Pacific Journal of Mathematics

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Vol. 5, No. 1 September 1955

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1. Introduction. Let α denote a fixed integer, $\alpha \geq 2$, and put $\omega = \exp(2\pi i/\alpha)$.

DEFINITION 1. The *Rademacher functions of order* α are defined by

$$\phi_0(x) = \omega^k \text{ if } k/\alpha \leq x < (k+1)/\alpha,\ k = 0, \cdots, \alpha - 1;$$

and for $n \geq 0$

$$\phi_n(x+1) = \phi_n(x) = \phi_0(\alpha^n x).$$

DEFINITION 2. The *Walsh functions of order* α are defined by

$$\psi_0(x) = 1,$$

and if $n = a_1 \alpha^{n_1} + \cdots + a_m \alpha^{n_m}$ where $0 < a_j < \alpha$ and $n_1 > n_2 > \cdots > n_m$, then

$$\psi_n(x) = \phi_{n_1}^{a_1}(x) \cdots \phi_{n_m}^{a_m}(x).$$

For convenience we let Ψ_α denote the set of Walsh functions of order α. We may observe that Ψ_2 is the orthonormal system of functions defined by Walsh [4]. R.E.A.C. Paley's proof that Ψ_2 is orthonormal and complete in $L(0, 1)$ may be modified by the reader to establish the same properties for Ψ_α, $\alpha = 3, 4, \cdots$. [3; pp. 242-244].

It is the purpose of this paper to study Fourier expansions in the sets Ψ_α. The results obtained here will include known results for ordinary Walsh Fourier series, most of which are contained in a paper of N. J. Fine [1]. In fact, most

Received May 26, 1953. Presented to the faculty of the University of Oregon in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics. The author wishes to express his gratitude to Professor Paul Civin for his stimulating guidance in the preparation of this paper.

Pacific J. Math. 5 (1955), 17-31

of the properties of Fourier expansions in Ψ_2 are shared by expansions in Ψ_α.

The system Ψ_α is in fact the character group of G_α, the countable product of cyclic groups of order α, transferred to the unit interval. The operation $\dot{+}$, introduced in § 2, is precisely the image of the group operation. Some of our results and many of our methods readily admit interpretations in G_α, although little mention of these will be made in the text. For example, in Lemma 1 we prove that the Haar integral in the group corresponds to the Lebesgue integral on $(0, 1)$.

Using an obvious abbreviation, we summarize our most important results: (i) The $W_\alpha FS$ of $f(x)$ converges to $f(x)$ a.e. if $f(x)$ is of bounded variation, and the convergence tests of Dini and Dini-Lipschitz are valid. (ii) If $f(x)$ has variation V and if c_k is the coefficient of $\psi_k(x)$ in the $W_\alpha FS$ of $f(x)$, then $|c_k| \leq Vk^{-1} \csc \pi/\alpha$. (iii) The continuity of $f(x)$ is a sufficient condition for the uniform $(C, 1)$ summability of the $W_\alpha FS$.

2. Notation and preliminary results. Define

$$I_{n,k} = I_{n,k}(\alpha) = \{x : k\,\alpha^{-n} \leq x < (k+1)\,\alpha^{-n}\},$$

$k = 0, \cdots, \alpha^n - 1$, $n = 1, 2, \cdots$. Then if $\phi_n(x)$ is the nth Rademacher function of order α, $\phi_n(x) = \omega^k$ if $x \in I_{n+1,k}$.

The term, *α-adic rational,* will denote any number of the form $k\alpha^{-n}$ where k and n are integers. Thus if x has the base α expansion

$$\sum_{j=1}^{\infty} x_j\,\alpha^{-j}, \quad 0 \leq x_j < \alpha,$$

where the terminating expansion is taken in case x is an α-adic rational, we see that $\phi_n(x) = \omega^{x_{n+1}}$.

