Lemma 6.3, $\pi(\mathfrak{G}^l)$ is closed in $\mathfrak{G}_1^l$. Similarly $\pi(\mathfrak{G}^r)$ is open and closed in $\mathfrak{G}_1^r$. Now

$$\mathfrak{G}_1 = \pi(\mathfrak{H}) = \pi(\mathfrak{H}_0) \cup \pi(\mathfrak{H}_-) \cup \pi(\mathfrak{H}_+)$$

and (as noted above) the latter sets are disjoint and also open by Lemma 6.6. But $\pi(\mathfrak{G}) = \pi(\mathfrak{H}_0)$ by Lemma 6.2. Thus $\pi(\mathfrak{G})$ is open and closed in $\mathfrak{G}_1$ and the proof of (3) is complete.

Only (4) remains to be shown. Either $\pi(\mathfrak{G}) = \mathfrak{G}_1$ or $\pi(\mathfrak{G})$ is properly contained in $\mathfrak{G}_1$. Suppose that the latter holds. By Lemma 6.2, $\pi(\mathfrak{H}_0) = \pi(\mathfrak{G})$. But $\pi(\mathfrak{H}) = \mathfrak{G}_1$. Thus $\mathfrak{H} \neq \mathfrak{H}_0$ and the function f defined on $\mathfrak{H}$ (and on $\pi(\mathfrak{H})$) is not identically zero. Since f is integral valued there is an integer $m > 0$ and $T \in \mathfrak{H}$ such that $|f(T)| = m$ and m is minimal with respect to this property. By Corollary 5.5, f is a homomorphism of $\pi(\mathfrak{H}) = \mathfrak{G}_1$ into the additive group J of integers. If we define f_1 on $\mathfrak{G}_1$ by the rule $f_1 = m^{-1}f$ then f_1 is a homomorphism of $\mathfrak{G}_1$ onto J. The kernel of this homomorphism is $\pi(\mathfrak{H}_0) = \pi(\mathfrak{G})$ (Lemma 6.2). If J is given the discrete topology then f_1 is an open mapping. Since the kernel is open in $\mathfrak{G}_1$ by (3), the inverse image under f_1 of any subset of J is open in $\mathfrak{G}_1$. Hence, [**11**, p. 64], $\mathfrak{G}_1/\pi(\mathfrak{G})$ is isomorphic, as a topological group, to J. This completes the proof of Theorem 6.1.

6.7. THEOREM. *The following statements are equivalent*:

(1) $\mathfrak{X}$ *is not stable-like*; (2) $\mathfrak{H} = \mathfrak{H}_0$; (3) $\pi(\mathfrak{G}) = \mathfrak{G}_1$.

Proof. The equivalence of (1) and (2) is given by Theorem 5.9. In the course of the proof of Theorem 6.1 it was shown that if $\pi(\mathfrak{G}) \neq \mathfrak{G}_1$ then $\mathfrak{H} \neq \mathfrak{H}_0$ so that (2) implies (3). If $\pi(\mathfrak{G}) = \mathfrak{G}_1$ then, by Lemma 6.2, $\pi(\mathfrak{H}_0) = \pi(\mathfrak{H})$. This shows that any element T of $\mathfrak{H}$ differs from an element of $\mathfrak{H}_0$ by a completely continuous transformation in $\mathfrak{G}(\mathfrak{X})$. Therefore, from Lemma 2.4, $\mathfrak{H} = \mathfrak{H}_0$.

6.8. DEFINITION. We say that $\mathfrak{X}$ is *projection-stable* if there exists an isomorphism in $\mathfrak{E}(\mathfrak{X})$ of $\mathfrak{X}$ onto a proper closed linear manifold $\mathfrak{N}$ where there is a (continuous) projection of $\mathfrak{X}$ on $\mathfrak{N}$.

Clearly if $\mathfrak{X}$ is stable-like then $\mathfrak{X}$ is projection-stable. Whether or not the converse is true is an open question. The notion just defined is connected with the notions of Theorem 6.1 by the following result.

6.9. THEOREM. *The following statements are equivalent*:

(1) $\mathfrak{X}$ *is not projection-stable*;

(2) $\mathfrak{G}^l = \mathfrak{G}^r = \mathfrak{G}$;

(3) $\pi(\mathfrak{G}) = \mathfrak{G}_1$ *and* $\mathfrak{G}_1 = \mathfrak{G}_1^l = \mathfrak{G}_1^r$.

Proof. If $\mathfrak{X}$ is not projection-stable then, by [**14**, Theorem 3.15], $\mathfrak{G}^l \subset \mathfrak{G}$ so that $\mathfrak{G}^l = \mathfrak{G}$. But then also $\mathfrak{G} = \mathfrak{G}^r$; for if $T \in \mathfrak{G}^r$, $TU = I$, then $U \in \mathfrak{G}$ and $T = U^{-1} \in \mathfrak{G}$. Thus (1) implies (2). Assume (2). By Theorem 6.1 we see that

$$\pi(\mathfrak{G}) = \pi(\mathfrak{G}^l \cup \mathfrak{G}^r) = \mathfrak{G}_1^l \cup \mathfrak{G}_1^r.$$

But $\pi(\mathfrak{G}) \subset \mathfrak{G}_1$. Hence

$$\mathfrak{G}_1^l = \mathfrak{G}_1^r = \mathfrak{G}_1 \quad \text{and} \quad \pi(\mathfrak{G}) = \mathfrak{G}_1.$$

Assume (3). If $\mathfrak{X}$ were projection-stable then by [**14**, Theorem 3.15] there exists $T \in \mathfrak{G}^l$, $T \notin \mathfrak{G}$. But $\pi(T) \in \mathfrak{G}_1^l = \mathfrak{G}_1$. Hence $T \in \mathfrak{H}$. By its nature $f(T) > 0$. However, from Theorem 6.7, $\mathfrak{H} = \mathfrak{H}_0$, which is a contradiction.

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