We introduce a binary operation, denoted by $\dot{+}$, and defined as follows: If $0 \leq a < 1$ and $0 \leq x < 1$, and if a and x have base α expansions

$$\sum_{1}^{\infty} a_j\alpha^{-j} \text{ and } \sum_{1}^{\infty} x_j\alpha^{-j}$$

respectively, then $a \dot{+} x$ will denote the number

$$\sum_{1}^{\infty} y_j \,\alpha^{-j}$$

where $y_j \equiv a_j + x_j \pmod{\alpha}$, $0 \le y_j < \alpha$. If we agree to take the terminating expansions for α-adic rationals whenever possible, it follows that for any fixed a and all $n \ge 0$ $\phi_n(a \dotplus x) = \phi_n(a)\,\phi_n(x)$, a.e. The exceptional values occur when $a \dotplus x$ is the infinite expansion of an α-adic rational. It is also true that $\psi_n(a \dotplus x) = \psi_n(a)\,\psi_n(x)$, a.e.

LEMMA 1. *If* $f(x) \in L(0,1)$ *then* $f(a \dotplus x) \in L(0,1)$ *and*

$$\int_0^1 f(x)\,dx = \int_0^1 f(a \dotplus x)\,dx.$$

The reader will have no difficulty in modeling a proof after the proof in the case $\alpha = 2$ [**1**, p. 379].

If $f(x) \in L(0,1)$ and if

$$c_n = \int_0^1 f(t)\,\overline{\psi}_n(t)\,dt$$

we say that $\sum_0^\infty c_n\,\psi_n(x)$ is the $W_\alpha FS$ of $f(x)$. Let $s_k(x)$ denote the kth partial sum of this series, so that

$$s_k(x) = \int_0^1 f(t) \sum_{0}^{k-1} \psi_j(x)\,\overline{\psi}_j(t)\,dt = \int_0^1 f(t)\,D_k(x,t)\,dt$$

where the kernel $D_k(x,t)$ is defined accordingly. We will write $D_k(t) = D_k(0,t)$. Observe that for all $k \le \alpha^n$, $D_k(x,t) = D_k(x', t')$ provided only that x and x' are in the same $I_{n,r}$ and that t and t' are in the same $I_{n,r'}$.

Let $z = z(x,n)$ be that number satisfying

$$(2.1) \qquad x \dotplus z = 0$$

except when this relation determines z as the nonterminating expansion of an α-adic rational. In these cases let the first n digits in the expansion of z be determined by (2.1), and let the remaining digits be zeros. For all $k \le \alpha^n$ we have for almost all t

$$(2.2)\quad D_k(x,t) = \sum_0^{k-1} \psi_j(x)\,\overline{\psi}_j(t) = \sum \overline{\psi}_j(z)\,\psi_j(t) = \sum \overline{\psi}_j(z \dot{+} t) = D_k(z \dot{+} t).$$

If we use Lemma 1 we have the following useful result.

$$(2.3)\qquad s_k(x) = \int_0^1 D_k(z \dot{+} t)\, f(t)\, dt$$

$$= \int_0^1 D_k(x \dot{+} z \dot{+} t) f(x \dot{+} t)\, dt = \int_0^1 D_k(t) f(x \dot{+} t)\, dt.$$

Unless otherwise stated all functions will be assumed to be periodic and integrable on $(0, 1)$.

3. Convergence.

LEMMA 2.

$$D_{\alpha^n}(t) = \begin{cases} \alpha^n \text{ if } t \in I_{n,0}, \\ 0 \quad \text{otherwise.} \end{cases}$$

Proof. We have from the definitions

$$(3.1)\qquad D_{\alpha^n}(t) = \sum_{r=0}^{\alpha^n - 1} \overline{\psi}_r(t) = \prod_{r=0}^{n-1} [1 + \overline{\phi}_r(t) + \cdots + \overline{\phi}_r^{\alpha-1}(t)].$$

If $t \in I_{n,0}$ each $\phi_r(t) = 1$, while if $t \notin I_{n,0}$ at least one factor in the product vanishes. (The pth factor is zero if $\phi_p(t) \neq 1$.)

By translating under $\dot{+}$ we see that Lemma 2 has the following equivalent form: *If $\rho = \rho(x, n)$ is such that $x \in I_{n,\rho}$ then*

$$D_{\alpha^n}(x,t) = \begin{cases} \alpha^n \quad \text{if } t \in I_{n,\rho}, \\ 0 \quad \text{otherwise}. \end{cases}$$

As an immediate consequence we have

THEOREM 1. *If $f(x) \in L(0,1)$ then $\lim_{n\to\infty} s_{\alpha^n}(x) = f(x)$* a.e. *In particular, $s_{\alpha^n}(x) \longrightarrow f(x)$ at a point of continuity of $f(x)$ and the convergence is uniform in a closed interval of continuity. If x is an α-adic rational then $s_{\alpha^n}(x) \longrightarrow f(x)$ provided x is a point of right hand continuity of $f(x)$.*

Additional usefulness of Lemma 2 is seen from the identity

$$(3.2)\quad D_n(x,t) = \sum_{j=1}^{m} \Big\{ \phi_{n_1}^{a_1}(x)\bar{\phi}_{n_1}^{a_1}(t) \cdots \phi_{n_{j-1}}^{a_{j-1}}(x)\bar{\phi}_{n_{j-1}}^{a_{j-1}}(t)$$

$$D_{\alpha^{n_j}}(x,t)\left[1 + \phi_{n_j}(x)\,\bar{\phi}_{n_j}(t) + \cdots + \phi_{n_j}^{a_j-1}(x)\,\bar{\phi}_{n_j}^{a_j-1}(t)\right]\Big\},$$

where the base α expansion of n is given in Definition 2. To prove (3.2) notice that

$$(3.3)\quad D_n(x,t) = D_{\alpha^{n_1}}(x,t) + \sum_{r=0}^{n-\alpha^{n_1}-1} \psi_{\alpha^{n_1}+r}(x)\,\bar{\psi}_{\alpha^{n_1}+r}(t)$$

$$= D_{\alpha^{n_1}}(x,t) + \phi_{n_1}(x)\,\bar{\phi}_{n_1}(t)\,D_{n-\alpha^{n_1}}(x,t).$$

By using (3.3) recursively we obtain (3.2).

The usual method of establishing convergence of the full sequence of partial sums of the $W_\alpha FS$ will be to reduce the convergence of $s_n(x)$ to that of $s_{\alpha^{n_1}}(x)$ by showing that $s_{\alpha^{n_1}}(x) - s_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$, where $\alpha^{n_1} \le n < \alpha^{n_1+1}$. In the following lemma we use the notation of Definition 2, with the additional convention of writing N for n_1.

LEMMA 3. *Let ν be a fixed positive integer and let $x \in I_{\nu,\rho}$. Then if $\sigma \ne \rho$*

$$(3.4)\quad \lim_{n\to\infty} \int_{I_{\nu,\sigma}} [D_n(x,t) - D_{\alpha^N}(x,t)] f(t)\,dt = 0.$$

If also $y \in I_{\nu,\rho}$ and $N \ge \nu$, then

$$(3.5)\quad \left| \int_y^{(\rho+1)\alpha^{-\nu}} [D_n(x,t) - D_{\alpha^N}(x,t)]\,dt \right| < \alpha,$$

and in case $y = \rho\alpha^{-\nu}$, the integral (3.5) vanishes.

Proof. In proving (3.4) we may suppose $N \ge \nu$. Let r be chosen so that $n_r \ge \nu > n_{r+1}$; in case $n_m \ge \nu$ take $r = m$. By Lemma 2 all $D_{\alpha^k}(x,t) = 0$ for $t \in I_{\nu,\sigma}$ and $k \ge \nu$. Thus $D_n(x,t) = D_n(x,t) - D_{\alpha^N}(x,t)$ and by (3.2) this is a sum of $m - r$ terms, each of which is, for $t \in I_{\nu,\sigma}$, a constant multiple of

$$\bar{\phi}_{n_1}^{a_1}(t)\cdots\bar{\phi}_{n_r}^{a_r}(t)=\bar{\psi}_{M(n)}(t),$$

say. A careful inspection of (3.2) shows that the sum of the moduli of the coefficients of $\bar{\psi}_{M(n)}(t)$ is bounded independent of n. Also, $M(n)\longrightarrow\infty$ as $n\longrightarrow\infty$. We have now reduced (3.4) to a theorem of Mercer [2, p. 17].

The inequality (3.5) is proved by writing $I_{\nu,\rho}$ as a sum of $I_{N,s}$. On each $I_{N,s}$ the integrand is a linear combination of $\bar{\phi}_N^b(t)$, $0<b<\alpha$. On each complete $I_{N,s}$ contained in $(y,(\rho+1)\alpha^{-\nu})$ the integral vanishes. The remainder of the interval of integration has length less than α^{-N}, and from (3.3) we see that the integrand is numerically less than α^{N+1}.

THEOREM 2. *If $f(x)$ is of bounded variation and continuous from the right on $[0,1)$, then as $n\longrightarrow\infty$, $s_n(x)\longrightarrow f(x)$ at every point of continuity and at every α-adic rational. If x is an α-adic irrational which is a point of discontinuity, $s_n(x)$ does not converge.*

Proof. To prove convergence it is sufficient to show that for $f(t)$ monotonic

$$s_n(x)-s_{\alpha^N}(x)=\int_0^1[D_n(x,t)-D_{\alpha^N}(x,t)]f(t)\,dt\longrightarrow 0.$$

Write this integral as

$$\int_{I_{\nu,\rho}}+\int_{CI_{\nu,\rho}}[D_n(x,t)-D_{\alpha^N}(x,t)]f(t)\,dt=J_1+J_2,$$

where C denotes the complement taken with respect to $(0,1)$. By the second theorem of the mean, there is $y\in I_{\nu,\rho}$ such that

$$J_1=f(\rho\alpha^{-\nu}+0)\int_{\rho\alpha^{-\nu}}^{y}[D_n-D_{\alpha^N}]\,dt$$

$$+f((\rho+1)\alpha^{-\nu}-0)\int_y^{(\rho+1)\alpha^{-\nu}}[D_n-D_{\alpha^N}]\,dt.$$

By (3.5)

(3.6) $$|J_1|\le\alpha|f((\rho+1)\alpha^{-\nu}-0)-f(x)|+\alpha|f(x)-f(\rho\alpha^{-\nu}+0)|<\epsilon/2$$

for ν sufficiently large and for $n\ge\alpha^\nu$, since $f(x+0)=f(x)=f(x-0)$. If

x is an α-adic rational, first choose ν large enough so that $\rho\alpha^{-\nu} = x$, so that only right hand continuity is involved in (3.6). With ν fixed, $J_2 \longrightarrow 0$ as $n \longrightarrow \infty$ by (3.4).

Notice that for convergence at x, the hypothesis of bounded variation is needed only in a neighborhood of x.

The proof of the second part of Theorem 2 will be omitted, except to note that it is sufficient to consider the $W_\alpha FS$ of $f(x)$, $f(x) = 0$ if $0 \leq x < a$, $f(x) = 1$ if $a < x \leq 1$, where a is an α-adic irrational. The partial sums of the $W_\alpha FS$ of $f(x)$ may be explicitly written in terms of the digits in the base α expansion of a, and the assertion follows directly.

Lemmas 2 and 3 provide a direct proof of the theorem of localization for $W_\alpha FS$.

THEOREM 3. *If $f(x) = g(x)$ a.e. for $a - \epsilon < x < a + \epsilon$, then the $W_\alpha FS$ of $f(x)$ and $g(x)$ are equiconvergent at a. If a is an α-adic rational it is sufficient that $f(x) = g(x)$ a.e. for $a < x < a + \epsilon$.*

LEMMA 4. *The kernel $D_k(x, t)$ satisfies*

$$\int_0^1 D_k(x, t)\,dt = 1, \tag{3.7}$$

and for $0 < t < 1$

$$|D_k(t)| < \alpha/t. \tag{3.8}$$

Proof. The first assertion is obvious.

For a proof of (3.8) the reader is referred to Fine's paper [**1**; pp. 391, 392].

THEOREM 4. *If for a fixed x,*

$$\frac{f(t) - c}{t - x} \in L(x - \delta, x + \delta) \text{ for some } \delta > 0,$$

then $s_n(x) \longrightarrow c$.

Proof. Suppose the base α expansion of x does not end in an infinite sequence of ones. Let z be determined by (2.1). Then we have, using (2.2) and (3.7)

$$s_n(x) - c = \int_{|t-x|<h<\delta} [f(t) - c] D_n(z \dotplus t)\, dt$$

$$+ \int_{|t-x|>h} [f(t) - c] D_n(x, t)\, dt = J_1 + J_2 .$$

One may verify that

$$|x - t| \leq \alpha (z \dotplus t). \tag{3.9}$$

Thus, with (3.8), we have

$$|J_1| \leq \alpha^2 \int_{|t-x|<h} \frac{|f(t) - c|}{|t - x|}\, dt < \epsilon$$

for h sufficiently small. With h fixed, $J_2 \longrightarrow 0$ by Theorem 3 and the remark below equation (3.6).

In case x is of the form excluded in the argument above, the proof must be modified. We put $z = z(x, n)$ where $z(x, n)$ is defined in § 2. Inequality (3.9) may not be satisfied on a set $F_n \subset (x - \delta,\ x + \delta)$. One may show that F_n is a subset of an interval of length α^{-n}, so

$$|J_1| \leq \alpha^2 \int_{|t-x|<h} \frac{|f(t) - c|}{|t - x|}\, dt + n \int_{F_n} |f(t) - c|\, dt = J_1' + J_1'' .$$

$J_1' < \epsilon$ as before, and with h fixed,

$$J_1'' \leq n\alpha^{-n} \int_{F_n} \frac{|f(t) - c|}{|t - x|}\, dt \longrightarrow 0$$

and $J_2 \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 1 and equation (2.2) provide a proof that

$$\int_0^1 |D_k(x, t)|\, dt = \int_0^1 |D_k(t)|\, dt \quad \text{for all} \quad x \in (0, 1).$$

We put $L_k = \int_0^1 |D_k(t)|\, dt$, the kth Lebesgue constant of the system Ψ_α.

LEMMA 5. *The Lebesgue constants satisfy* $L_k = O(\log k)$, *where the* O *depends upon* α.

Proof. By Lemma 4, $|D_k(t)| \le \min(\alpha/t, k)$. Thus

$$L_k \le \int_0^{\alpha/k} k\, dt + \int_{\alpha/k}^1 \alpha/t\, dt = O(\log k).$$

In the statement of the next theorem, $W(\delta; f)$ is the modulus of continuity of $f(x)$;

$$W(\delta; f) = \sup_{|h| \le \delta,\ 0 \le x < 1} |f(x+h) - f(x)|.$$

THEOREM 5. *If $f(x)$ satisfies $W(\delta; f) = o((\log \delta^{-1})^{-1})$ as $\delta \longrightarrow 0$, then $s_n(x) \longrightarrow f(x)$ uniformly.*

Proof. For this proof, write $n = a\alpha^k + k'$ where $0 < a < \alpha$, $0 \le k' < \alpha^k$. Since

$$s_n - s_{\alpha^k} = (s_n - s_{a\alpha^k}) + (s_{a\alpha^k} - s_{\alpha^k}) = S_1 + S_2,$$

it is sufficient to show that $S_1 \longrightarrow 0$ and $S_2 \longrightarrow 0$ uniformly. By using Lemma 2 and (3.3) we obtain

$$S_2 = \int_{I_{k,\rho}} [\phi_k(x)\, \overline{\phi}_k(t) + \cdots + \phi_k^{a-1}(x)\, \overline{\phi}_k^{a-1}(t)]\alpha^k f(t)\, dt,$$

where ρ is chosen so that $x \in I_{k,\rho}$. Since $f(x)$ is uniformly continuous, $S_2 \longrightarrow 0$ as $k \longrightarrow \infty$. Again using (3.3),

$$S_1 = \int_0^1 \phi_k^a(x)\, \overline{\phi}_k^a(t)\, D_{k'}(x, t)\, f(t)\, dt.$$

Replacing t by $t \dotplus b\alpha^{-k-1}$, we have

$$S_1 = \omega^{-ab} \int_0^1 \phi_k^a(x)\, \overline{\phi}_b^a(t) D_{k'}(x, t) f(t \dotplus b\alpha^{-k-1})\, dt,$$

so by subtraction

$$S_1(1 - \omega^{ab}) = \phi_k^a(x) \int_0^1 D_{k'}(x, t)\, \overline{\phi}_k^a(t)[f(t) - f(t \dotplus b\alpha^{-k-1})]\, dt.$$

If b is chosen so that $|1 - \omega^{ab}| \ge 3^{1/2}$, this becomes

$$|S_1|3^{1/2} \leq W(\alpha^{-k}; f)\, L_k' = o(1),$$

where we have used Lemma 5.

4. Fourier coefficients.

THEOREM 6. *If*

$$f(x) \sim \sum_0^\infty c_n \psi_n(x),$$

then

$$f(a \dotplus x) \sim \sum_0^\infty d_n \psi_n(x)$$

where $d_n = c_n \psi_n(a)$.

Proof. This is a consequence of Lemma 1 and the relation $\psi_n(a \dotplus x) = \psi_n(a)\, \psi_n(x)$, a.e.

By using Theorem 6 and the scheme from the proof of Theorem 5 we may establish the following.

THEOREM 7. *If*

$$f(x) \sim \sum_0^\infty c_j\, \psi_j(x),$$

then

$$|c_n| \leq 3^{1/2}\, W((\alpha - 1)/n; f).$$

There is a similar result with W replaced by the integral modulus of continuity.

As a corollary to Theorem 7 there is the following.

THEOREM 8. *If* $f(x) \in \mathrm{Lip}(\eta)$, *then* $c_n = O(n^{-\eta})$ *where the* O *depends upon* α.

For the next lemma we define

$$J_n(x) = \int_0^x \psi_n(t)\,dt$$

and we write $n = a\alpha^k + k'$, where $0 < a < \alpha$, $0 \leq k' < \alpha^k$.

LEMMA 6. *For $n \geq 0$ and all x,*

$$|J_n(x)| < n^{-1} \csc \pi/\alpha .$$

Proof. If $x \in I_{k,\rho}$ we have, from elementary properties of $\psi_n(x)$,

$$(4.1) \qquad |J_n(x)| = \left|\int_{\rho\alpha^{-k}}^x \psi_n(t)\,dt\right| = \left|\psi_{k'}(\rho\alpha^{-k})\int_{\rho\alpha^{-k}}^x \phi_k^a(t)\,dt\right| .$$

If τ is defined by the relation $x \in I_{k+1,\tau}$, we have by a direct calculation

$$\left|\int_{\rho\alpha^{-k}}^x \phi_k^a(t)\,dt\right| \leq \max\left\{\left|\int_{\rho\alpha^{-k}}^{\tau\alpha^{-k-1}} \phi_k^a(t)\,dt\right|, \left|\int_{\rho\alpha^{-k}}^{(\tau+1)\alpha^{-k-1}} \phi_k^a(t)\,dt\right|\right\}$$

$$\leq \max\left\{\alpha^{-k-1}\left|\frac{1-\omega^{a\tau}}{1-\omega}\right|, \alpha^{-k-1}\left|\frac{1-\omega^{a(\tau+1)}}{1-\omega}\right|\right\}$$

$$\leq \alpha^{-k-1} \csc \pi/\alpha < n^{-1} \csc \pi/\alpha .$$

THEOREM 9. *If $f(x)$ has total variation V then*

$$|c_n| \leq V n^{-1} \csc \pi/\alpha .$$

Proof. Since $J_n(0) = J_n(1) = 0$,

$$(4.2) \qquad c_n = -\int_0^1 \overline{J_n}(x)\,df(x),$$

and the theorem is now seen to be a consequence of Lemma 6.

For $\alpha = 2$, Theorem 9 was proved by N. J. Fine [1, p. 383] and in this case $\csc \pi/\alpha = 1$. That this factor is necessary when $\alpha > 2$ is seen from the following example. For an arbitrary positive integer k define $n = \alpha^{k+1} - 1$. Let β denote the integral part of $\alpha/2$ and put $\zeta = \beta\alpha^{-k-1}$ and $\xi = \zeta + \beta/\alpha$. Let $f(x)$ represent the characteristic function of the interval $[\zeta, \xi)$. By using (4.1) and (4.2) we may calculate c_k. It turns out that

$$|c_k| = [B(\alpha)/2]^2 \ \alpha^{-n-1} \csc \pi/\alpha \ V,$$

where $B(\alpha) = \max_{0 < b < \alpha} |1 - \omega^b|$ so that $3^{1/2} \leq B(\alpha) \leq 2$.

5. $(C, 1)$ **summability.** Let $\sigma_k(x)$ represent the kth $(C, 1)$ mean of $\{s_n(x)\}$, and define the kernel,

$$F_k(x, t) = k^{-1} \sum_1^k D_r(x, t).$$

We will write $F_k(0, t) = F_k(t)$.

LEMMA 7. *For $k \geq 1$, $\int_0^1 F_k(x, t)\,dt = 1$, and for $0 < t < 1$, $|F_k(t)| < \alpha/t$.*

Proof. These properties follow directly from the corresponding properties of $D_k(x, t)$.

LEMMA 8. *There is a constant M such that for all $k \geq 0$*

$$\int_0^1 |F_{\alpha^k}(x, t)|\,dt \leq M.$$

Proof. Write n in the form $n = a\alpha^k + k'$ where $0 < a < \alpha$ and $0 \leq k' \leq \alpha^k$. By a somewhat tedious calculation involving repeated use of (3.2) we obtain

$$(5.1) \quad nF_n(t) = [1 + \cdots + \bar{\phi}_k^{a-1}(t)]\alpha^k F_{\alpha^k}(t) + \bar{\phi}_k^a(t) k' F_{k'}(t)$$
$$+ \{1 + [1 + \bar{\phi}_k(t)] + \cdots + [1 + \cdots + \bar{\phi}_k^{a-2}(t)]\}\alpha^k D_{\alpha^k}(t)$$
$$+ [1 + \cdots + \bar{\phi}_k^{a-1}(t)] k' D_{\alpha^k}(t).$$

If we take $k' = \alpha^k$ and $a = \alpha - 1$ in (5.1) we obtain

$$(5.2) \qquad \alpha^{k+1} F_{\alpha^{k+1}}(t) = R_k(t)\,\alpha^k F_{\alpha^k}(t) + Q_k(t)\,\alpha^k D_{\alpha^k}(t)$$

where

$$(5.3) \qquad R_k(t) = \begin{cases} \alpha & \text{if } \phi_k(t) = 1, \\ 0 & \text{otherwise}, \end{cases}$$

and

(5.4) $$Q_k(t) = \begin{cases} \alpha(\alpha-1)/2 & \text{if } \phi_k(t) = 1, \\ \alpha/(1-\overline{\phi}_k(t)) & \text{otherwise}. \end{cases}$$

By applying a simple induction argument to (5.2) we obtain

(5.5) $$\alpha^{k+1} F_{\alpha^{k+1}}(t) = Q_k(t)\,\alpha^k D_{\alpha^k}(t) + \sum_{r=1}^{k} R_k(t) R_{k-1}(t) \cdots R_r(t) Q_{r-1}(t)\,\alpha^{r-1} D_{\alpha^{r-1}}(t) + \prod_{r=0}^{k} R_r(t).$$

Let

$$S = \sum_{r=1}^{\alpha-1} |1-\omega^r|^{-1},$$

then equations (5.3)-(5.5) enable us to show that

$$\alpha^{k+1}\int_0^1 |F_{\alpha^{k+1}}(t)|\,dt \le \alpha^k[(\alpha-1)/2+S]+1+[(\alpha-1)/2-S]\sum_1^k \alpha^{r-1},$$

from which the lemma follows.

Observe that by setting $k=0$ in (5.2) we see that for $\alpha > 2$ the kernels $F_{\alpha^k}(t)$ are not positive. Fine showed that in case $\alpha = 2$, $F_{\alpha^k}(t) \ge 0$ [**1**, p. 396].

LEMMA 9. *If t is not of the form $t = d\alpha^{-m}$, $m \ge 1$, $0 \le d < \alpha$, then* $\lim_{k\to\infty} F_k(t) = 0$.

Proof. Let t be given and choose n so that $\alpha^{-n} < t < \alpha^{-n+1}$. Write $k = p\alpha^n + q$ where $0 \le q < \alpha^n$. Then

$$kF_k(t) = \sum_{r=0}^{p-1}\sum_{s=1}^{\alpha^n} D_{r\alpha^n+s}(t) + \sum_{s=1}^{q} D_{p\alpha^n+s}(t).$$

One can show that $D_{r\alpha^n+s}(t) = D_{\alpha^n}(t)D_r(\alpha^n t) + \psi_r(\alpha^n t)D_s(t)$. This gives

$$D_{r\alpha^n+s}(t) = \overline{\psi}_r(\alpha^n t)D_s(t),$$

so that

$$kF_k(t) = \alpha^n F_{\alpha^n}(t) D_p(\alpha^n t) + \overline{\psi}_p(\alpha^n t) qF_q(t).$$

Put b equal to the integral part of $\alpha^n t$. Since $0 < \alpha^n t - b < 1$, we have by Lemma 4

$$|D_p(\alpha^n t)| \le \alpha(\alpha^n t - b)^{-1}.$$

Using Lemma 7 we obtain

$$|kF_k(t)| \le \alpha^{n-2} t^{-1}(\alpha^n t - b)^{-1} + q\alpha t^{-1},$$

from which the conclusion follows.

THEOREM 10. *If $f(x)$ is continuous then $\sigma_{\alpha^k}(x) \longrightarrow f(x)$ uniformly.*

Proof. It follows from (2.3) and Lemma 7 that

$$(5.6) \qquad \sigma_n(x) - f(x) = \int_0^1 F_n(t)[f(x \dotplus t) - f(x)]dt.$$

By applying Lemmas 7-9 together with a standard argument we can show that

$$\int_0^1 |F_{\alpha^k}(t)|\ |f(x \dotplus t) - f(x)|\, dt \longrightarrow 0 \text{ uniformly}.$$

THEOREM 11. *If $f(x)$ is continuous then $\sigma_n(x) \longrightarrow f(x)$ uniformly.*

Proof. Let the base α expansion of n be given in Definition 2. From (5.1) we obtain the estimate

$$(5.7) \qquad |nF_n(t)| \le \sum_{r=1}^{m} \{a_r \alpha^{n_r} |F_{\alpha^{n_r}}(t)| + \frac{1}{2} a_r(a_r + 1)\alpha^{n_r} D_{\alpha^{n_r}}(t)\}.$$

Let $\epsilon_k = \epsilon_k(x)$ represent the larger of

$$\int_0^1 |F_{\alpha^k}(t)|\ |f(x \dotplus t) - f(x)|\, dt$$

and

$$\int_0^1 D_{\alpha k}(t)\,|f(x \dotplus t) - f(x)|\,dt,$$

so that by Theorems 1 and 10 $\epsilon_k \longrightarrow 0$ uniformly. Using (5.6) and (5.7)

$$|\sigma_n(x) - f(x)| \leq \alpha \sum_{r=1}^{m} a_r \alpha^{n_r} n^{-1} \epsilon_{n_r} = \delta_n, \text{ say.}$$

One may readily verify that the transformation which sends $\{\epsilon_k\}$ into $\{\delta_n\}$ is regular, so that $\delta_n \longrightarrow 0$ uniformly, and the theorem is proved.

It is interesting to note that by virtue of a well known consequence of the Banach-Steinhaus theorem [**5**, p. 99], Theorem 11 implies that $\int_0^1 |F_n(t)|\,dt \leq M$.

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Pacific Journal of Mathematics

Vol. 5, No. 1 September, 1955